ALEX CAMERON: RESEARCH STATEMENT

1. Introduction

My research is in extremal problems for graphs, hypergraphs, and other combinatorial structures as well as the closely related field of Ramsey theory. Loosely speaking, these areas are all about finding the threshold of certain parameters at which some large and unstructured object is forced to contain a small and highly-structured subobject.

For example, if a graph on n vertices has more than $n^2/4$ edges, then it must contain a 3-clique, three vertices that are all pairwise adjacent, no matter its structure. The complete bipartite graph with equal (or nearly equal) parts (see Figure 1) demonstrates that we can have at least this many edges without a 3-clique. The fact that this is the best that we can do is called Mantel's Theorem, and we say that $n^2/4$ is the nth extremal number for the 3-clique K_3, $\mathrm{ex}(n, K_3) = n^2/4$. Turán's Theorem generalized this result for cliques of any size. It states that the maximum number of edges that a graph can have before it is forced to contain a clique of k vertices is the same as the number of edges found in the complete balanced $(k-1)$-partite graph.

Questions that ask for such an extremal number of allowed edges before some forbidden subgraph (or family of subgraphs) is forced are known as Turán-type problems after Paul Turán due to his important early results and conjectures concerning forbidden complete graphs and hypergraphs [31, 32, 33]. These kinds of questions are difficult to answer in general for hypergraphs and other combinatorial structures. Even for 3-uniform hypergraphs, the extremal number of a 4-clique is unknown.

Ramsey theory is similar to extremal Turán-type problems. The simplest case in classical Ramsey theory for graphs asks the following question: "If we take the complete graph on n vertices, can we color each of the edges either red or blue in such a way that avoids a monochromatic K_3?" The answer depends on the number of vertices, n. If $n \leq 5$, then yes, this is possible. However, if $n \geq 6$, then a monochromatic K_3 is always forced no matter how the edges are colored.

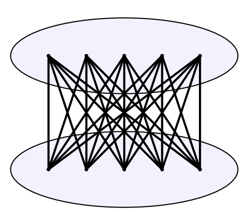

Figure 1. The complete balanced bipartite graph contains no 3-clique.

In general, we can ask the following: "Given k colors and positive integers $s_1, \ldots, s_k \geq 2$, what is the minimum number of vertices N such that for every $n \geq N$, no matter how you color the edges of the complete graph on n vertices, for some i there will exist a set of s_i vertices whose induced edges are all given color i?" This minimum N is called the Ramsey number $R(s_1, \ldots, s_k)$. Ramsey numbers are notoriously difficult to find even for small cases.

I can categorize my current research into two main ongoing projects. My first project is about a variant of the Ramsey problem originally proposed by Erdős and Shelah [15, 16]. My second project extends the notions of Turán-type problems to a class of combinatorial structures called directed hypergraphs. Such structures are useful models when studying propositional logic and other areas. The solutions to extremal problems frequently use methods from unexpected areas of mathematics including probability, linear algebra, analysis, and model theory.

2. Project 1: The Erdős-Gyárfás problem of generalized Ramsey numbers

2.1. **Background.** Let K_n denote the complete graph on n vertices. Fix positive integers p and q such that $1 \leq q \leq \binom{p}{2}$. A (p, q)-coloring of K_n is any coloring of its edges such that every p vertices span edges of at least q distinct colors. Let $f(n, p, q)$ denote the minimum number of colors needed to give a (p, q)-coloring of K_n. This is known as the Erdős-Gyárfás function. Erdős and Shelah [15, 16] originally introduced the function in 1975, but it was not studied systematically until 1997 when Erdős and Gyárfás [17] looked at the growth rate of $f(n, p, q)$ as $n \to \infty$ for fixed values of p and q.

