Partitions, part 2

Vanderbilt Math Circle

February 25, 2019

Generating Functions

"Infinite polynomials"

Generating Functions

"Infinite polynomials"

- $GF(1) = 1 + x + x^2 + x^3 + \cdots$

Generating Functions

"Infinite polynomials"

- $GF(1) = 1 + x + x^2 + x^3 + \cdots$
- $GF(2^{-n}) = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots$

Generating Functions

"Infinite polynomials"

- $GF(1) = 1 + x + x^2 + x^3 + \cdots$
- $GF(2^{-n}) = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots$
- $GF(F_{n+1}) = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$

Partitions

A *partition* is a way of writing a positive number as a sum of positive numbers.

Partitions

A *partition* is a way of writing a positive number as a sum of positive numbers.

Partitions of 4:

Partitions

A *partition* is a way of writing a positive number as a sum of positive numbers.

Partitions of 4:

$$4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$$

Partitions

A *partition* is a way of writing a positive number as a sum of positive numbers.

Partitions of 4:

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1$$

Hence, $p(4) = 5$.

Partitions

n	$p(n)$	n	$p(n)$
4	5	54	386155
9	30	59	831820
14	135	64	1741630
19	490	69	3554345
24	1575	74	7089500
29	4565	79	13848650
34	12310	84	26543660
39	31185	89	49995925
44	75175	94	92669720
49	173525	99	169229875

Infinite Products

We considered infinite products like

$$\prod_{i=1}^{\infty}(1+q^n)$$

by writing down finite products first.

Infinite Products

We considered infinite products like

$$\prod_{i=1}^{\infty}(1+q^n)$$

by writing down finite products first.

- $\prod_{n=1}^{1}(1+q^n) = 1+q$
- $\prod_{n=1}^{2}(1+q^n) = 1+q+q^2+q^3$
- $\prod_{n=1}^{3}(1+q^n) = 1+q+q^2+2q^3+q^4+q^5+q^6$

Infinite Products

We considered infinite products like

$$\prod_{i=1}^{\infty}(1+q^n)$$

by writing down finite products first.

- $\prod_{n=1}^{1}(1+q^n) = 1+q$
- $\prod_{n=1}^{2}(1+q^n) = 1+q+q^2+q^3$
- $\prod_{n=1}^{3}(1+q^n) = 1+q+q^2+2q^3+q^4+q^5+q^6$

How did we find the 10^{th} coefficient of the infinite product?

Take Home Questions

Ferrers Diagrams

Goal:

Ferrers Diagrams

Goal: Represent partitions by pictures.

Ferrers Diagrams

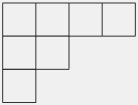
Goal: Represent partitions by pictures.

Example: $4 + 2 + 1$ is a partition of 7.

Ferrers Diagrams

Goal: Represent partitions by pictures.

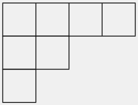
Example: $4+2+1$ is a partition of 7.



Ferrers Diagrams

Goal: Represent partitions by pictures.

Example: $4 + 2 + 1$ is a partition of 7.



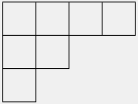
Problems:

- Draw a Ferrers diagram for the partition $8 + 4 + 2 + 2 + 1 + 1$ of 18.

Ferrers Diagrams

Goal: Represent partitions by pictures.

Example: $4 + 2 + 1$ is a partition of 7.



Problems:

- Draw a Ferrers diagram for the partition $8 + 4 + 2 + 2 + 1 + 1$ of 18.
- Draw all Ferrers diagrams for partitions of 6.

Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

Creating New Diagrams

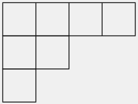
Open-ended Question: How do we create new diagrams/partitions from a given one?

Example: Move a block from the last row to the end of the first row.

Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

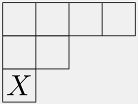
Example: Move a block from the last row to the end of the first row.



Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

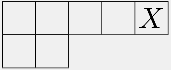
Example: Move a block from the last row to the end of the first row.



Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

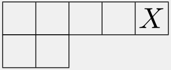
Example: Move a block from the last row to the end of the first row.



Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

Example: Move a block from the last row to the end of the first row.

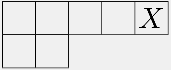


- Does this always give a new partition?

Creating New Diagrams

Open-ended Question: How do we create new diagrams/partitions from a given one?

Example: Move a block from the last row to the end of the first row.



- Does this always give a new partition?
- Is this process "invertible"?

Creating New Diagrams

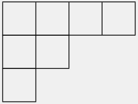
Another Example:

Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.

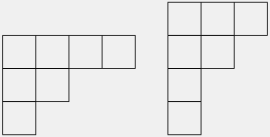
Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.



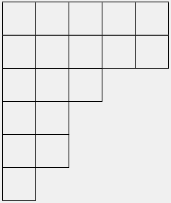
Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.



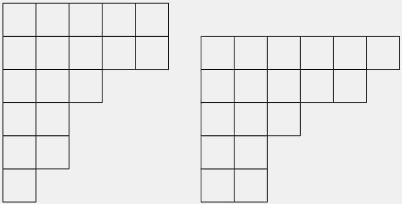
Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.



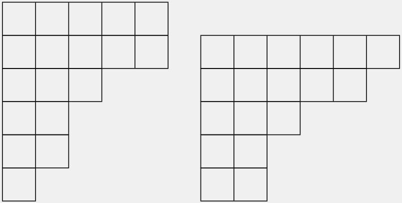
Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.



Creating New Diagrams

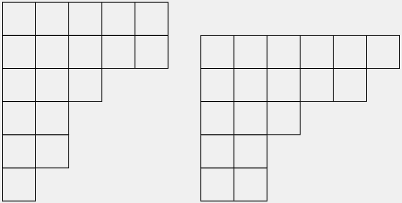
Another Example: Turn the rows into columns and vice versa.



- Is this process "invertible"?

Creating New Diagrams

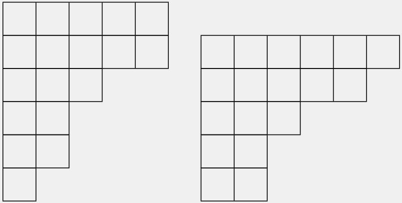
Another Example: Turn the rows into columns and vice versa.



- Is this process "invertible"?
- Does this always give a new partition?

Creating New Diagrams

Another Example: Turn the rows into columns and vice versa.



- Is this process “invertible”?
- Does this always give a new partition?

We call this process **conjugation**.

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?
- Why do we care about this?

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?
- Why do we care about this?

Similar Question: What about if a partition has at most five parts? What does this tell us about the conjugate?

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?
- Why do we care about this?

Similar Question: What about if a partition has at most five parts? What does this tell us about the conjugate?

- Draw some examples to try to figure it out.

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?
- Why do we care about this?

Similar Question: What about if a partition has at most five parts? What does this tell us about the conjugate?

- Draw some examples to try to figure it out.
- If the parts of a partition are at most five, does its conjugate have at most five parts?

Conjugation

Question: If a partition has a one in it, what does this mean about the conjugate partition?

- Draw some examples to try to figure it out.
- If a partition has a unique largest part, does this mean the conjugate has a one in it?
- Why do we care about this?

Similar Question: What about if a partition has at most five parts? What does this tell us about the conjugate?

- Draw some examples to try to figure it out.
- If the parts of a partition are at most five, does its conjugate have at most five parts?
- What does this tell us about certain types of partitions?

Partition Properties

Instead of saying $4 + 2 + 1$ is a partition of 7, we can also say $\lambda = (4, 2, 1)$ is a partition of 7.

Partition Properties

Instead of saying $4 + 2 + 1$ is a partition of 7, we can also say $\lambda = (4, 2, 1)$ is a partition of 7. This allows us to define properties of partitions more easily.

Partition Properties

Instead of saying $4+2+1$ is a partition of 7, we can also say $\lambda = (4, 2, 1)$ is a partition of 7. This allows us to define properties of partitions more easily.

Example: $\#(\lambda)$ denotes the number of parts of λ.

Partition Properties

Instead of saying $4 + 2 + 1$ is a partition of 7, we can also say $\lambda = (4, 2, 1)$ is a partition of 7. This allows us to define properties of partitions more easily.

