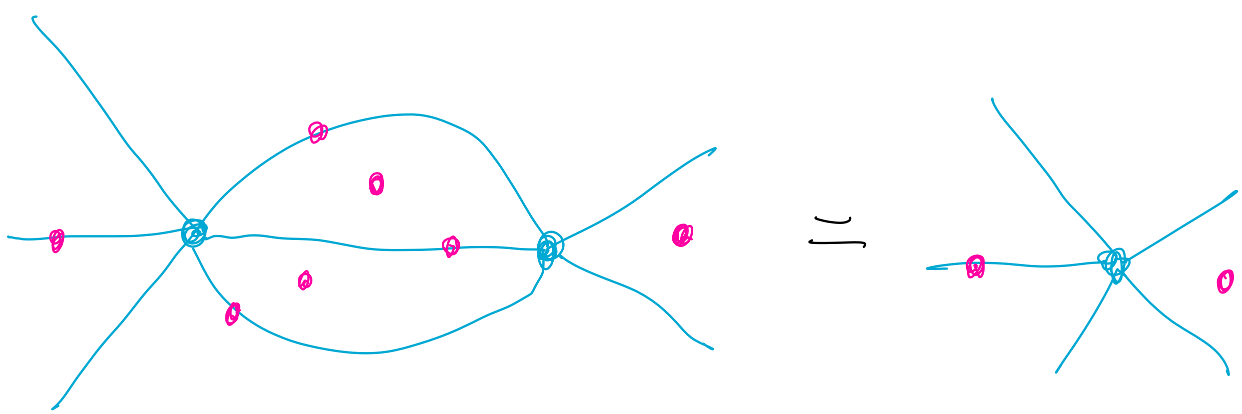
Completions of n-cats etc.

Completions of n-categories,
Ostrik's thm for n-cats,
SETs

(joint with Scott Morrison)



Outline:

0. review definition of completion
1. Some properties of the completion
2. regular representation $C \to Rep(C)$
3. TQFTs with defects
4. Proof of Ostrik thm for n-categories
5. other stuff, as time permits

$\boxed{0}$ Recall defn of completion

C : pivotal n-category

$\overline{C}$: completion of C

$\overline{C}^{\circ} = \{$ ways of uniformly labeling stratifications so that evaluation of n-dim'l diagrams is the same for all full stratifications $\}$

0 Recall defn of completion

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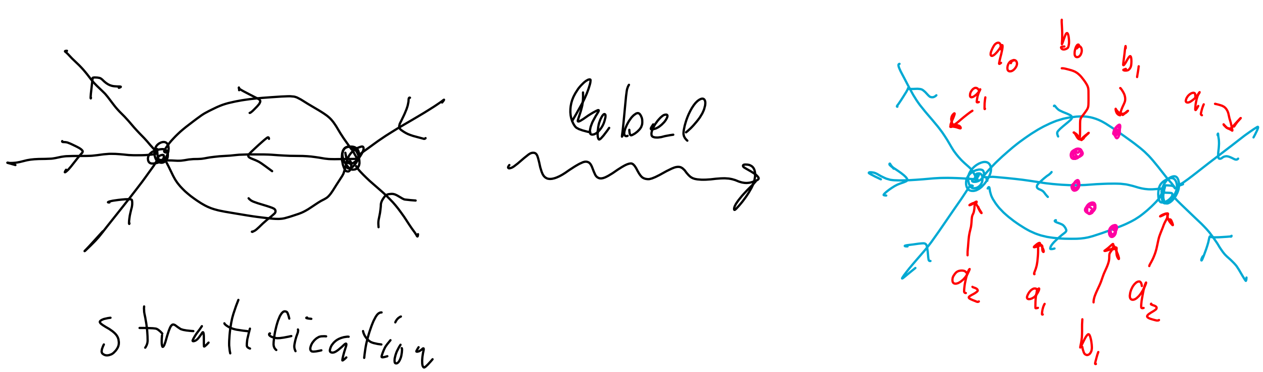
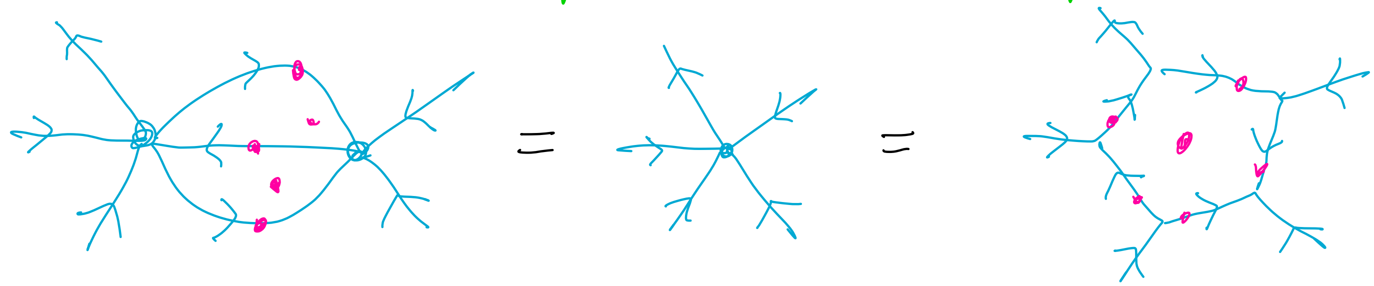
$\overline{C}$: completion of C

$\overline{C}^\circ = \{$ways of uniformly labeling stratifications so that evaluation of n-dim'l diagrams is the same for all full stratifications$\}$

$\overline{C}^\circ \ni \overline{a} = (a_0, a_1, \dots a_n, b_0, \dots, b_{n-1})$

