1 Take-Home Problems

1. Prove that \( p(n) < p(n+1) \) for all \( n \geq 1 \).

2. a). If \( \lambda \) is a partition of \( n \) with no distinct parts (in other words, every part that appears in the sum shows up at least twice), what can we say about its conjugate? (Hint: Think about the example where we found that having one as a part meant that the conjugate had a distinct largest part and vice versa. Try to find a similar meaning here.)
   b). Write down a product formula that gives the generating function for the number of partitions of \( n \) with no distinct parts.

3. An overpartition of a number \( n \) is a partition of \( n \) where the first occurrence of any part may be overlined. For example, the overpartitions of 2 are
   \[ 2, \bar{2}, 1 + 1, \bar{1} + 1, \]
   and the overpartitions of 3 are
   \[ 3, \bar{3}, 2 + 1, 2 + \bar{1}, 2 + \bar{1}, 2 + 1, 1 + 1, 1 + 1, 1 + 1. \]
   a). What are the overpartitions of 4?
   b). Let \( \bar{p}(n) \) be the number of overpartitions of \( n \). What is the generating function for \( \bar{p}(n) \)? (Hint: Recall how we found the generating function for \( p(n) \).)

4. A partition \( \lambda \) is said to be self-conjugate if the conjugate of \( \lambda \) is \( \lambda \) itself. Using Ferrers Diagrams, prove that the number of self-conjugate partitions of a number \( n \) is the same as the number of partitions of \( n \) into distinct odd parts. (Hint: try “folding” Ferrers diagrams of partitions with distinct odd parts.)

5. Let \( n \) be a positive whole number, and for any number \( x \), let \([x]\) be the largest whole number less than or equal to \( x \). For instance, \( \pi = 3.14..., \) so \([\pi]\) = 3, and \([5]\) = 5 since 5 is already a whole number.
   a). Find a formula for the sum of the numbers \( 1 + 2 + ... + [\sqrt{n}] \). Conclude that this sum is less than \( n \).
b). Let $S = \{s_1, s_2, \ldots, s_k\}$ be a collection of numbers from the set $\{1, 2, \ldots, [\sqrt{n}]\}$. Show that there is a partition of $n$ which starts with $s_1 + s_2 + \ldots + s_k$. For instance, if $n = 101$, so that $[\sqrt{101}] = 10$, if $S$ is the collection $S = 1, 3, 4, 9$ of whole numbers less than or equal to 10, then $1 + 3 + 4 + 9 + 84$ is a partition of 101 which “starts” with $1 + 3 + 4 + 9$.

c). Recall from an earlier math club meeting that the number of collections of numbers from $\{1, 2, \ldots, [\sqrt{n}]\}$, that is, the number of subsets, is $2^{[\sqrt{n}]}$. Conclude that the number of partitions of $n$ grows exponentially with $n$, and give a function $f(n)$ such that $f(n) < p(n)$.

d). Can you find better approximations to $p(n)$? How many partitions are “missed” by this procedure? Can you find an upper bound for $p(n)$, that is, a function $g(n)$ such that $p(n)$ is always less than $g(n)$?