We will use the standard asymptotic notation in what follows: For two functions, $f(n)$ and $g(n)$, we write $f = O(g)$ if there exists some constant c and some integer N such that $f(n) \leq cg(n)$ for all $n \geq N$. We write $f = o(g)$ if $f/g \to 0$ as $n \to \infty$. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \omega(g)$ if $g = o(f)$. Finally, we write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Erdős and Gyárfás [17] proved that for fixed p and q,

$$f(n, p, q) = O\left(n^a\right) \tag{1}$$

where $a = \frac{p-2}{1-q+\binom{p}{2}}$. Additionally, they gave upper and lower bounds on $f(n, p, q)$ for various small values of p and q. They also showed that for each fixed $p \geq 3$, $f(n, p, p) = \Omega\left(n^{1/(p-2)}\right)$ and posed the question of whether $f(n, p, p-1)$ is subpolynomial in n for all p. Mubayi and Eichorn [14, 24] showed that this is true for $p = 4, 5$. In 2015, Conlon, Fox, Lee, and Sudakov [12] showed that it is true in general by giving an explicit $(p, p-1)$-coloring of K_n for any $p \geq 3$ (the "CFLS" coloring). This shows that the threshold value for q at which $f(n, p, q)$ first becomes polynomial in n is at $q = p$.

Since Erdős and Gyárfás's work, better upper and lower bounds for this function in terms of n have been found for various cases of small fixed values for (p, q) by Mubayi [25], Axenovich [2], and Krop [19]. Specifically, Mubayi [25] showed that the

$$f(n, 4, 4) \leq n^{1/2+o(1)},$$

almost matching the order of the known lower bound. However, the best known general upper bound when $p = q$ is

$$f(n,p,p) = O\left(n^{2/(p-1)}\right),$$

which comes from Equation 1 [17].

2.2. **Research in progress.** Recently, Heath and I [11] improved this upper bound in the case when $q = p = 5$ by giving an explicit $(5,5)$-coloring that uses only $n^{1/3+o(1)}$ colors. This comes close to matching the order of the lower bound, $n^{1/3}$.

Theorem 2.1 (Cameron-Heath, [11])**.** *As $n \to \infty$,*

$$f(n,5,5) \leq n^{1/3} 2^{O\left(\sqrt{\log n}\log\log n\right)}.$$

The construction used to prove Theorem 2.1 uses a modified version of the CFLS coloring and pairs it with an "algebraic" coloring, extending many of the ideas behind Mubayi's $(4,4)$-edge coloring [25]. The algebraic part of our coloring views the vertices of the complete graph as three-dimensional vectors over a finite field, $\mathbb{F}$. The explicit definition of the color between any two such vectors is a bit technical, but is essentially a map to the base field

$$\chi : \mathbb{F}^3 \times \mathbb{F}^3 \to \mathbb{F},$$

giving about $n^{1/3}$ colors. This construction allows for interesting general arguments about affine geometry when proving that the coloring is actually $(5,5)$.

Similarly, I [10] was able to show that some modifications to Mubayi's $(4,4)$-coloring along with the modified version of the CFLS coloring we used in [11], gives a good $(5,6)$-coloring.

Theorem 2.2 (Cameron, [10])**.** *As $n \to \infty$,*

$$\left(\frac{5}{6}n - \frac{95}{144}\right)^{1/2} \leq f(n,5,6) \leq n^{1/2} 2^{O\left(\sqrt{\log n}\log\log n\right)}.$$

This again beats the best general upper bound from Equation 1 and provides another success for the general strategy of using finite field constructions.

2.3. **Proposed research.**

(1) (p,p)**-colorings in general.** The method of combining a variation of the CFLS coloring with a general algebraic construction using vectors from a space of dimension $p-2$ has the potential to show that

$$n^{1/(p-2)} \leq f(n,p,p) = n^{1/(p-2)+o(1)}$$

for $p \geq 6$. I have already shown that the variation the CFLS coloring (when generalized) only leaves p-cliques with exactly $p-1$ colors behind that are highly structured and easily definable. I believe that these remaining "bad" cliques could be avoided with a general algebraic coloring.

(2) **A better bound for $f(n,5,7)$.** The $(5,5)$ and $(5,6)$-colorings that I developed in [10, 11] have left $q = 7$ as the only remaining value for which a polynomial gap (in the order) between the known upper and lower bounds exists when $p = 5$. In this case we know that there are positive constants c_1 and c_2 such that

$$c_1 n^{2/3} \leq f(n,5,7) \leq c_2 n^{3/4}.$$

I would like to tighten this gap.

I have been attempting to extend the methods I have used so far to give an explicit $(5,7)$-coloring using only $n^{2/3+o(1)}$ colors. Several ideas have come close, but nothing has been successful yet. I believe that if such a construction does eventually succeed, then it would shed much more light on how these specific algebraic constructions might be generalized in a way to give colorings for (p,q) when $q \geq p$.