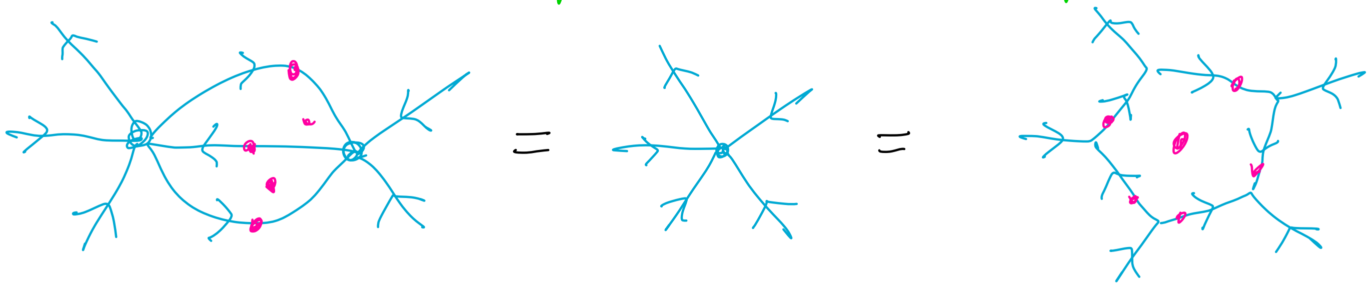


Generalized idempotent condition:



etc

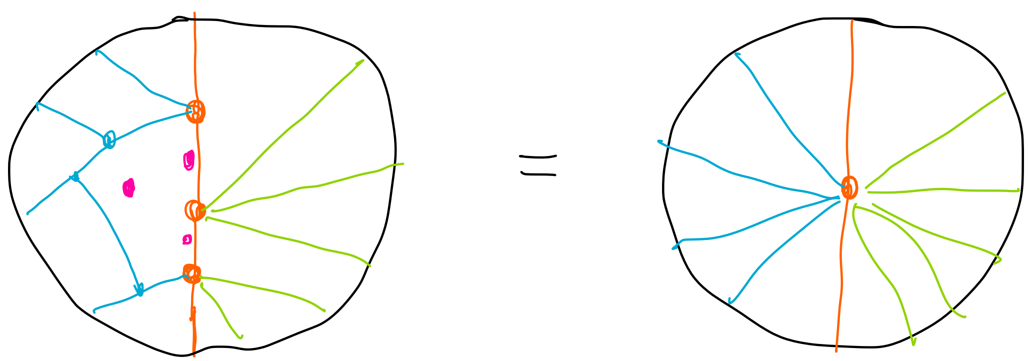
Generalized idempotent condition:



etc

$\bar{C}^1(\bar{a} \mid \bar{a}') = \{$ ways of uniformly labeling stratifications of $\bar{a} \mid \bar{a}'$ so that evaluations of full diargams are all equal $\}$

$\bar{x} : \bar{a} \to \bar{a}'$ $\qquad \bar{x} = (x_1, x_2, \dots)$

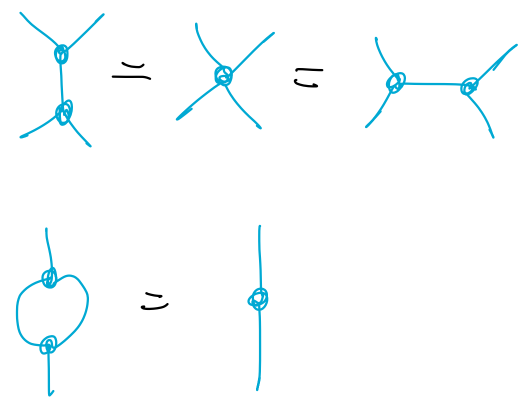


- for $n=1$, we can renormalize to eliminate bubble factors (•), and the completion is the usual idempotent completion

- for $n=1$, we can renormalize to eliminate bubble factors ($\circ$), and the completion is the usual idempotent completion

- for $n=2$, we can again renormalize to eliminate bubble factors, and the completion is the familiar 2-category:

$$\left[\begin{array}{l} 0 - \text{special Frob. alg. objects in } C \\ 1 - \text{bimodule objects in } C \\ 2 - \text{intertwiners} \end{array}\right.$$



- for $n \geq 3$, not possible to eliminate bubble factors

More examples:

If C is a braided category ($n=3$), then

$\overline{C}^{\circ}$ contains {commutative alg. objects in C}

$\overline{C}'(* \to *)$ contains {associative alg. objects in C}

If $C = G_{[n]} \simeq \Pi_{\leq n}(BG)$ (G: finite group), then $\overline{C}^{\circ}$ contains $\{(H, \omega) \mid \omega \text{ an } n\text{-cocycle on } H,\ H \subset G\}$

☆ But above lists are far from exaustive:

$\overline{\mathrm{Vec}_{[3]}}^{\circ} \supset$ {fusion categories}

$\boxed{1}$ Properties of the completion

ⓐ $\mathcal{C} \underset{\substack{\uparrow \\ \text{u-Morita}}}{\simeq} \overline{\mathcal{C}}$ ($\Rightarrow$ $\mathcal{C}$ and $\overline{\mathcal{C}}$ lead to isomorphic TQFTs, hamiltonians, etc.)

ⓑ $\mathcal{C} \underset{\substack{\uparrow \\ \text{morita}}}{\simeq} \mathcal{D} \Rightarrow \overline{\mathcal{C}} \underset{\substack{\uparrow \\ \text{functor}}}{\cong} \overline{\mathcal{D}}$

1 Properties of the completion

(a) $C \underset{\substack{\uparrow \\ \text{u-Morita}}}{\simeq} \bar{C}$ ($\Rightarrow$ C and $\bar{C}$ lead to isomorphic TQFTs, hamiltonians, etc.)

