Egyptian and Continued Fractions

Nashville Math Club

September 10, 2020

Fractions

Question

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- If you don't like fractions, maybe today will change your mind.

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- `https://www.youtube.com/watch?v=hXsjwq1Q6HE`

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- `https://www.youtube.com/watch?v=hXsjwq1Q6HE`
- What if I told you there was a way to write down π as accurately as you want, with **no memorization required**?

What is a fraction

Definition

A **fraction** is a number of the form $\frac{a}{b}$ where $b \neq 0$ and a, b are whole numbers. A number is called **rational** if it can be written as a fraction. Otherwise, the nubmer is **irrational**.

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$$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

- Each person should take 1/2 a pizza (4 slices) then 1/4 a pizza (2 slices) then 1/8 a pizza (1 slice).

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Fact

Every rational number is an Egyptian fraction.

Computing Egyptian Fractions

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Take a fraction $\frac{a}{b} < 1$ *in lowest terms. Ceiling function:* $\lceil x \rceil$ *is smallest whole number bigger than* x*; e.g.,* $\lceil \pi \rceil = 4$ *(round up).*

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Exercise

Compute $\frac{4}{23}$ *as an Egyptian fraction.*

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$$\frac{4}{23} = \frac{1}{6} + \frac{1}{138}$$

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Compute $\frac{5}{22}$ *as an Egyptian fraction.*

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$\frac{5}{22} = \frac{1}{5} + \frac{1}{47} + \frac{1}{4070}$.

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- How do you know this algorithm will ever stop?

An open problem

Conjecture (Erdős–Straus (1948))

If $n \geq 2$, then there are whole numbers $x, y, z > 0$:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

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$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$$

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This is true for $n \leq 10^{17}$*.*

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Fact

This is true for $n \leq 10^{17}$. Its **much** *easier if we allow negatives:*

$$\frac{4}{4k+1} = \frac{1}{k} - \frac{1}{k(4k+1)} = \frac{1}{2k} + \frac{1}{2k} - \frac{1}{k(4k+1)}.$$

Rope puzzles

Setup

You have a bunch of ropes and a lighter. Each burns in 60 *minutes. How can you make timers for different numbers of minutes?*

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How can you time 45 minutes?

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A final rope puzzle

Question

How can you time 40 *minutes?*

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Answer

We write $\frac{40}{60} = \frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

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We write $\frac{40}{60} = \frac{2}{3} = \frac{1}{2} + \frac{1}{6}$. *So you can light one rope at both ends, then when that's done (30 min later), light a 2nd rope at both ends and two points in between, giving three segments, each with both ends burning.*

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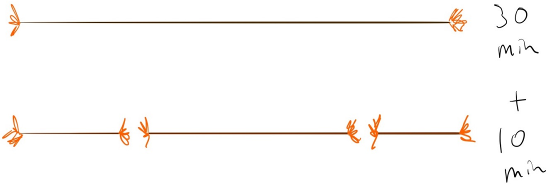

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We write $[a_0; a_1, a_2, \ldots, a_n]$ *for the above.*

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Example

$$\frac{13}{11} = 1 + \frac{1}{5 + \frac{1}{2}} = [1; 5, 2]$$

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- An example:

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$$15 = 10 \cdot 1 + 5$$

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- Why should we care? The number 5 here, the last "r" before we stopped, is the **greatest common divisor** $\gcd(100, 15)$.

Relation to continued fractions

- This is super fast for computers. Breaking into primes $100 = 2^2 \cdot 5^2$, $15 = 3 \cdot 5$ is **super slow eventually**. If you could factor numbers into primes fast, you could break a lot of security on the internet, and many bank accounts would be insecure.

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Let's use the steps above to write $\frac{100}{15}$ as a continued fraction: $100 = 15 \cdot 6 + 10 \implies \frac{100}{15} = 6 + \frac{10}{15}$ Now $15 = 10 \cdot 1 + 5 \implies \frac{15}{10} = 1 + \frac{5}{10} \implies \frac{10}{15} = \frac{1}{1+\frac{5}{10}} \implies \frac{100}{15} = 6 + \frac{1}{1+\frac{5}{10}}$.

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Let's use the steps above to write $\frac{100}{15}$ as a continued fraction:
$100 = 15 \cdot 6 + 10 \implies \frac{100}{15} = 6 + \frac{10}{15}$ Now $15 = 10 \cdot 1 + 5 \implies \frac{15}{10} = 1 + \frac{5}{10} \implies \frac{10}{15} = \frac{1}{1+\frac{5}{10}} \implies \frac{100}{15} = 6 + \frac{1}{1+\frac{5}{10}}$. Then
$10 = 5 \cdot 2 + 0 \implies \frac{10}{5} = 2 \implies \frac{5}{10} = \frac{1}{2} \implies \frac{100}{15} = 6 + \frac{1}{1+\frac{5}{10}} = 6 + \frac{1}{1+\frac{1}{2}} = [6; 1, 2]$.

What's going on

- General observation: Our Euclidean algorithm can be rearranged to give

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- General formula:

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \frac{1}{q_N}}}} = [q_0; q_1, \ldots, q_N].$$

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Thus, $\frac{1071}{462} = [2; 3, 7]$.

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$$\implies x = \frac{135}{999} = \frac{5}{37}.$$

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Thus, x is the Golden Ratio $\frac{1+\sqrt{5}}{2}$, since the negative answer **doesn't make sense**.

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- This is **really** close to $\sqrt{3}$ for a fraction with denominator only 2131.

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- A number is **quadratic** (built out of square roots) if and only if its continued fraction **eventually repeats**.

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- But $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots,]$, so maybe **February** 12**th** should be e day!