Currently, I know of successful algebraic colorings for (p,q) cases: $(3,3)$, $(4,4)$, $(4,5)$, $(5,5)$, $(5,6)$, and $(5,8)$. Equation 1 gives a linear upper bound on $f(n,5,8)$, but I have shown that a variation of the algebraic coloring gives a linear upper bound with a better coefficient. So there is reason to believe that such constructions might apply to more cases.

(3) **The hypergraph version.** Let $f_k(n,p,q)$ denote the minimum number of colors needed to color the edges of the complete k-uniform hypergraph on n vertices in such a way so that every p vertices span at least q colors. To date, little work has been done on this hypergraph version of the problem. There appear to be only two papers published on the topic, one by Conlon, Fox, Lee, and Sudakov [13] and one by Mubayi [26].

The main problem in the area is to determine for fixed p the threshold values for q at which there are large jumps in the order of the $f(n,p,q)$. For $p > k \geq 3$ and $0 < i < k$, Conlon, Fox, Lee, and Sudakov [13] showed that there exists a constant c dependent on k, p, and i for which

$$f_k\left(n, p, \binom{p-i}{k-i} + 1\right) = \Omega\left(log_{(i-1)}(n)^c\right)$$

where we define $log_0(x) = x$ and $\log_i(x) = \log\left(\log_{i-1}(x)\right)$. They conjecture that this value of q is such a jump in the order. Is it true that

$$f_k\left(n, p, \binom{p-i}{k-i}\right) = \left(\log_{(i-1)} n\right)^{o(1)}?$$

Mubayi and I have spent some time working on the upper and lower bounds on the Erdős-Gyárfás function for 3-graphs for small cases of p and q. It seems likely that algebraic constructions have a place in this area as well.

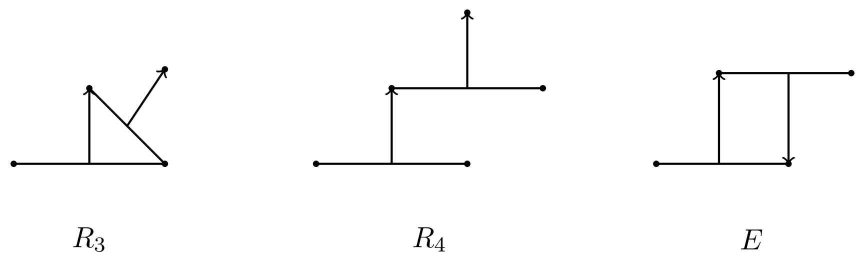

FIGURE 2. Three of the $(2 \to 1)$-graphs with two edges and extremal numbers that are cubic in n.

3. Project 2: Extremal problems on directed hypergraphs

3.1. **Background.** In addition to graphs and hypergraphs, Turán-type problems have been considered for directed graphs and multigraphs by Brown, Erdős, Harary, and Simonovits [4, 5]. Brown and Simonovits [6] studied the more general directed multi-hypergraphs, directed hypergraphs with r-uniform edges such that the vertices of each edge are given a linear ordering. The properties of a different, nonuniform definition of directed hypergraph were studied by Gallo, Longo, Pallottino, and Nguyen [18]. They defined a directed hyperedge as some subset of vertices with a partition into head vertices and tail vertices.

Yet another type of directed hypergraph arises as a model used to represent definite Horn clauses in the study of propositional logic and knowledge representation [1, 28]. This kind of directed hypergraph has nonuniform edges each with exactly one "head" vertex. Some combinatorial properties of a limited version of this model were recently studied by Langlois, Mubayi, Sloan, and Turán [21, 22]. In this uniform version, each edge of a directed hypergraph has exactly three vertices - one "head" vertex and two "tail" vertices.

They studied the extremal numbers for two specific $2 \to 1$ directed hypergraphs, the 4-resolvent (R_4) and the 3-resolvent (R_3) configurations, named after their behavior as Horn clauses (see Figure 2). In what follows, we extend the use of $\mathrm{ex}(n, F)$ to denote the maximum number of edges that a $(2 \to 1)$-graph on n vertices can have without containing a copy of the forbidden $(2 \to 1)$-graph F. Langlois, Mubayi, Sloan, and Turán [21, 22] determined $\mathrm{ex}(n, R_4)$ exactly for sufficiently large n and found good upper and lower bounds on $\mathrm{ex}(n, R_3)$.