(b) $C \underset{\substack{\uparrow \\ \text{Morita}}}{\simeq} D \Rightarrow \bar{C} \underset{\substack{\uparrow \\ \text{functor}}}{\cong} \bar{D}$

(c) In semi-simple case,

$\bar{C}^{\circ} \longleftrightarrow C$-modules (i.e. gapped boundaries for C)

More generally,

$$\boxed{\bar{C} \underset{\text{functor}}{\cong} \operatorname{Rep}(C)}$$

(d) Equivariantization arises naturally from the completion $\overline{G}_{(n)}$

Deequivariantization arises naturally from the completion $\overline{\mathrm{Rep}(G)}_{(n)}$

(More generally, get twisted [de]equivariantizations)

(e) $\overline{\mathrm{Vec}}_{(n+1)} \rightsquigarrow$ state sum models for $(n+1)$-dim'l TQFTs (eg. Turaev-Viro sum)

② The regular representation(s)

Let $x \in C^0$. We want to construct a C-module (i.e. functor from C to [(n-1)-cats, functors, nat. trans, ...])

Algebraically,

[1...n]-morphisms

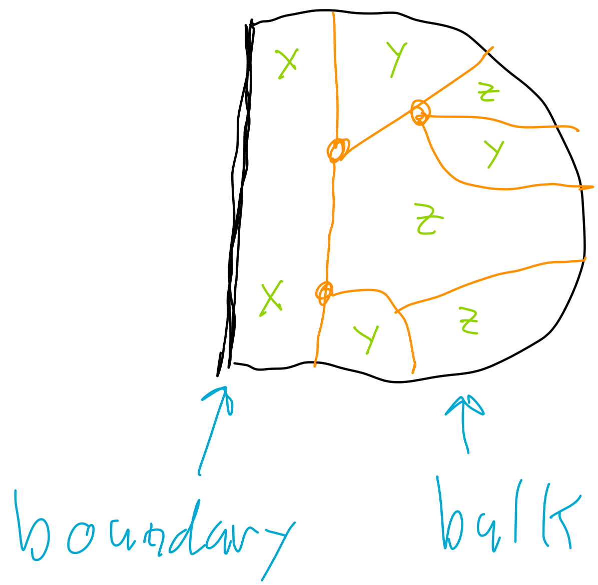
$$y \in C^0 \longmapsto (n-1)\text{-cat } \mathrm{mor}_C(x \to y)$$

$$(e : y \to y') \in C^1 \longmapsto \text{functor } \mathrm{mor}(x \to y) \to \mathrm{mor}(x \to y')$$

$$\alpha \longmapsto \alpha \circ e$$

(and so on)

In terms of string diagrams, the above C-module corresponds to a "smooth" boundary condition labeled by x:



Continuing in this way, we get a functor

$$C \to \big[\, C\text{-modules},\ \underbrace{C\text{-module-bimodules},\ \cdots\cdots}_{\text{bimodules all the way up}} \,\big] =: \mathrm{Rep}(C)$$

Since $\overline{\mathcal{C}}$ is Morita equivalent to $\mathcal{C}$, we have the composite map

$$\overline{\mathcal{C}}^{0} \xrightarrow[\text{regular rep'n}]{} \mathrm{Rep}(\overline{\mathcal{C}})^{0} \xrightarrow[\text{restriction}]{} \mathrm{Rep}(\mathcal{C})^{0}$$

☆ The above map is the easy half of the n-Ostrik thm. We want an inverse to this map

$n=1$ example: minimal idempotent in $\mathbb{C}[G] \mapsto$ irrep. of G

$n=2$ example: alg. object A in $\otimes$-cat $\mathcal{C}$
$\mapsto$ $\mathcal{C}$-module A-mod

3 TQFTs with defects

Thm $n+\varepsilon$: From any pivotal n-cat, we can construct an $(n+\varepsilon)$-dim'l TQFT.

Proof: String diagrams.

③ TQFTs with defects

Thm $n+\varepsilon$: From any pivotal n-cat, we can construct an $(n+\varepsilon)$-dim'l TQFT.

Proof: String diagrams.

Thm $n+1$: Let C be a pivotal n-cat and let $z: C(S^n) \to \mathbb{C}$ be such that the induced inner products on $C(B^n; s)$ are positive definite for all ∂-conditions $s \in C(S^{n-1})$. The $\exists!$ $(n+1)$-dim'l TQFT with $Z(B^{n+1}) = z$.

Proof: Induct on handle index.

Def'n: C is semisimple if it satisfies above hypotheses.

Thm $n+1$ w/ defects. Let $\mathcal{C}$ and $\mathcal{D}$ be semisimple pivotal n-cats (as above). Let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$ bimodule. Suppose $\exists\ y: \mathcal{M}\Big[$ 

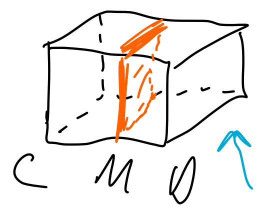
$\Big]$ such that the

S^n

induced inner products on $\mathcal{M}\Big[$ 

$; s \Big]$ are positive-definite for all

B^n

∂-conditions s. The there $\exists!$ TQFT-with-defects such that $Z\Big($ 

$\Big) = y$.

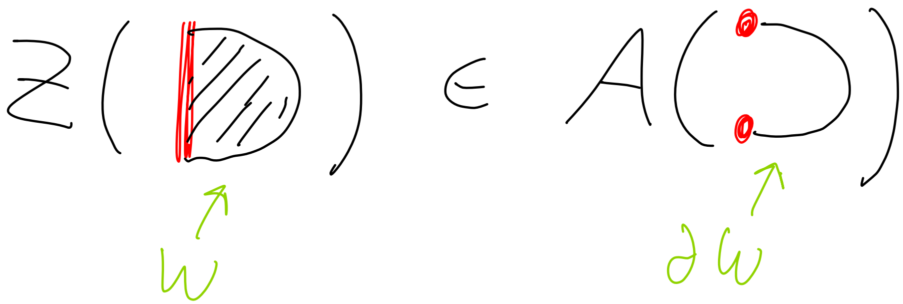
B^{n+1}

Proof: Induct on index of defect-handles.