Example: $\#(\lambda)$ denotes the number of parts of λ.

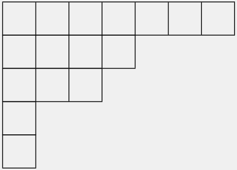
- What is $\#(4, 4, 3, 3, 3, 1)$?

Partition Properties

Instead of saying $4+2+1$ is a partition of 7, we can also say $\lambda = (4,2,1)$ is a partition of 7. This allows us to define properties of partitions more easily.

Example: $\#(\lambda)$ denotes the number of parts of λ.

- What is $\#(4,4,3,3,3,1)$?
- What is $\#(\lambda)$ if λ is represented by the following Ferrers Diagram?



Partition Properties

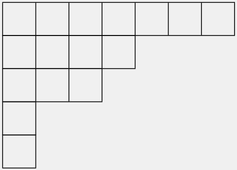
Example: $\ell(\lambda)$ denotes the largest part of λ.

- What is $\ell(4,4,3,3,3,1)$?

Partition Properties

Example: $\ell(\lambda)$ denotes the largest part of λ.

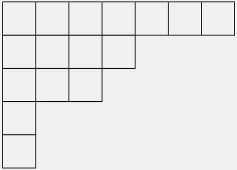
- What is $\ell(4, 4, 3, 3, 3, 1)$?
- What is $\ell(\lambda)$ if λ is represented by the following Ferrers Diagram?



Partition Properties

Example: $\ell(\lambda)$ denotes the largest part of λ.

- What is $\ell(4, 4, 3, 3, 3, 1)$?
- What is $\ell(\lambda)$ if λ is represented by the following Ferrers Diagram?



We can restate what we noticed earlier: the number of partitions λ of n with $\ell(\lambda) \leq 5$ is the same as the number of partitions λ' of n with $\#(\lambda') \leq 5$.

Further Partition Properties

Example: $s(\lambda)$ denotes the smallest part of λ.

Further Partition Properties

Example: $s(\lambda)$ denotes the smallest part of λ.

Consider partitions with distinct parts.
Example: $\sigma(\lambda)$ denotes the number of consecutive parts of a partition at the beginning of the partition.

Further Partition Properties

Example: $s(\lambda)$ denotes the smallest part of λ.

Consider partitions with distinct parts.
Example: $\sigma(\lambda)$ denotes the number of consecutive parts of a partition at the beginning of the partition.

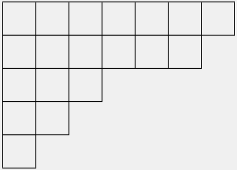
- What is $\sigma(8, 7, 6, 5, 3, 2, 1)$?

Further Partition Properties

Example: $s(\lambda)$ denotes the smallest part of λ.

Consider partitions with distinct parts.
Example: $\sigma(\lambda)$ denotes the number of consecutive parts of a partition at the beginning of the partition.

- What is $\sigma(8, 7, 6, 5, 3, 2, 1)$?
- What is $\sigma(\lambda)$ if λ is represented by the following Ferrers Diagram?



Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

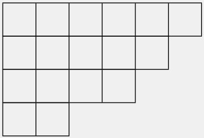
If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.

Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.

Example:

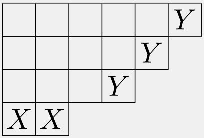


Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.

Example:

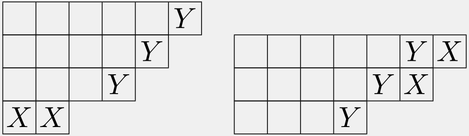


Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.

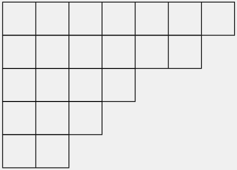
Example:



Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

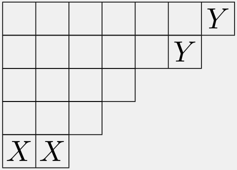
If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.
Example:



Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.
Example:

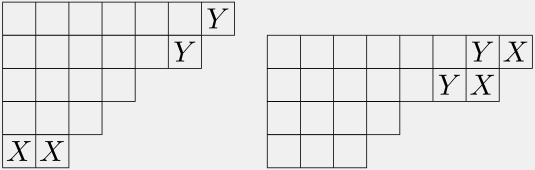


Another Way of Creating Diagrams

Let's continue to only consider partitions with distinct parts.

