# Holography and Entanglement

**Matthew Headrick**
*Brandeis University*

NCGOA Spring Institute 2019
May 6, 2019

*It from Qubit*
Simons Foundation

NSF

U.S. DEPARTMENT OF ENERGY | Office of Science

TRUTH · EVEN UNTO ITS INNERMOST PARTS
Brandeis

# Entropy and area

Bekenstein-Hawking '74:

$$S = \frac{k_{\mathrm{B}} c^3 \mathrm{area(horizon)}}{4 G_{\mathrm{N}} \hbar} = k_{\mathrm{B}} \frac{\mathrm{area(horizon)}}{4 l_{\mathrm{P}}^2}$$

Planck length
(≈$10^{-33}$ cm)


Event Horizon Telescope '19

$G_{\mathrm{N}}$ → gravity

$\hbar$ → quantum mechanics

$k_{\mathrm{B}}$ → statistical mechanics


What are the "atoms" of the black hole?
Why is $S \propto$ area?

# Entropy and area

If space has $d$ dimensions, $G_{\rm N}\hbar$ has units of area ($L^{d-1}$)

*Planck area:* basic unit in quantum gravity, translates into unit of *entropy*

Generalizations of Bekenstein-Hawking:

De Sitter spacetime (Gibbons-Hawking '77):

$$S = \frac{\text{area(horizon)}}{4G_{\rm N}\hbar}$$

Holographic entropy bounds (Bekenstein '81, Bousso '99):
for arbitrary closed surface in arbitrary spacetime

$$S \leq \frac{\text{area}}{4G_{\rm N}\hbar}$$

Jacobson '95: area-entropy relation *implies* Einstein equation

How general is the area-entropy relation? What is its origin?

A clue: *Holographic entanglement entropy* (Ryu-Takayanagi '06)

Vast (but also limited) generalization of Bekenstein-Hawking

To understand it, we first need to extend our notion of entropy…

# Entanglement Entropy

Classical mechanics:

definite state → certain outcome for any measurement

Quantum mechanics:

definite state → uncertain outcomes for some measurements

Example: $|\uparrow\rangle$

measurement of $S_z$ definitely gives $+\frac{1}{2}\hbar$

measurement of $S_x$ gives $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ with equal probability

When only certain kinds of measurements are allowed, a definite (pure) state will *effectively* be indefinite (mixed)

Suppose a system has two parts, but we can only measure one part

Spin singlet state: $$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\underbrace{\uparrow\rangle}_{A}|\underbrace{\downarrow\rangle}_{B} - |\downarrow\rangle|\uparrow\rangle\right) \qquad S_{AB} = 0$$

To see that this is a pure state (superposition, not mixture, of $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$) requires access to both $A$ and $B$

For an observer who only sees $A$, effective state is mixed:

$$\rho_A = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \qquad S_A = \ln 2$$

*Indefiniteness* from *entanglement*

# Entanglement Entropy

In general: if $\rho$ is state of full system, then effective state for subsystem $A$ is

$$\rho_A = \mathrm{Tr}_{A^c}\,\rho$$

*Entanglement entropy* is defined as von Neumann entropy of subsystem:

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A$$

(*Any* entropy can be viewed as due to entanglement with environment)

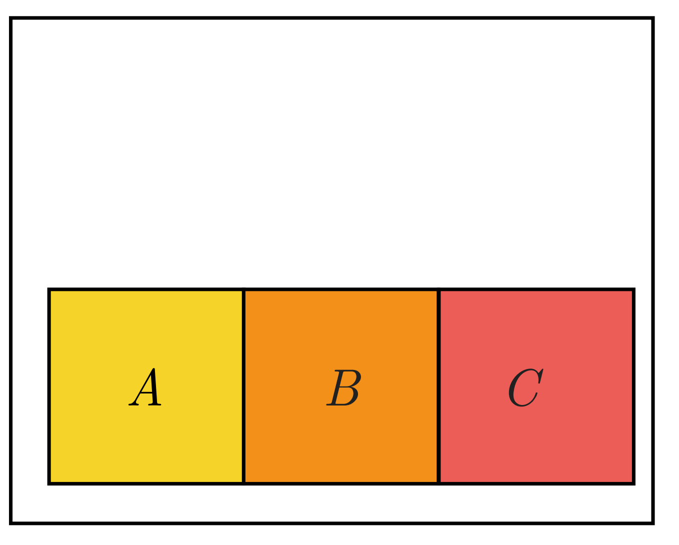
Entanglement entropies obey many important properties, such as:

Subadditivity: $S_{AB} \leq S_A + S_B$

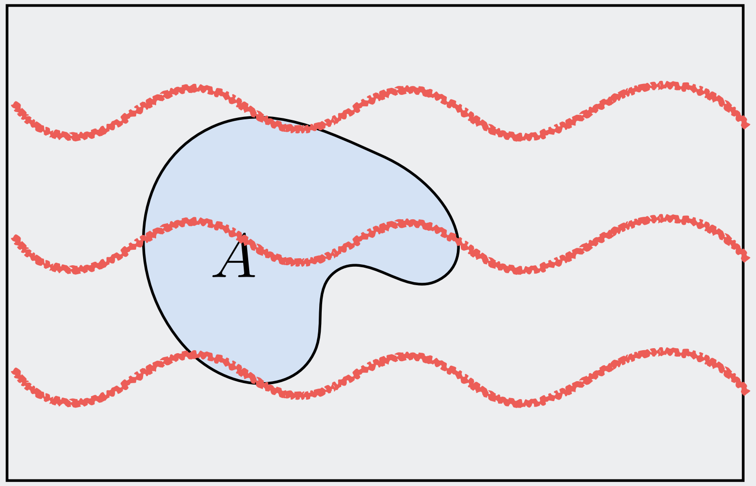
Mutual information: $I_{A:B} := S_A + S_B - S_{AB} \geq 0$

Strong subadditivity: $S_{AB} + S_{BC} \geq S_B + S_{ABC}$

(Lieb-Ruskai '73) $(I_{A:BC} \geq I_{A:B})$



# Entanglement Entropy in QFT



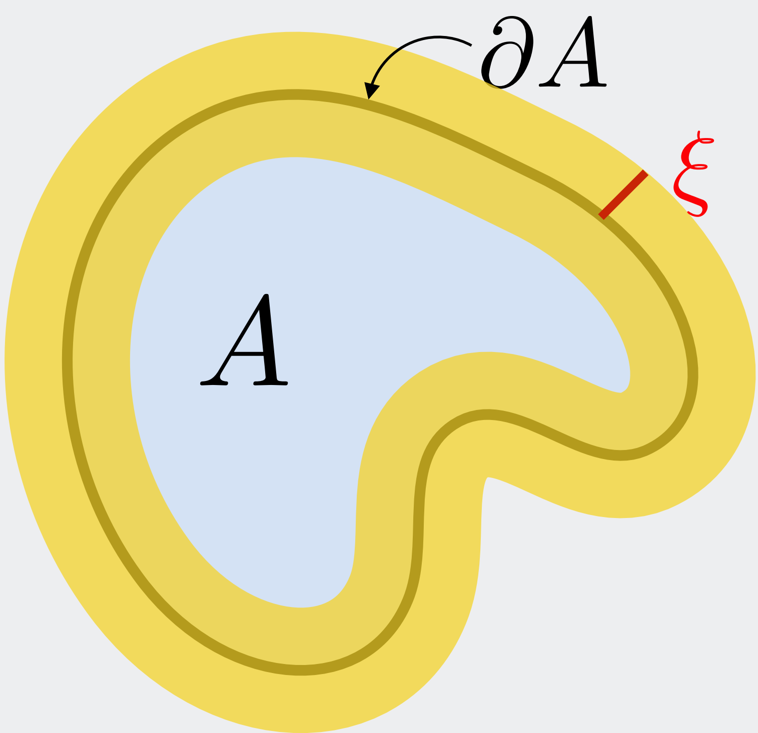

In quantum field theories (& many-body systems), spatial regions are highly entangled with each other

Consider microwave cavity
Even in vacuum, electromagnetic field fluctuates:
zero-point quantum fluctuations of modes
Each mode is distributed in space
=> fluctuations are spatially correlated
=> any part $A$ of cavity is entangled with rest

$S_A$ is ultraviolet divergent due to entanglement of short-wavelength modes across $\partial A$

Massive (gapped) field: entanglement extends out to *correlation/Compton length*

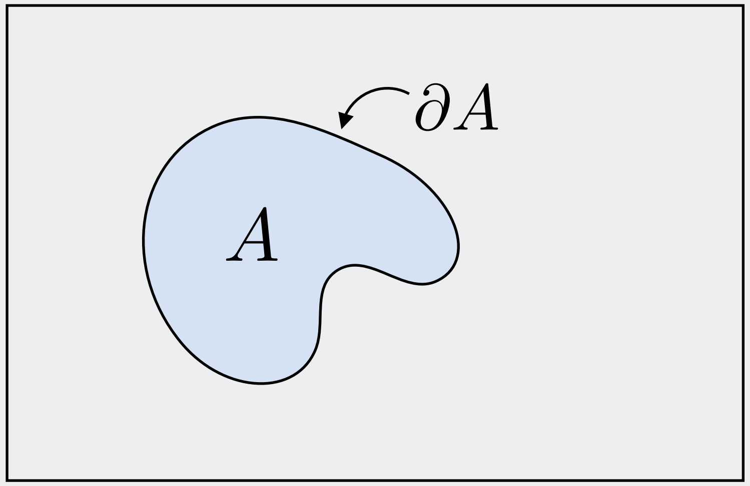
$$\xi \sim 1/m$$




$$S_A = \mathrm{area}(\partial A)\left(\frac{1}{\epsilon^{d-1}} - \frac{1}{\xi^{d-1}}\right) + \cdots$$

