Generating Functions and Partitions

Vanderbilt Math Circle

February 11, 2019
Generating Functions

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Another way to think about it: “Infinite polynomial”
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Example: $1 + x + x^2 + x^3 + \ldots$
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- Example: \( 1 + x + x^2 + x^3 + \ldots = GF(1) \)
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- Example: $GF(n)$
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- Example: $GF(2^{-n})$
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Applications:
- create formulas,
- make estimations,
- establish divisibility properties,
- and more
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A problem to work on

**Question:** What is the value of 0.9999...?
A problem to work on

**Question:** What is the value of $0.9999\ldots$?

**Answer:** Set $x = 0.9999\ldots$. 

Do you believe this?
A problem to work on

**Question:** What is the value of 0.9999...?

**Answer:** Set \( x = 0.9999... \). Then \( 10x = 9.999... \),
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**Question:** What is the value of 0.9999...?

**Answer:** Set $x = 0.9999...$. Then $10x = 9.999...$, so $10x - 9 = 0.999\cdots = x$. 
Question: What is the value of 0.9999...?

Answer: Set $x = 0.9999....$ Then $10x = 9.999...$, so $10x - 9 = 0.999... = x$. Solving for $x$ tells us that $x = 1$. 
A problem to work on

**Question:** What is the value of 0.9999 . . . ?

**Answer:** Set \( x = 0.9999 . . . \). Then \( 10x = 9.999 . . . \), so \( 10x - 9 = 0.999 . . . = x \). Solving for \( x \) tells us that \( x = 1 \).

Do you believe this?
Finding Infinite Sums

Problem 1:
\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

Problem 2:
\[ 1 + a + a^2 + a^3 + \ldots \]

Does your answer for Problem 2 always work?

Problem 3:
\[ \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}} \]

Problem 4:
\[ 2 + \frac{1}{1 + 1} + \frac{1}{2 + 1} + \frac{1}{1 + 1} + \frac{1}{2 + 1} + \ldots \]
Finding Infinite Sums

Problem 1: \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \)
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\[
2 \div \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \ldots}}}}}}
\]
Recap

From Problem 2, we have: $GF(1) = \frac{1}{1-x}$. 
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- Remember:
Recap

From Problem 2, we have: $GF(1) = \frac{1}{1-x}$.

*Remember:* This is formal.
Rabbit problem

It takes one month for rabbits to mature, and after they have matured, every pair of rabbits produces another pair of rabbits, one boy and one girl.
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If we start with a pair of baby rabbits,

- how many pairs of rabbits are there after one month?
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- Example: $\frac{F_{10}}{F_9} = \frac{55}{34} \approx 1.6176470588 \ldots$
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- **Example:** $\frac{F_{100}}{F_{99}} = \frac{354224848179261915075}{218922995834555169026} \approx 1.6180339887 \ldots$
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The ratio of Fibonacci numbers $F_{n+1}/F_n$ approaches the golden ratio:
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The ratio of Fibonacci numbers $F_{n+1}/F_n$ approaches the golden ratio:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887 \cdots$$
Formula for $F_n$

Let $f(x) = GF(F_{n+1}) = F_1 + F_2x + F_3x^2 + F_4x^3 + \ldots$. 
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**Question:** Write down:
- $xf(x)$

What do you notice?

What is $f(x) - xf(x) - x^2f(x)$?

We can solve and get $GF(F_{n+1}) = f(x) = x^1 - x^2 - x^3$. 

What does this tell us about $F_n$?
Formula for $F_n$

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- How do we factor $1 - x - x^2$?
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- What are its roots?
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Let $\phi = \frac{1 + \sqrt{5}}{2}$, $\phi' = \frac{1 - \sqrt{5}}{2}$. Then

$$GF(F_n) = \frac{x}{1 - x - x^2} = \frac{x}{(1 - \phi x)(1 - \phi' x)}$$
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Comparing like terms on each side:
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Comparing like terms on each side:

$$F_{n+1} = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).$$
Lucas Numbers

Lucas numbers are defined similar to Fibonacci numbers:

\[
L_1 = 1, \quad L_2 = 3, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 3.
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Problem: Find a formula for \( L_n \) in the same way as for Fibonacci numbers.
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Fibonacci Problems

We have a formula for $F_n$. 