If $s(\lambda) \leq \sigma(\lambda)$, we can move the bottom row to the ends of the first rows.

Example:



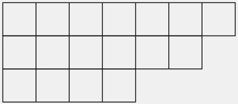
Another Way of Creating Diagrams

If $s(\lambda) > \sigma(\lambda)$, we can do "the opposite".

Another Way of Creating Diagrams

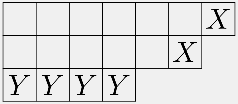
If $s(\lambda) > \sigma(\lambda)$, we can do "the opposite".

Example:



Another Way of Creating Diagrams

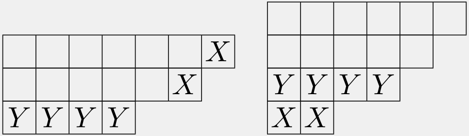
If $s(\lambda) > \sigma(\lambda)$, we can do “the opposite”.
Example:



Another Way of Creating Diagrams

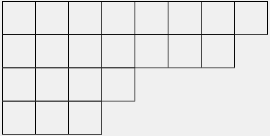
If $s(\lambda) > \sigma(\lambda)$, we can do “the opposite”.

Example:



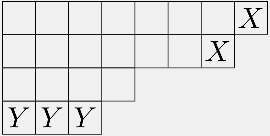
Another Way of Creating Diagrams

If $s(\lambda) > \sigma(\lambda)$, we can do “the opposite”.
Example:



Another Way of Creating Diagrams

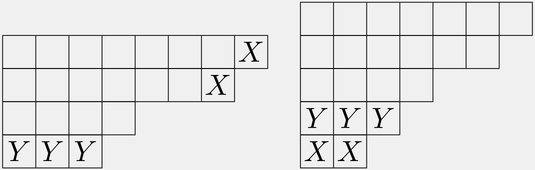
If $s(\lambda) > \sigma(\lambda)$, we can do “the opposite”.
Example:



Another Way of Creating Diagrams

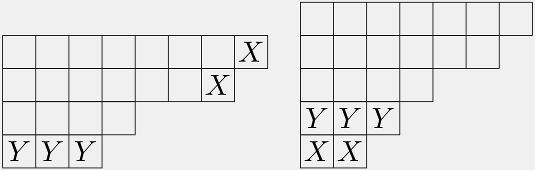
If $s(\lambda) > \sigma(\lambda)$, we can do “the opposite”.

Example:



Another Way of Creating Diagrams

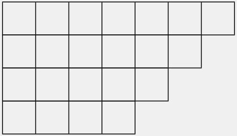
If $s(\lambda) > \sigma(\lambda)$, we can do "the opposite".
Example:



- Does this process always work?

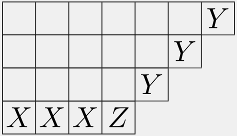
Pentagonal Numbers

Let's try this process on:



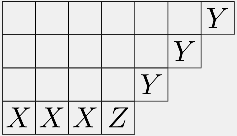
Pentagonal Numbers

Let's try this process on:



Pentagonal Numbers

Let's try this process on:



What goes wrong when we try to move the row?

Pentagonal Numbers

Let's try this process on:

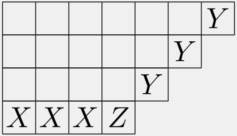
						Y
					Y	
				Y		
X	X	X	Z			

What goes wrong when we try to move the row?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r$ for some r.