short-distance cut-off

# Entanglement Entropy in QFT




In quantum field theories (& many-body systems), spatial regions are highly entangled with each other

$S_A$ depends on:

- parameters of theory (including $\epsilon$)
- state
- size and shape of region $A$

Contains a lot important physics

Examples:

length$(\partial A)$

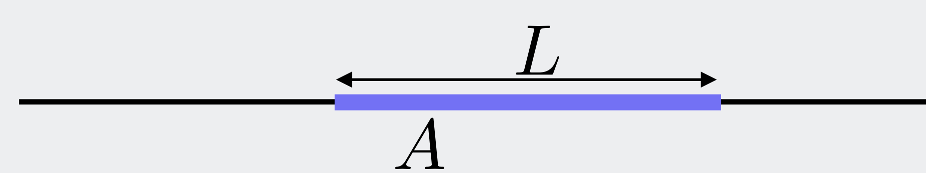
- Gapped theory in $d = 2$ : $S_A = \dfrac{L}{\epsilon} - \dfrac{L}{\xi} - \gamma$

topological entanglement entropy
(Kitaev-Preskill '05; Levin-Wen '05)

correlation length

central charge

- Critical (conformal) theory in $d = 1$: $S_A = \dfrac{c}{3} \ln \dfrac{L}{\epsilon}$



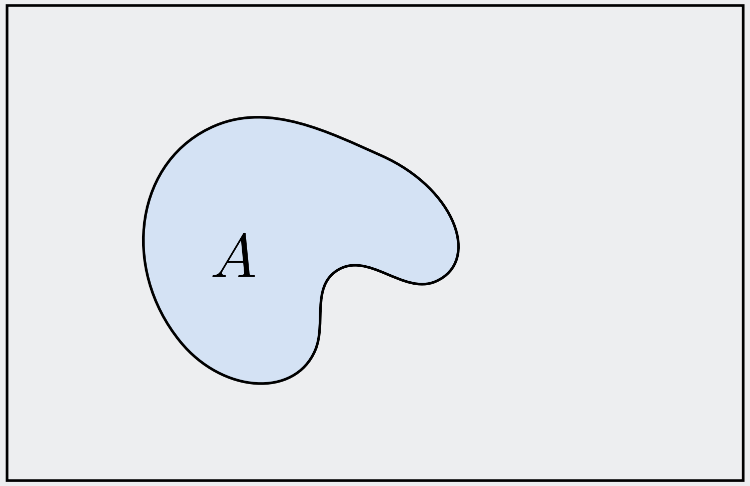

(Holzhey-Larsen-Wilczek '94; Calabrese-Cardy '03)

short-distance cutoff

- At finite temperature, also usual extensive entropy: $s(T) \times \mathrm{volume}(A)$

thermal entropy density

# Entanglement Entropy in QFT




Powerful probe of QFTs and many-body systems:

- quantum criticality
- topological order
- renormalization-group flows
- energy conditions
- many-body localization
- quenches
- much more…

However, usually very difficult to compute—even in free theories

*Simplifies* in certain theories with *many strongly-interacting* fields…

# Holographic dualities

Consider a QFT with $N$ interacting fields
for example $SU(n)$ Yang-Mills theory, $N \sim n^2$

When $N$ is large, these fields may admit a *collective* description in terms of a small number of degrees of freedom

- classical (think of hydrodynamics)
- usually complicated

However, in certain cases, when the fields are very *strongly* interacting, it simplifies dramatically:

General relativity in $d+1$ dimensions with cosmological constant $\Lambda < 0$
(plus some matter fields)
subject to certain boundary conditions: "universe in a box"
(Maldacena '97)