3.2. **Research in progress.** Up to this point my research has focused primarily on determining the extremal number of edges for several particular $(2 \to 1)$-graphs. Additionally, I have worked on generalizing the various ideas of a uniform "directed hypergraph" in terms of first-order logic. I have shown that some foundational results for hypergraphs extend to this entire class of structures.

As with digraphs, there are actually two notions of an extremal number for $(2 \to 1)$-graphs. We call a $(2 \to 1)$-graph *oriented* if any three distinct vertices span at most one edge. Given a directed hypergraph F let $\mathrm{ex}_o(n, F)$ denote the maximum number of edges

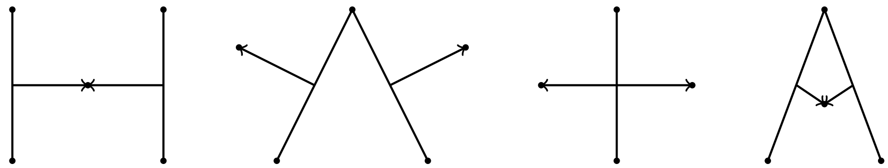

FIGURE 3. Four of the $(2 \to 1)$-graphs with two edges and extremal numbers that are not cubic in n.

that an oriented $(2 \to 1)$-graph on n vertices can have without containing a copy of F. This differs from the standard extremal number $\mathrm{ex}(n, F)$ studied by Langlois, Mubayi, Sloan, and Turán [21, 22] only in the additional restriction that the F-free $(2 \to 1)$-graphs be oriented. This restriction does not always change the extremal number, but sometimes the difference $\mathrm{ex}(n, F) - \mathrm{ex}_o(n, F)$ can be quite large (cubic in n) depending on F.

For sufficiently large n, I [7, 8] have found the exact extremal numbers, both standard and oriented, for each $(2 \to 1)$-graph with exactly two edges. The nontrivial cases are shown in Figure 2 and Figure 3. Of course, the standard extremal number for R_4 was already found by Langlois, Mubayi, Sloan, and Turán [21], but I have found a much shorter proof for this result and shown that the oriented extremal number is the same [7].

Additionally, I characterized the difference between degenerate $(2 \to 1)$-graphs, those with extremal numbers that are not cubic in n, and nondegenerate $(2 \to 1)$-graphs, those with cubic extremal numbers [9]. This result actually applies more generally to all generalized directed hypergraphs defined by the following definition from [9].

Definition Let $\mathcal{L} = \{E\}$, a language with one r-ary relation symbol E. Let T be an $\mathcal{L}$-theory that consists of a single sentence of the form

$$\forall x_1 \cdots x_r E(x_1, \ldots, x_r) \implies \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{\pi \in J_T} E(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

for some subgroup of the group of permutations on r elements, $J_T \subseteq S_r$. Call such a theory a *generalized directed hypergraph theory* and any finite model of T is a *generalized directed hypergraph (GDH).*

So for a fixed edge size r, the set of GDHs can be related through a poset depending on the subgroup of permutations associated with each. At the top of this lattice will always be the standard r-uniform hypergraph (associated with the group of all permutations on r elements S_r), and at the bottom is the totally ordered r-uniform hypergraph studied by Brown and Simonovits [6] (associated with the trivial group). I have shown that their supersaturation, continuity, and approximation results for this kind of structure [6] extend to all GDHs [9]. I have also extended many important notions from extremal hypergraph theory to this general setting including Turán density, blow-ups, blow-up density, edge-polynomials, and jumps/nonjumps.

3.3. **Proposed research.**

(1) **Enumeration and stability.** The two main areas under the umbrella of extremal combinatorial questions that I have yet to address for the $(2 \to 1)$-graphs are enumeration and stability. With respect to a forbidden subgraph F, an enumeration result asks for the number of different labelled F-free graphs on n vertices. Roughly speaking, a stability result shows that any F-free graph on n vertices with "almost" the maximum number of edges will differ from some canonical extremal F-free structure by "few" edges. These notions are closely related. It was already shown in [21] that the extremal construction for R_4 is stable, but I have yet to find a corresponding result for R_3 despite its similar extremal construction.