We are interested in the case $D = \mathrm{triv}$,

M : $\mathcal{C}$-module.

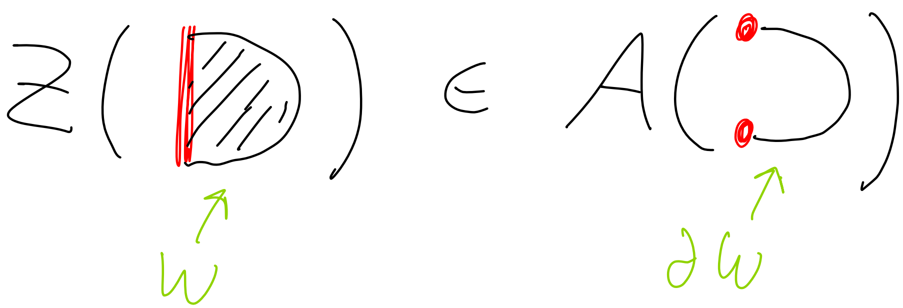
$$Z(W) \in A(\partial W)$$



We are interested in the case $D = \mathrm{triv}$,

M : C-module.

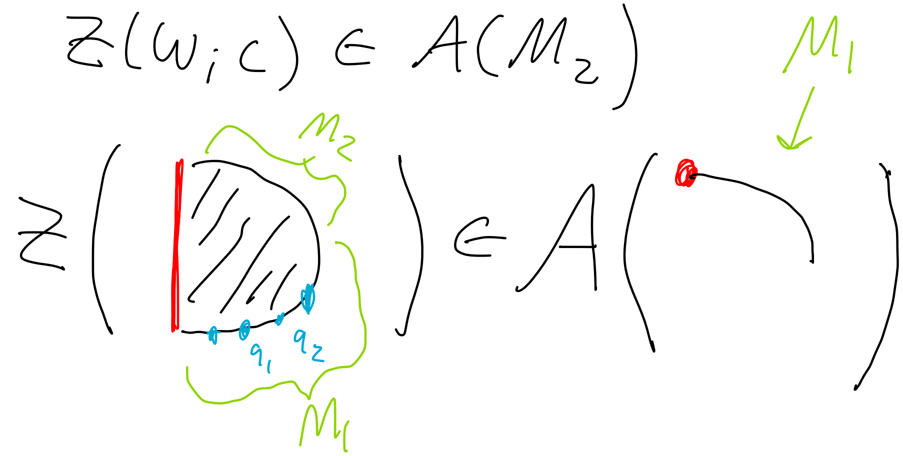
$Z(W) \in A(\partial W)$



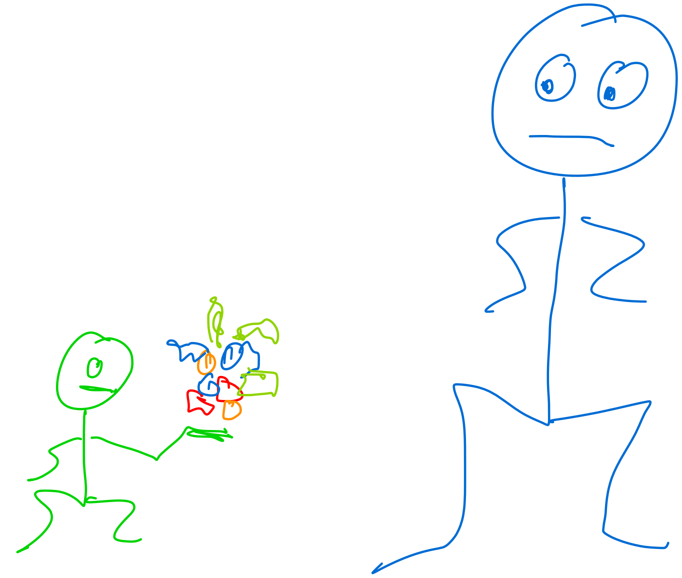