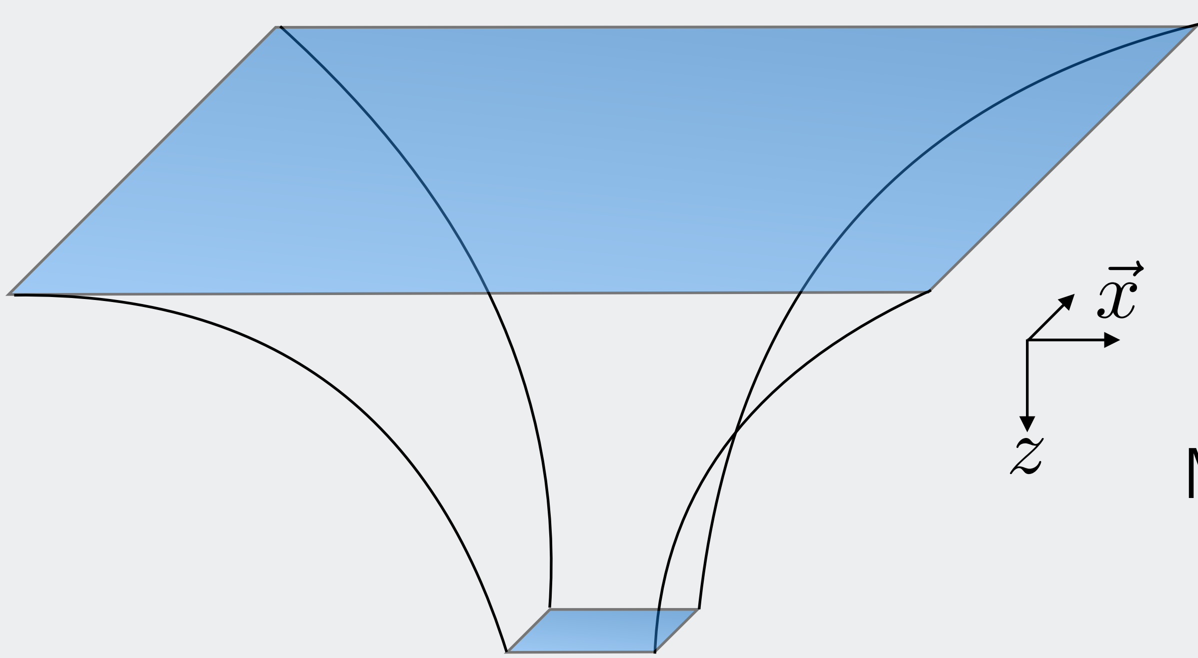
# Holographic dualities

If QFT is conformal (scale-invariant), ground state is anti-de Sitter (AdS) spacetime:

AdS radius $\quad\quad\quad\quad$ $d$ dimensions of QFT

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dz^2 + d\vec{x}^2\right)$$

extra dimension




$z = \epsilon$ boundary of AdS, where field theory "lives"

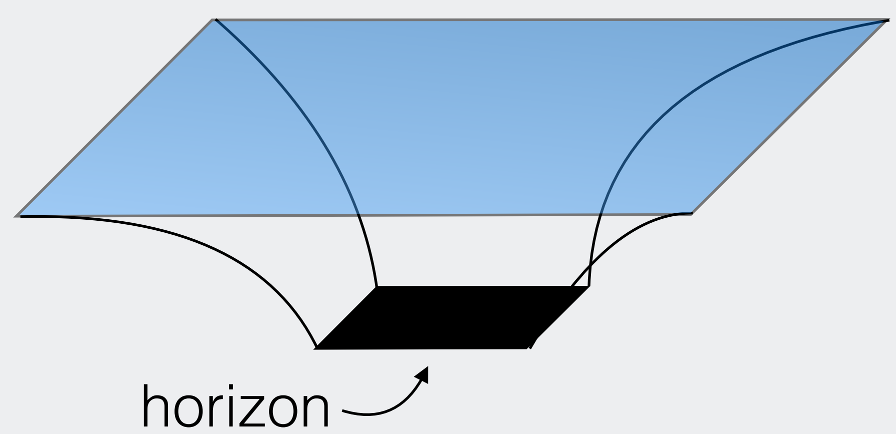
Map between boundary and bulk is non-local

If QFT is gapped (massive), space ends on wall at $z_{\max} \sim \xi$ (correlation length)

Many specific examples known in various dimensions
(mostly supersymmetric, derived from string theory)

# Holographic Dualities

| QFT | GR |
|---|---|
| $N$ | AdS radius $\dfrac{R^{d-1}}{G_{\mathrm{N}}\hbar}$ Planck area |
| thermodynamic limit $N \to \infty$ | classical limit $\hbar \to 0$ |
| statistical fluctuations | quantum fluctuations |
| collective modes | gravitational waves, etc. |
| deconfined plasma | black hole |
| $S \propto N$ | $S = \dfrac{\mathrm{area(horizon)}}{4G_{\mathrm{N}}\hbar}$ |