A recently developed method in extremal combinatorics, hypergraph containers [3, 29], has been applied toward getting these kinds of results in a variety of combinatorial structures with great success. Kühn, Osthus, Townsend, and Zhao [20] used hypergraph containers to get enumeration results for certain forbidden digraphs. Terry [30] used hypergraph containers to obtain some general results about enumeration of any hereditary property (including F-freeness) for general first-order relational structures.

Currently, Turán and I are discussing the various ways that we could extend the concept of an "extremal problem" to structures with more than one relation in a way that is meaningful. We have several ideas, but we are also aware that such generalizations have been considered by Razborov [27] and others.

(2) **Difference between oriented and standard extremal numbers.** Gerbner, Keszegh, Turán, and I have been looking into characterizing the set of forbidden $(2 \to 1)$-graphs F which give $\mathrm{ex}(n, F) - \mathrm{ex}_o(n, F) = \Theta(n^3)$. Interestingly, this question seems to connect back with the stability question.

(3) **Exponents.** Another question from extremal graph theory asks the following: "For which r does there exist a graph F such that $\mathrm{ex}(n, F) = \Theta(n^r)$?" This question can be just as easily asked in the $(2 \to 1)$-graph setting. For the set of $2 \to 1$ standard extremal exponents, it is easy to see that if r is an exponent for some undirected graph G, then $1 + r$ is an exponent for the $(2 \to 1)$-graph, $G^{\to}$, defined by $V(G^{\to}) = V(G) \cup \{x\}$ and

$$E(G^{\to}) = \{ab \to x : ab \in E(G)\}.$$

It is not difficult to show that

$$\mathrm{ex}(n, G^{\to}) = n \cdot \mathrm{ex}(n-1, G).$$

It is also interesting to note that while the extremal number for two undirected 3-edges that intersect in exactly one vertex is linear in n, any $2 \to 1$ orientation given to these two edges makes the extremal number quadratic in n.

(4) **Other extremal numbers.** I also have found upper and lower bounds for various extremal numbers for some other $(2 \to 1)$-graphs as well as a few from other types of GDHs including $(r \to 1)$-uniform directed hypergraphs, $(1 \to 1 \to 1)$-uniform directed hypergraphs, and a model where each edge has three vertices with one of

two cyclic orders: $a \to b \to c \to a$ or $a \to c \to b \to a$. This last model relates to research on d-simplex structures by Leader and Tan [23].

(5) **Jumping constant conjecture.** For hypergraphs, we can define a *jump* roughly as a number $\alpha \in [0, 1)$ such that any hypergraph with edge density slightly more than α must contain an arbitrarily large subhypergraph with edge density $\alpha + c$ for some fixed c. A big area of research is to determine the subset of $[0, 1)$ that are jumps for r-uniform hypergraphs where $r \geq 3$.

This definition can be applied to GDHs in a natural way. Moreover, for a fixed edge size r, there is a relation between the subgroup lattice of GDHs and the set of jumps for each. I have shown [9] that jumps always pass up the lattice, but not always (if ever) back down. That is, if T' and T are GDH theories with associated groups $J_{T'} \subseteq J_T$, then the set of jumps for T' is a subset of the set of jumps for T. However, if the order of J_T is at least three times the order of $J_{T'}$, then this subset is necessarily proper. I would like to look further into the case where the order of J_T is twice as much as the order of $J_{T'}$.

4. Other Research Interests

In general, I would be very happy to work on any combinatorial problems. I have spent some time working on open problems involving list colorings, decision tree complexity of graph and hypergraph properties, the sensitivity conjecture, forbidden posets, random threshold hypergraphs, and Turán numbers for various forbidden hypergraphs. I am always interested in working on a new problem. More broadly, I am interested in learning and working more in the in areas of graph limits, graph algebras, and finite model theory.

References

[1] Dana Angluin, Michael Frazier, and Leonard Pitt. Learning conjunctions of Horn clauses. *Machine Learning*, 9(2-3):147–164, 1992.

[2] Maria Axenovich. A generalized Ramsey problem. *Discrete Mathematics*, 222(1-3):247–249, 2000.

[3] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *Journal of the American Mathematical Society*, 28(3):669–709, 2015.

[4] W.G. Brown, Paul Erdős, and Miklós Simonovits. Extremal problems for directed graphs. *Journal of Combinatorial Theory, Series B*, 15(1):77–93, 1973.

[5] W.G. Brown and Frank Harary. Extremal digraphs. *Combinatorial Theory and Its Applications*, 4:135–198, 1969.