$\partial W = M_1 \cup M_2$

$c \in A(M_1)$

$Z(W; c) \in A(M_2)$

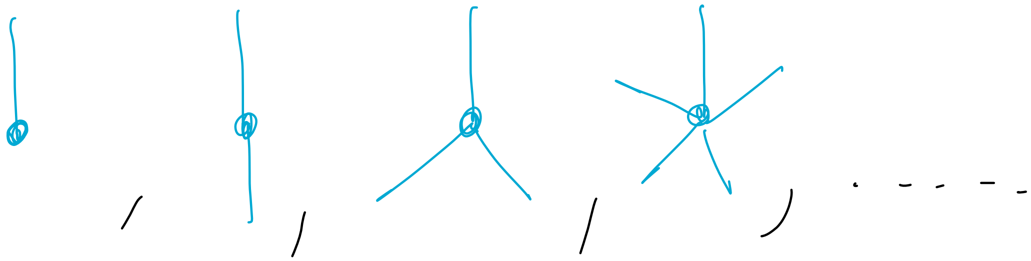


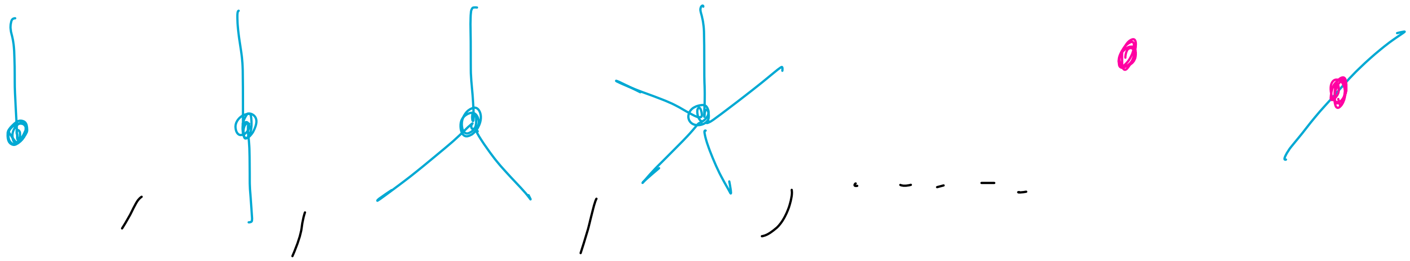
④ Proof of n-Ostrik thm

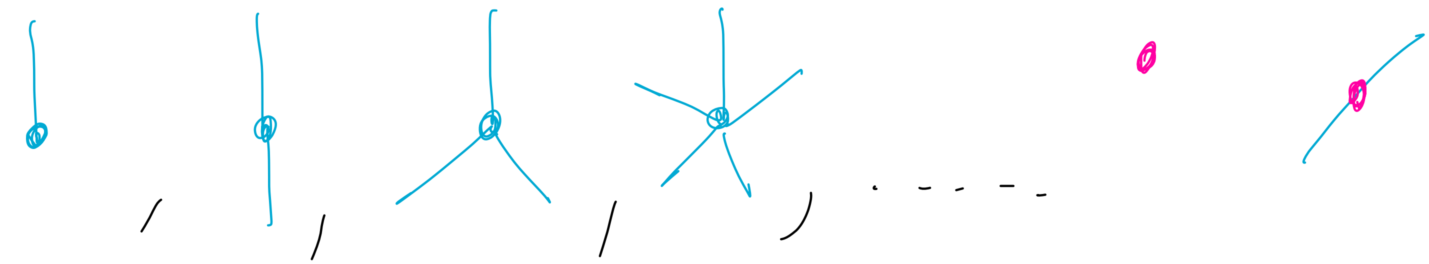


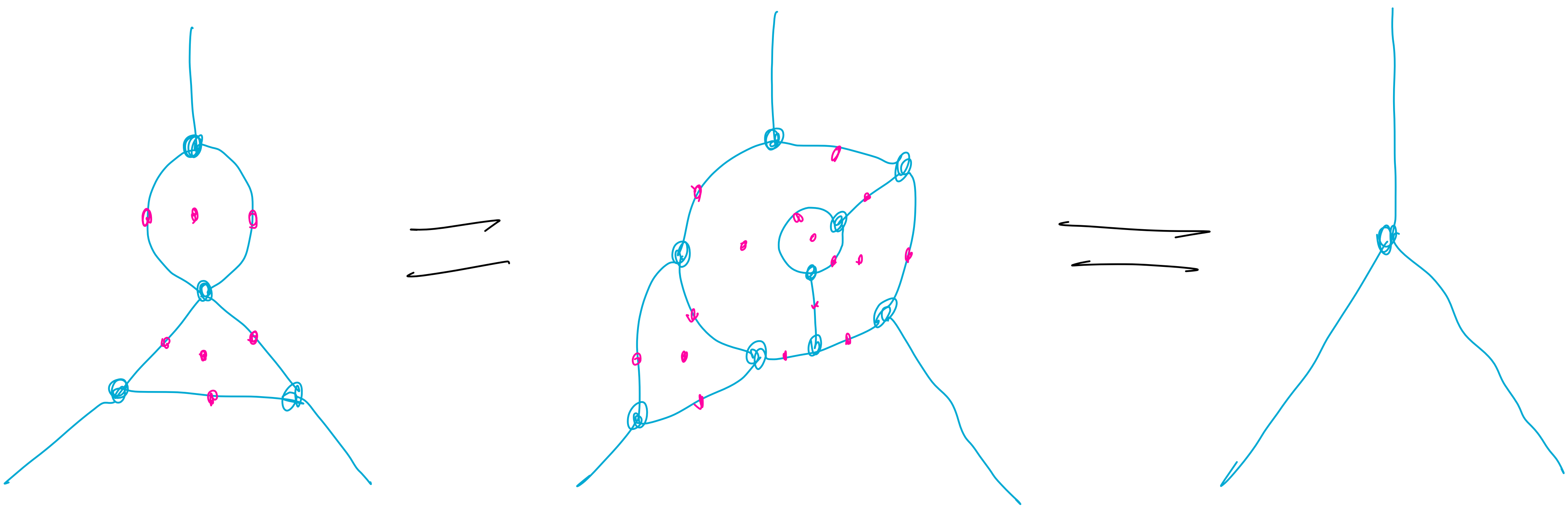
$\neq$

$\neq$



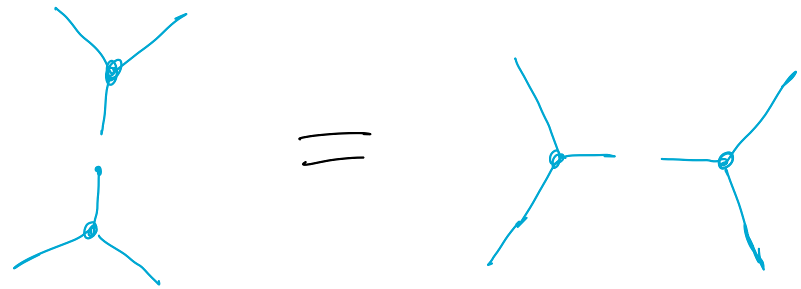