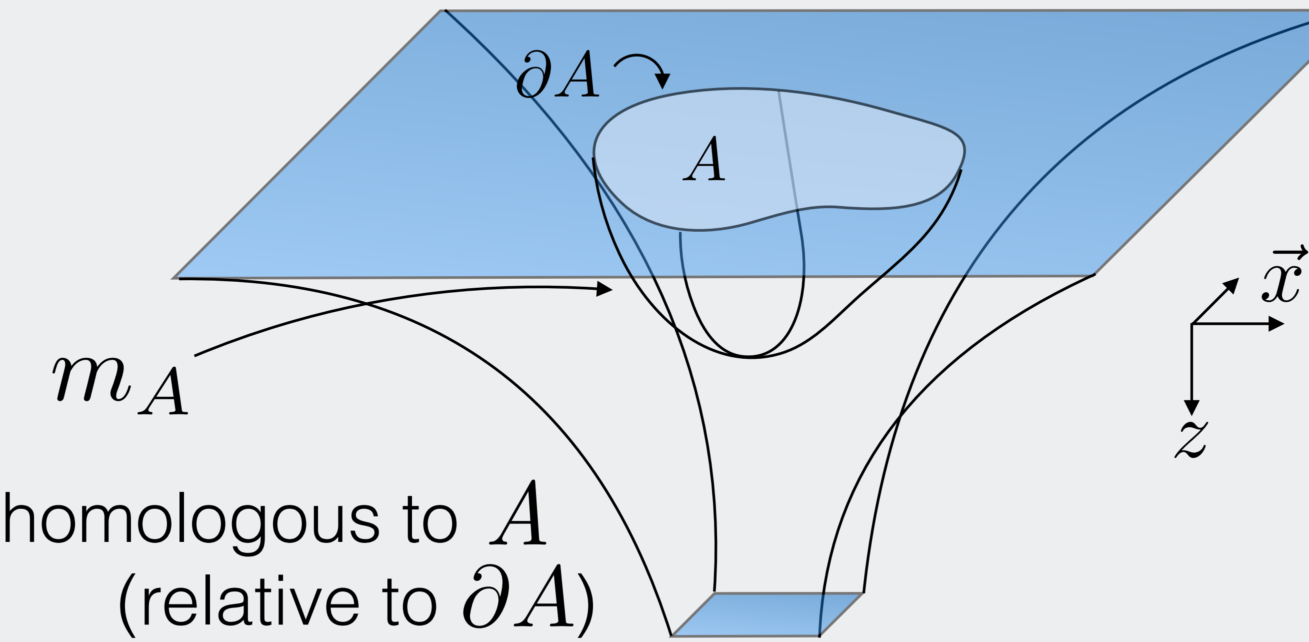

Holographic dualities are useful for computing *many* things in strongly interacting QFTs

Let's talk about entanglement entropies...

# Holographic Entanglement Entropy

Ryu-Takayanagi '06:

$$S_A = \frac{\mathrm{area}(m_A)}{4G_\mathrm{N}\hbar}$$




$m_A =$ minimal-area hypersurface homologous to $A$ (relative to $\partial A$)