[6] W.G. Brown and Miklós Simonovits. Digraph extremal problems, hypergraph extremal problems, and the densities of graph structures. *Discrete Mathematics*, 48(2):147–162, 1984.

[7] Alex Cameron. Extremal numbers for $2 \to 1$ directed hypergraphs with two edges part 1: The nondegenerate cases. arXiv preprint arXiv:1607.04930, 2016.

[8] Alex Cameron. Extremal numbers for $2 \to 1$ directed hypergraphs with two edges part 2: The degenerate cases. arXiv preprint arXiv:04932, 2016.

[9] Alex Cameron. Extremal problems on generalized directed hypergraphs. http://arxiv.org/pdf/1607.04927, 2016.

[10] Alex Cameron. An explicit edge-coloring of K_n with six colors on every K_5. *arXiv preprint arXiv:1704.01156*, 2017.

[11] Alex Cameron and Emily Heath. A $(5,5)$-coloring of K_n with few colors. *arXiv preprint arXiv:1702.06227*, 2017.
[12] David Conlon, Jacob Fox, Choongbum Lee, and Benny Sudakov. The Erdős–Gyárfás problem on generalized Ramsey numbers. *Proceedings of the London Mathematical Society*, 110(1):1–18, 2015.
[13] David Conlon, Jacob Fox, Choongbum Lee, and Benny Sudakov. On the grid Ramsey problem and related questions. *International Mathematics Research Notices*, 2015(17):8052–8084, 2015.
[14] Dennis Eichhorn and Dhruv Mubayi. Edge-coloring cliques with many colors on subcliques. *Combinatorica*, 20(3):441–444, 2000.
[15] Paul Erdős. Problems and results on finite and infinite graphs. *Recent Advances in Graph Theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 183–192, 1975.
[16] Paul Erdős. Solved and unsolved problems in combinatorics and combinatorial number theory. *European Journal of Combinatorics*, 2:1–11, 1981.
[17] Paul Erdős and András Gyárfás. A variant of the classical Ramsey problem. *Combinatorica*, 17(4):459–467, 1997.
[18] Giorgio Gallo, Giustino Longo, Stefano Pallottino, and Sang Nguyen. Directed hypergraphs and applications. *Discrete applied mathematics*, 42(2):177–201, 1993.
[19] Elliot Krop. How many edge-colors for almost-rainbow K_5s? *arXiv preprint arXiv:0803.0572*, 2008.
[20] Daniela Kühn, Deryk Osthus, Timothy Townsend, and Yi Zhao. On the structure of oriented graphs and digraphs with forbidden tournaments or cycles. *Journal of Combinatorial Theory, Series B*, 2017.
[21] Marina Langlois. *Knowledge representation and related problems*. PhD thesis, University of Illinois at Chicago, 2010.
[22] Marina Langlois, Dhruv Mubayi, Robert H. Sloan, and György Turán. Combinatorial problems for Horn clauses. In *Graph Theory, Computational Intelligence and Thought*, pages 54–65. Springer, 2009.
[23] Imre Leader and Ta Sheng Tan. Directed simplices in higher order tournaments. *Mathematika*, 56(01):173–181, 2010.
[24] Dhruv Mubayi. Edge-coloring cliques with three colors on all 4-cliques. *Combinatorica*, 18(2):293–296, 1998.
[25] Dhruv Mubayi. An explicit construction for a Ramsey problem. *Combinatorica*, 24(2):313–324, 2004.
[26] Dhruv Mubayi. Coloring triple systems with local conditions. *Journal of Graph Theory*, 81(3):307–311, 2016.
[27] Alexander A Razborov. Flag algebras. *The Journal of Symbolic Logic*, 72(04):1239–1282, 2007.
[28] S.J. Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2002.
[29] David Saxton and Andrew Thomason. Hypergraph containers. *Inventiones mathematicae*, 201(3):925–992, 2015.
[30] Caroline Terry. Structure and enumeration theorems for hereditary properties in finite relational languages. *arXiv preprint arXiv:1607.04902*, 2016.
[31] Paul Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48(137):436–452, 1941.
[32] Paul Turán. On the theory of graphs. *Colloq. Math.*, 3:19–30, 1954.
[33] Paul Turán. Research problems. *Magyar Tud. Akad. Kutató Int. Közl.*, 6:417–423, 1961.