Generating Functions and Partitions

Vanderbilt Math Circle

February 11, 2019

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Applications:

- create formulas,
- make estimations,
- establish divisibility properties,
- and more

A problem to work on

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Do you believe this?

Finding Infinite Sums

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Does your answer for Problem 2 always work?

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Problem 1: $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

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Does your answer for Problem 2 always work?

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Finding Infinite Sums

Problem 1: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Problem 2: $1 + a + a^2 + a^3 + \dots$
Does your answer for Problem 2 always work?

Problem 3: $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$

Problem 4:

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\dots}}}}}$$

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From Problem 2, we have: $GF(1) = \frac{1}{1-x}$.

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From Problem 2, we have: $GF(1) = \frac{1}{1-x}$.

- *Remember:* This is formal.

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The *ratio* of Fibonacci numbers F_{n+1}/F_n approaches the **golden ratio**:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$$

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What do you notice?

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We can solve and get $GF(F_{n+1}) = f(x) = \frac{x}{1-x-x^2}$.

- What does this tell us about F_n?

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&= \frac{1}{\phi - \phi'}\left(\frac{1}{1-\phi x} - \frac{1}{1-\phi' x}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(1+\phi x+\phi^2 x^2+\dots\right) - \left(1+\phi' x+(\phi')^2 x^2+\dots\right)\right).
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Comparing like terms on each side:

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Comparing like terms on each side:

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

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Problem: Find a formula for L_n in the same way as for Fibonacci numbers.

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Recently Solved Problem: Show that there are finitely many F_n that are of the form a^b for integers a, b with $b > 1$.

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$$4, 3+1, 2+2, 2+1+1, 1+1+1+1$$

$p(n)$ is the number of partitions of n, so $p(4)=5$.

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Look at the sequence $p(n)$ so far.

- Is there a pattern?

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Look at the sequence $p(n)$ so far.

- Is there a pattern?
- What is $p(5)$?

More partitions

Problem: What is $p(7)$?

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n	$p(n)$	n	$p(n)$
4	5	54	386155
9	30	59	831820
14	135	64	1741630
19	490	69	3554345
24	1575	74	7089500
29	4565	79	13848650
34	12310	84	26543660
39	31185	89	49995925
44	75175	94	92669720
49	173525	99	169229875

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19	490	69	3554345
24	1575	74	7089500
29	4565	79	13848650
34	12310	84	26543660
39	31185	89	49995925
44	75175	94	92669720
49	173525	99	169229875

Do you notice anything about these values?

Euler Products

Figure: Leonard Euler (1707-1783)

Euler's Product

Generating functions consider formal infinite sums.

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- $\prod_{n=1}^{1}(1+q^n) = 1+q$

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- $\prod_{n=1}^{1}(1+q^n) = 1+q$
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Generating functions consider formal infinite sums.

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Let's start by writing down finite products:

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Hence, $\prod_{n=1}^{\infty}(1+q^n) = 1 + q + \cdots + 7q^{10} + \ldots.$