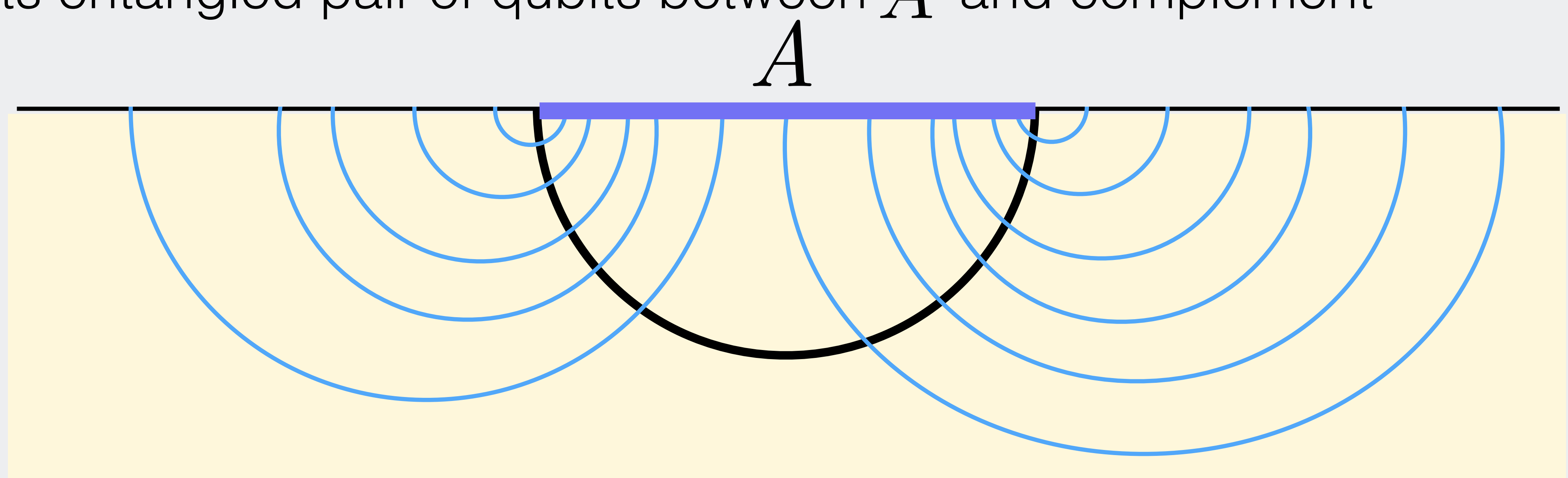
hangs down in order to minimize area

MH-Freedman '16: $S_A =$ max # "bit threads" connecting $A$ to rest of boundary

(equivalence to minimal surface by Riemannian max flow-min cut theorem Federer '74, ...)

Each bit thread has cross section of 4 Planck areas

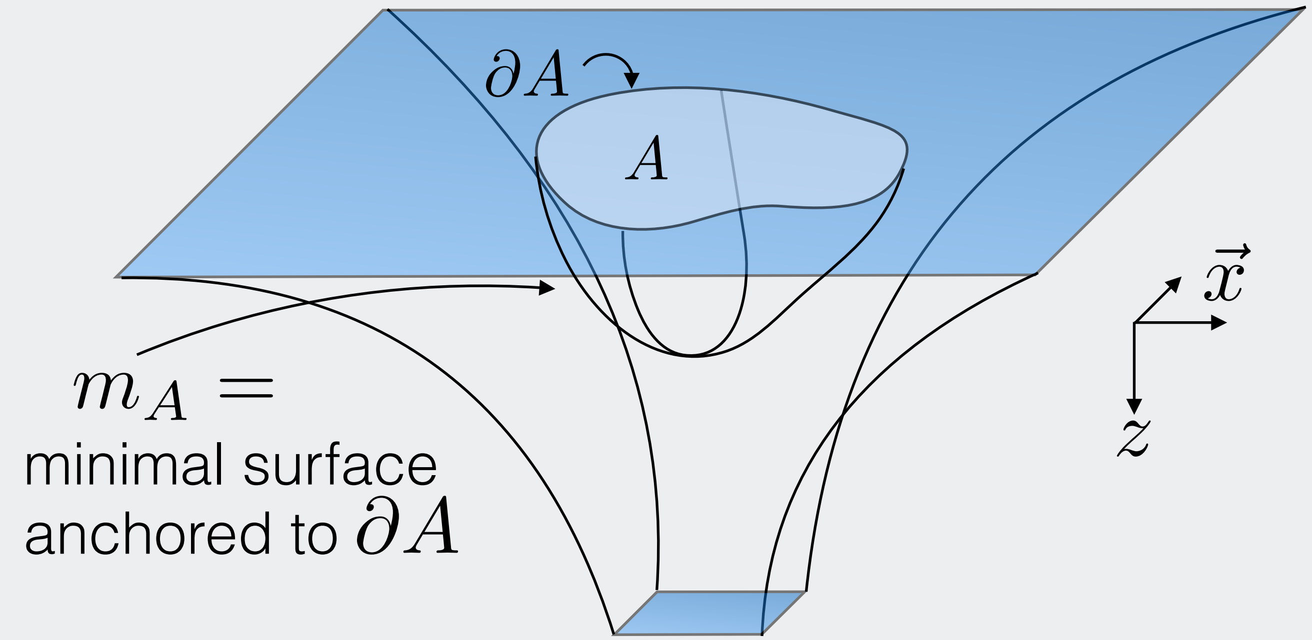
Represents entangled pair of qubits between $A$ and complement



# Holographic Entanglement Entropy

Ryu-Takayanagi '06:

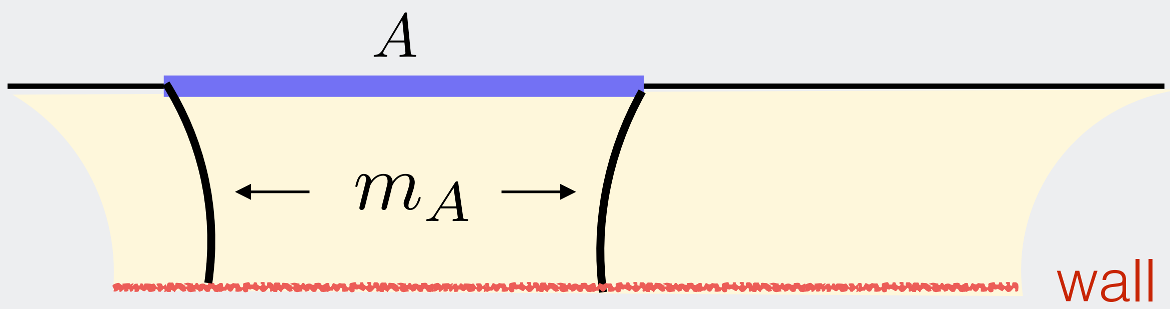
$$S_A = \frac{\text{area}(m_A)}{4G_{\mathrm{N}}\hbar}$$



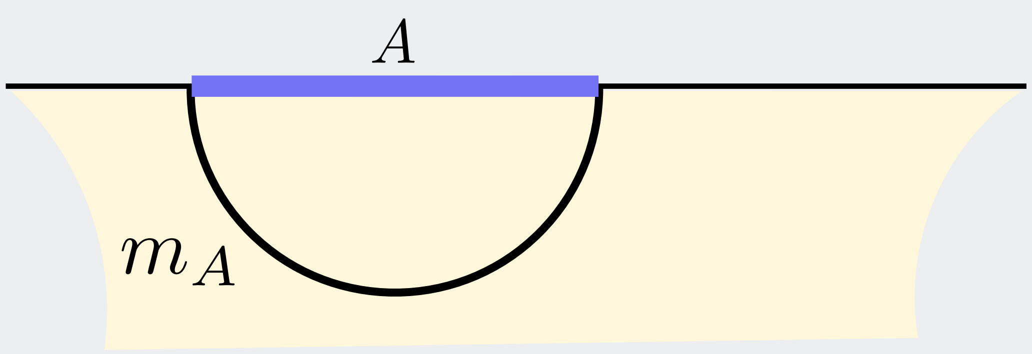

Geometrizes entanglement:

- Area-law UV divergence due to infinite area of $m_A$ near boundary
- Gapped theory:
  minimal surface extends to wall
  (Klebanov, Kutasov, Murugan '07)

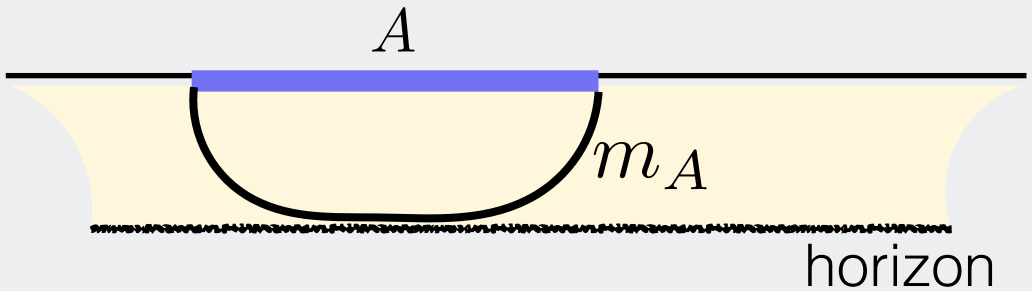
$$S_A = \text{area}(\partial A)\left(\frac{1}{\epsilon^{d-1}} - \frac{1}{\xi^{d-1}}\right) + \cdots$$




- Conformal theory in $d = 1$ : $S_A = \frac{c}{3} \ln \frac{L}{\epsilon}$




- Finite temperature:
  minimal surface hugs horizon
  => extensive entropy $s(T)\text{volume}(A)$




Democratizes Bekenstein-Hawking: not about horizons!