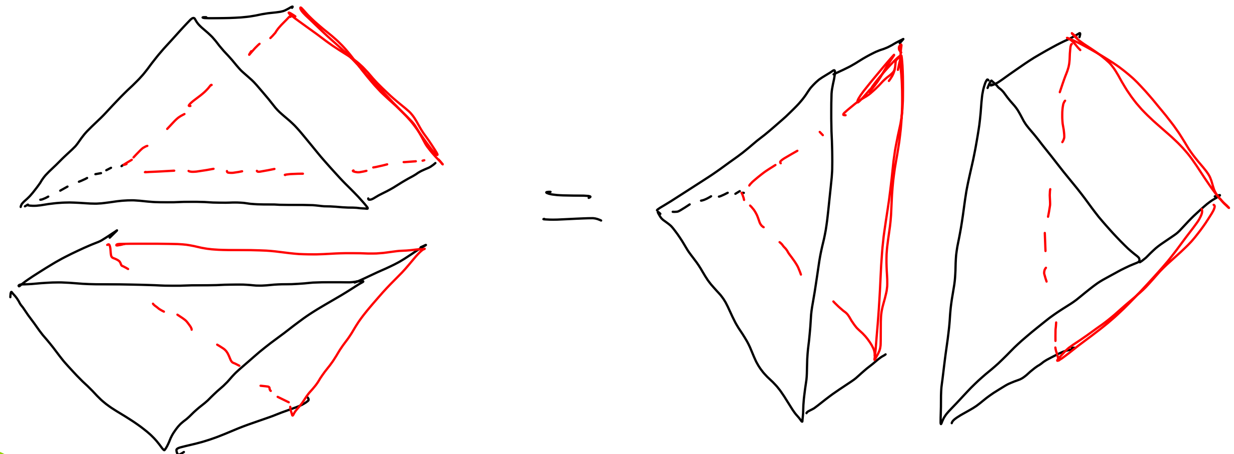


(Alternative definition: n-alg = maximally useless lego blocks)

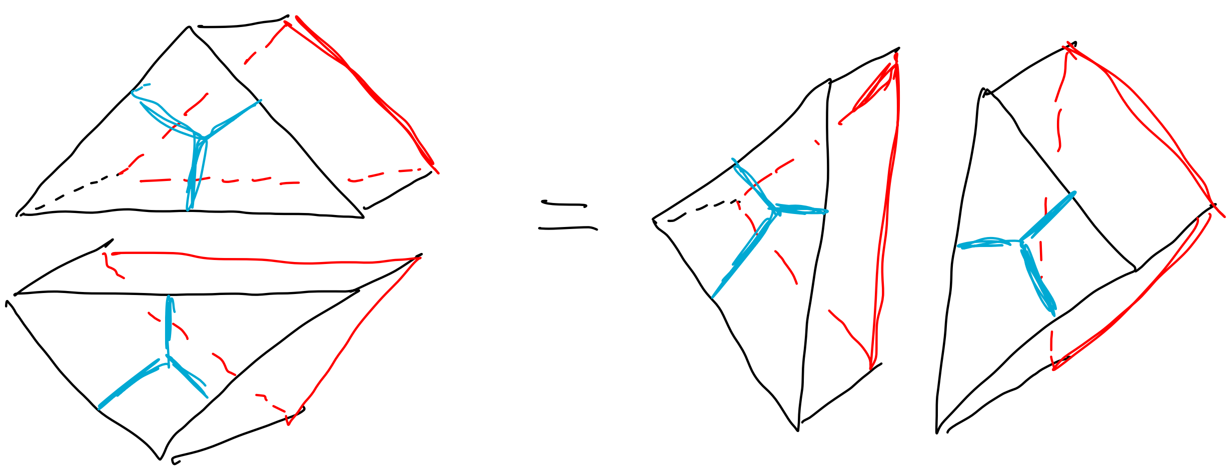
What we want:

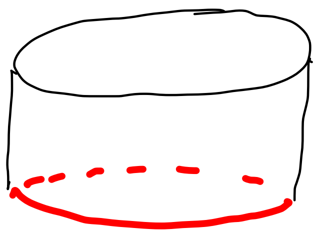
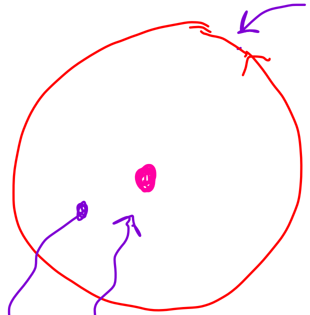
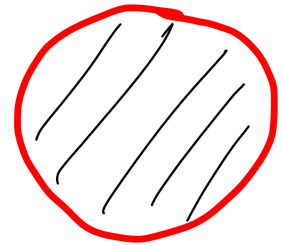


What we have:



IDEA:



$Z\Big[$  $\Big] =$  $\in A\Big[$  $\Big]$

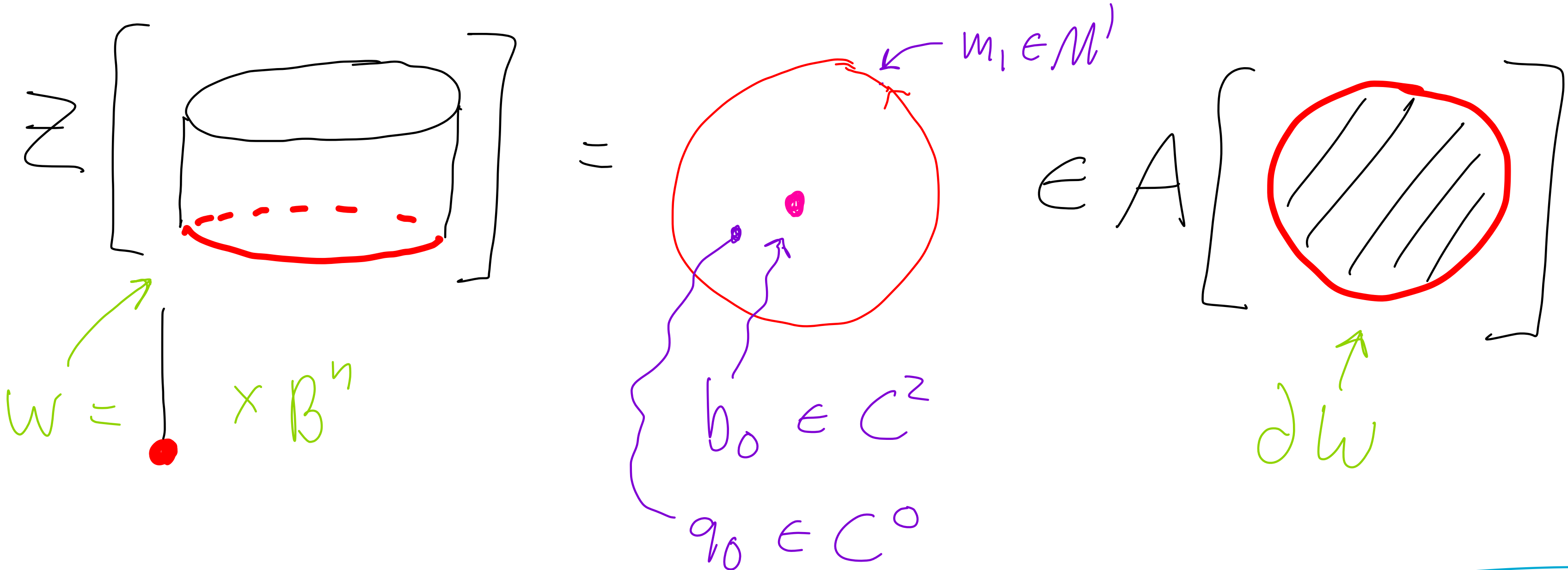
$m_1 \in M^1$

$W =$  $\times B^n$

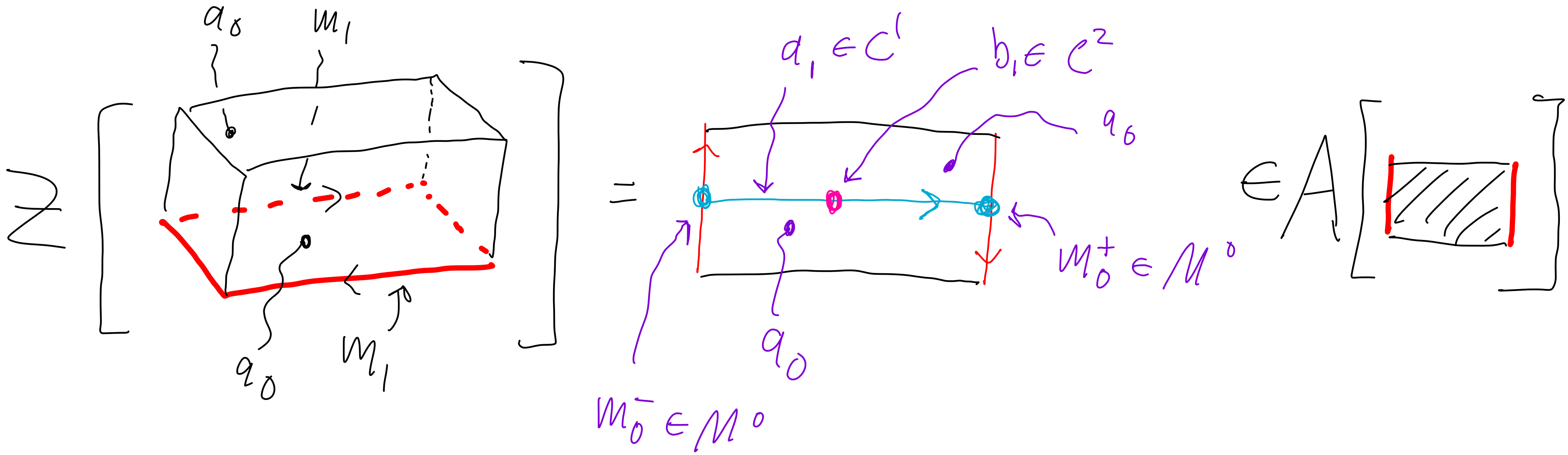
$b_0 \in C^2$

$q_0 \in C^0$

∂W

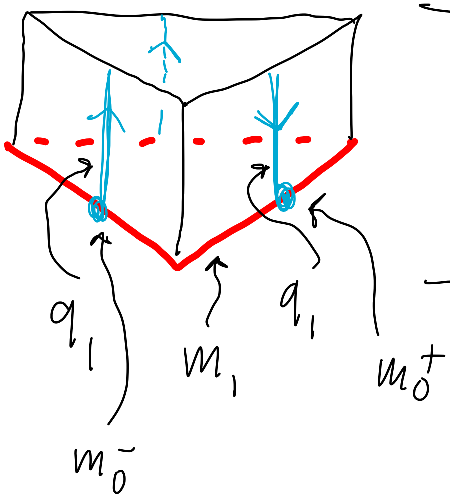


Z
=
$m_1 \in M^1$
$\in A$
$W = | \times B^n$
$b_0 \in C^2$
$a_0 \in C^0$
∂W

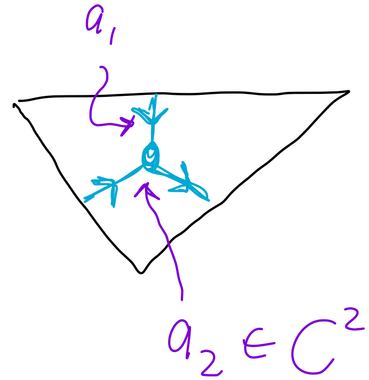


Z
a_0
m_1
a_0
m_1
=
$a_1 \in C^1$
$b_1 \in C^2$
a_0
$\in A$
$m_0^+ \in M^0$
a_0
$m_0^- \in M^0$