Pentagonal Numbers

Let's try this process on:



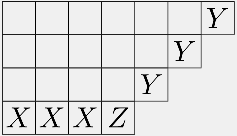
What goes wrong when we try to move the row?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r$ for some r. In which case,

$$n = r + (r+1) + \cdots + (2r-1)$$

Pentagonal Numbers

Let's try this process on:



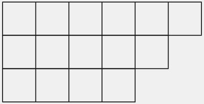
What goes wrong when we try to move the row?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r$ for some r. In which case,

$$n = r + (r+1) + \cdots + (2r-1) = \frac{1}{2}r(3r-1).$$

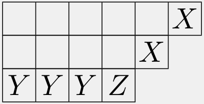
Pentagonal Numbers

Similarly, we try the process on:



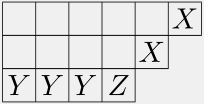
Pentagonal Numbers

Similarly, we try the process on:



Pentagonal Numbers

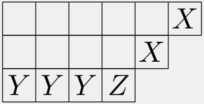
Similarly, we try the process on:



What goes wrong when we try to move the end elements?

Pentagonal Numbers

Similarly, we try the process on:



What goes wrong when we try to move the end elements?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$ for some r.

Pentagonal Numbers

Similarly, we try the process on:

					X
				X	
Y	Y	Y	Z		

What goes wrong when we try to move the end elements?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$ for some r. In which case,

$$n = (r + 1) + (r + 2) + \cdots + 2r$$

Pentagonal Numbers

Similarly, we try the process on:

					X
				X	
Y	Y	Y	Z		

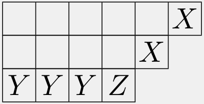
What goes wrong when we try to move the end elements?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$ for some r. In which case,

$$n = (r + 1) + (r + 2) + \cdots + 2r = \frac{1}{2}r(3r + 1).$$

Pentagonal Numbers

Similarly, we try the process on:



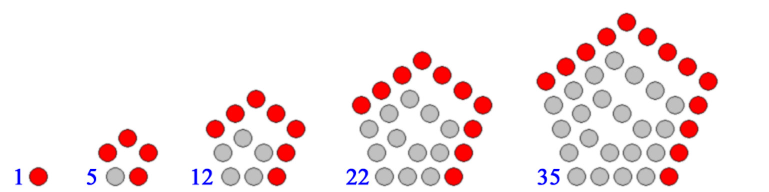
What goes wrong when we try to move the end elements?

This happens exactly when $\sigma(\lambda) = r$ and $s(\lambda) = r + 1$ for some r. In which case,

$$n = (r+1) + (r+2) + \cdots + 2r = \frac{1}{2}r(3r+1).$$

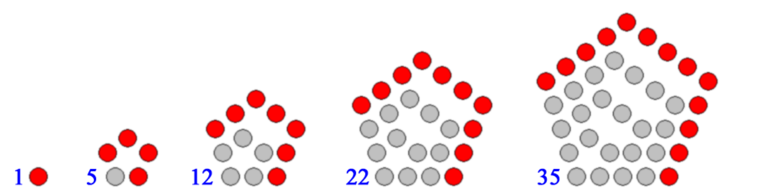
Numbers of the form $\frac{1}{2}r(3r \pm 1)$ are called **pentagonal numbers**.

Another Way of Thinking of Pentagonal Numbers



Source: (Author: Aldoaldoz)

Another Way of Thinking of Pentagonal Numbers



Source: (Author: Aldoaldoz)
Formula:

$$\frac{r \cdot (3r-1)}{2}.$$

Thinking Again about Infinite Products

- What does this teach us about partitions?

Thinking Again about Infinite Products

- What does this teach us about partitions?
- Recall that the generating function for $p(n)$ is

$$f(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}.$$

Thinking Again about Infinite Products

- What does this teach us about partitions?
- Recall that the generating function for $p(n)$ is

$$f(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}.$$

- Thus,

$$\frac{1}{f(q)} = (1-q)(1-q^2)(1-q^3)\dots,$$

and so

Thinking Again about Infinite Products

- What does this teach us about partitions?
- Recall that the generating function for $p(n)$ is

$$f(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}.$$

- Thus,

$$\frac{1}{f(q)} = (1-q)(1-q^2)(1-q^3)\dots,$$

and so

$$f(q) \cdot \left[(1-q)(1-q^2)(1-q^3)\dots\right] = 1 + 0q + 0q^2 + \dots.$$

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

- What would an interpretation for

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

  be?