Fibonacci Problems

We have a formula for $F_n$. Does that tell us everything about these numbers?
Fibonacci Problems

We have a formula for $F_n$.

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**Open Problem:** Show that there are infinitely many $F_n$ that are prime.
Fibonacci Problems

We have a formula for $F_n$.

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**Open Problem:** Show that there are infinitely many $F_n$ that are prime.

**Recently Solved Problem:** Show that there are finitely many $F_n$ that are of the form $a^b$ for integers $a, b$ with $b > 1$. 
Partition Function $p(n)$

A *partition* of a positive integer $n$ is a way of writing $n$ as a sum of positive integers.
Partition Function \( p(n) \)

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- *Example:* \( 4 + 1 \) is a partition of 5
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- *Example:* $1 + 4$ is the same partition
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$4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$

$p(n)$ is the number of partitions of *n*, so $p(4) = 5$. 
Partition Function $p(n)$

**Problem:** Find:

- $p(0)$
- $p(1) = 1$
- $p(2) = 2$
- $p(3) = 3$
- $p(4) = 5$

Look at the sequence $p(n)$ so far. Is there a pattern? What is $p(5)$?
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- What is $p(5)$?
More partitions

**Problem:** What is $p(7)$?
More partitions

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More partitions

**Problem:** What is $p(7)$? $p(9)$? $p(99)$?
More partitions

**Problem:** What is $p(7)$? $p(9)$? $p(99)$?

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More partitions

**Problem:** What is $p(7)$? $p(9)$? $p(99)$?

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</tr>
</tbody>
</table>

Do you notice anything about these values?
Euler Products

Figure: Leonard Euler (1707-1783)
Euler’s Product

Generating functions consider formal infinite sums.
Euler’s Product

Generating functions consider formal infinite sums.

- What about infinite products?
Euler’s Product

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- What does $\prod_{n=1}^{\infty} (1 + q^n)$ mean?
Euler’s Product

Generating functions consider formal infinite sums.

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Let’s start by writing down finite products:
Euler’s Product

Generating functions consider formal infinite sums.

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\[
\prod_{n=1}^{1} (1 + q^n)
\]

\[
\prod_{n=1}^{2} (1 + q^n)
\]

\[
\prod_{n=1}^{3} (1 + q^n)
\]
Euler’s Product

Generating functions consider formal infinite sums.

- What about infinite products?
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Let’s start by writing down finite products:

- $\prod_{n=1}^{1} (1 + q^n) = 1 + q$
Euler’s Product

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Let’s start by writing down finite products:

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Euler’s Product

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- \( \prod_{n=1}^{2} (1 + q^n) = 1 + q + q^2 + q^3 \)
- \( \prod_{n=1}^{3} (1 + q^n) = 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6 \)
Euler’s Product

**Question:** How do we determine the coefficient of $q^{10}$ for \( \prod_{n=1}^{\infty} (1 + q^n) \) without multiplying out the first 10 terms?
Euler’s Product

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\[
\prod_{n=1}^{\infty} (1 + q^n)
\]

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**Counting Interpretation:** How many partitions are there of 10 with each number in the sum distinct?
**Euler’s Product**

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10, 9 + 1,
Euler’s Product

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\[
10, 9 + 1, 8 + 2,
\]
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**Question:** How do we determine the coefficient of $q^{10}$ for 
\[
\prod_{n=1}^{\infty} (1 + q^n)
\] without multiplying out the first 10 terms?

**Counting Interpretation:** How many partitions are there of 10 with each number in the sum distinct?

10, $9 + 1$, $8 + 2$, $7 + 3$, 
...
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$$\prod_{n=1}^{\infty} (1 + q^n)$$ without multiplying out the first 10 terms?

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$$10, 9 + 1, 8 + 2, 7 + 3, 7 + 2 + 1,$$
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10, 9 + 1, 8 + 2, 7 + 3, 7 + 2 + 1, 6 + 4, 6 + 3 + 1

Hence, \[\prod_{n=1}^{\infty} (1 + q^n) = 1 + q + \cdots + 7q^{10} + \ldots.\]