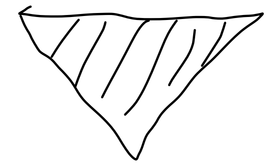
$Z\Big[$



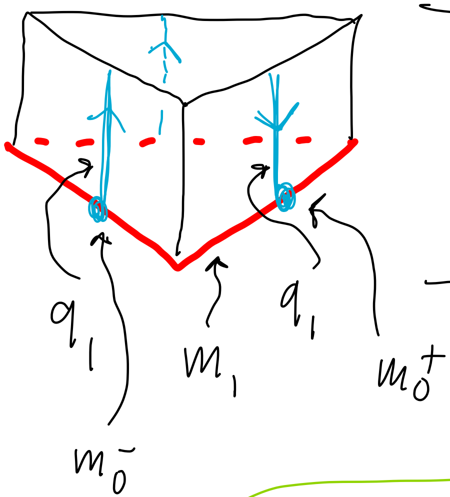
$\Big] =$

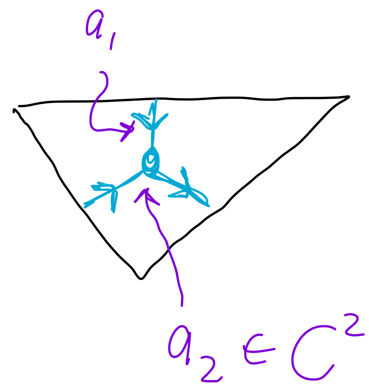


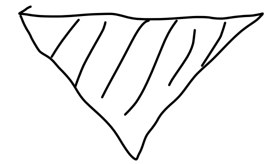
$\in A\Big[$

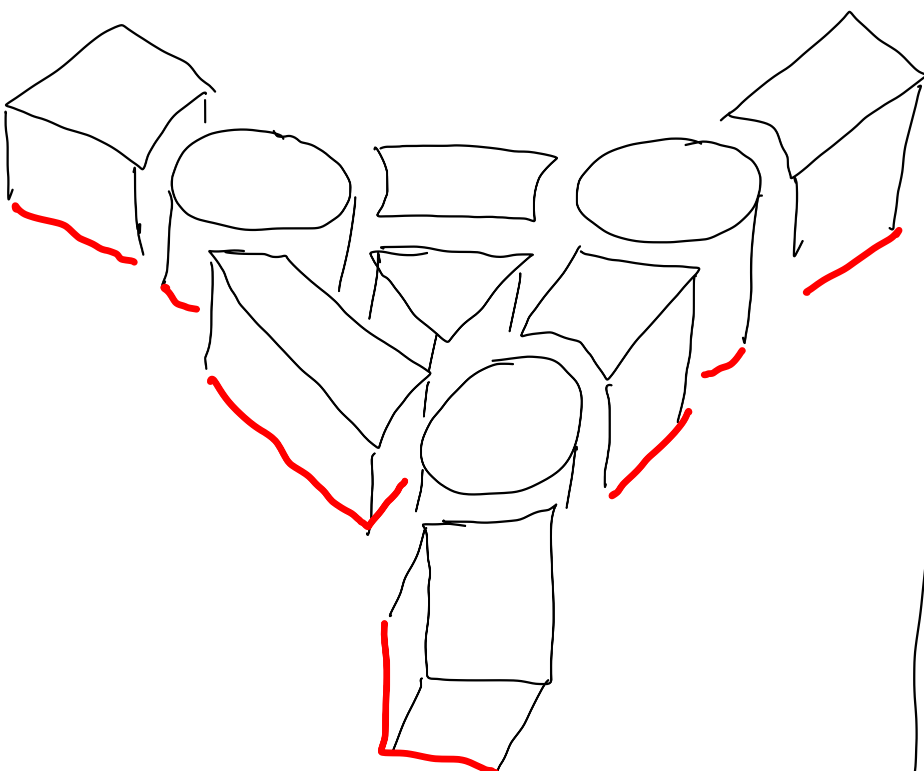
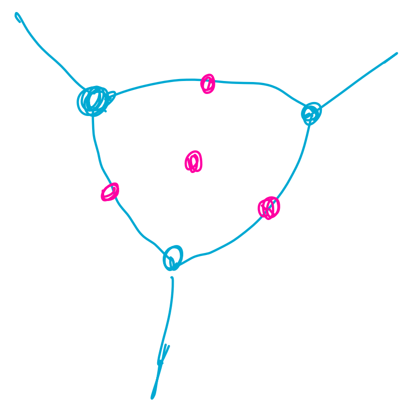


$\Big]$

$Z\Big[$ 

$\Big] =$ 

$\in A\Big[$  $\Big]$

$Z\Big[$  $\Big] =$ 

SET classification (?)

SET $\longleftrightarrow$ gapped boundary for $\mathcal{G}_{(n)}$

~ Def'n

$\longleftrightarrow$ n-alg object in $\mathcal{G}_{(n)}$

n-Ostrik thm

$\underline{n=2}$: pairs (H, ω), $H \subset G$, $\omega \in H^2(H, U(1))$

(All of them)

$\underline{n \geq 3}$: pairs (H, ω), $H \subset G$, $\omega \in H^n(H, U(1))$

NOT all of them, even when $G \cong$ triv

☆ semisimple 1-cats: $\oplus$ triv

NOT true for $(n \geq 2)$-cats