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

- What would an interpretation for

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

  be?
- **Exercise:** Find the number of partitions of 11 into distinct parts.

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

- What would an interpretation for

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

be?

- **Exercise:** Find the number of partitions of 11 into distinct parts. How many of these have an even number of parts? How many of these have an odd number of parts?

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

- What would an interpretation for

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

  be?
- **Exercise:** Find the number of partitions of 11 into distinct parts. How many of these have an even number of parts? How many of these have an odd number of parts?
- **Exercise:** Find the number of partitions of 12 into distinct parts.

The Pentagonal Number Theorem

- Last time, we talked about a "counting" interpretation of

$$(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots.$$

- What would an interpretation for

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

  be?
- **Exercise:** Find the number of partitions of 11 into distinct parts. How many of these have an even number of parts? How many of these have an odd number of parts?
- **Exercise:** Find the number of partitions of 12 into distinct parts. How many of these have an even number of parts? How many of these have an odd number of parts?

The Pentagonal Number Theorem

- What does this phenomenon tell us?

The Pentagonal Number Theorem

- What does this phenomenon tell us?
- **Answer:** The n^{th} coefficient of

$$(1-q)(1-q^2)(1-q^3)\dots$$

is 0 except for pentagonal numbers.

The Pentagonal Number Theorem

- What does this phenomenon tell us?
- **Answer:** The n^{th} coefficient of

$$(1-q)(1-q^2)(1-q^3)\dots$$

  is 0 except for pentagonal numbers.
- The coefficient for the pentagonal number $\frac{1}{2}r(3r \pm 1)$ is $(-1)^r$.

The Pentagonal Number Theorem

- What does this phenomenon tell us?
- **Answer:** The n^{th} coefficient of

$$(1-q)(1-q^2)(1-q^3)\dots$$

  is 0 except for pentagonal numbers.
- The coefficient for the pentagonal number $\frac{1}{2}r(3r \pm 1)$ is $(-1)^r$.
- In other words,

$$(1-q)(1-q^2)(1-q^3)\dots = 1 - q - q^2 + q^5 + q^7 - \dots.$$

Recursive Relation for $p(n)$

- Combining all the steps above,

  $$(1 + p(1) + p(2)q^2 + p(3)q^3 + \ldots)\cdot(1 - q - q^2 + q^5 + q^7 - \ldots) = 1$$

  where $1, 2, 5, 7, 12, 15, \ldots$ are the pentagonal numbers.

Recursive Relation for $p(n)$

- Combining all the steps above,

$$(1 + p(1) + p(2)q^2 + p(3)q^3 + \ldots)\cdot(1 - q - q^2 + q^5 + q^7 - \ldots) = 1$$

where $1, 2, 5, 7, 12, 15, \ldots$ are the pentagonal numbers.

- **Exercise:** Use this to compute $p(n)$ for $n = 1, 2, \ldots, 10$. What patterns do you notice?

Recursive Relation for $p(n)$

- Combining all the steps above,

$$(1 + p(1) + p(2)q^2 + p(3)q^3 + \dots) \cdot (1 - q - q^2 + q^5 + q^7 - \dots) = 1$$

where $1, 2, 5, 7, 12, 15, \dots$ are the pentagonal numbers.

- **Exercise:** Use this to compute $p(n)$ for $n = 1, 2, \dots, 10$. What patterns do you notice?
- Can you compute $p(50)$ now without finding all of the partitions? What about $p(100)$? Could you teach a computer to compute big values of $p(n)$?

What We've Learned

- How to represent partitions with diagrams

What We've Learned

- How to represent partitions with diagrams
- How to change these diagrams to create new types of partitions

What We've Learned

- How to represent partitions with diagrams
- How to change these diagrams to create new types of partitions
- Relations between different properties of partitions

What We've Learned

- How to represent partitions with diagrams
- How to change these diagrams to create new types of partitions
- Relations between different properties of partitions
- Pentagonal numbers and what they tell us about distinct partitions

What We've Learned

- How to represent partitions with diagrams
- How to change these diagrams to create new types of partitions
- Relations between different properties of partitions
- Pentagonal numbers and what they tell us about distinct partitions
- A way to compute $p(n)$