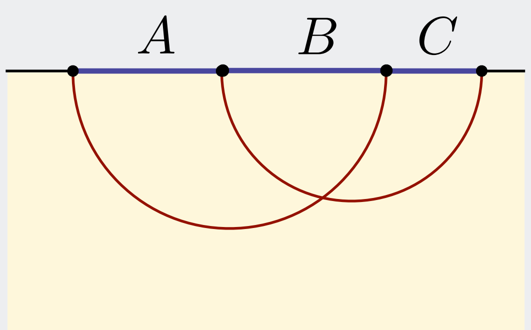
# Holographic Entanglement Entropy

*Quantum information theory* is built into *classical spacetime geometry*

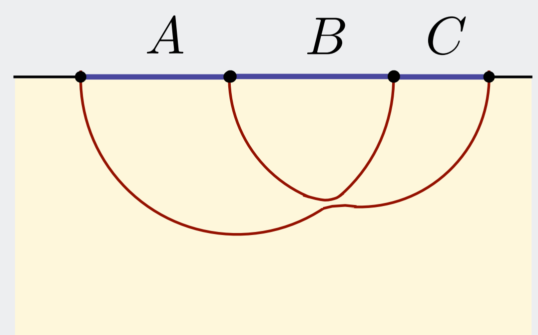
Example: Strong subadditivity

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

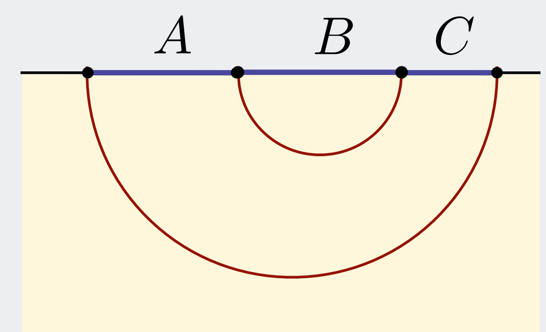
$S_{AB} + S_{BC} =$




$=$




$\geq$




$= S_{ABC} + S_B$

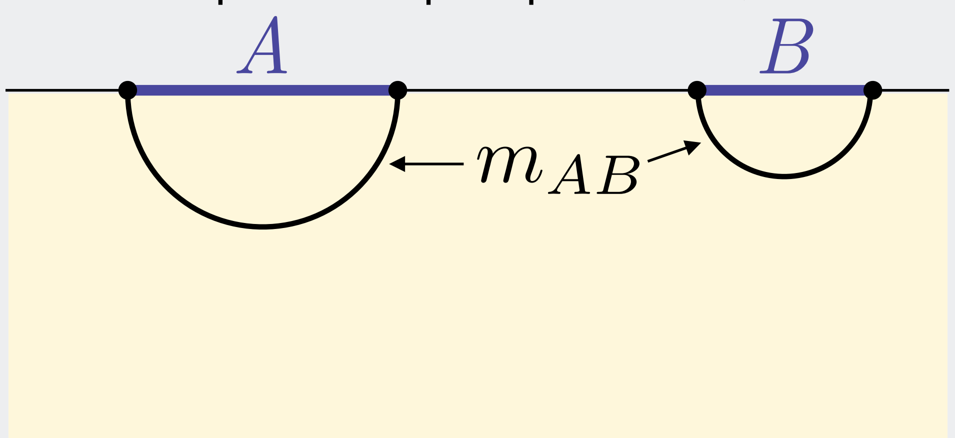
(MH-Takayanagi '07)

(Dual proof using bit threads Freedman-MH '16)

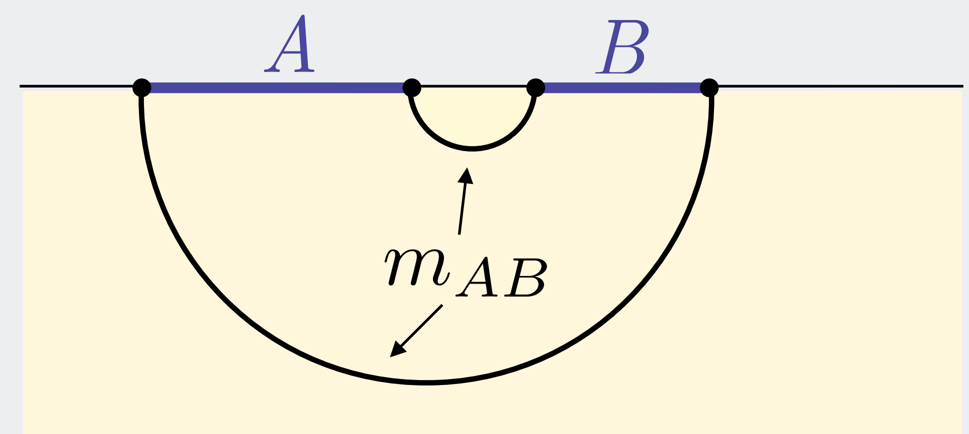
In fact, RT formula obeys *all* general properties of entanglement entropies
(Hayden-MH-Maloney '11; MH '13)

# Holographic Entanglement Entropy

Also has special properties, such as phase transitions (MH '10)




$S_{AB} = S_A + S_B \quad \Rightarrow \quad I_{A:B} = 0$




$S_{AB} < S_A + S_B \quad \Rightarrow \quad I_{A:B} > 0$

*Monogamy of mutual information* inequality (Hayden-MH-Maloney '11; MH '13):

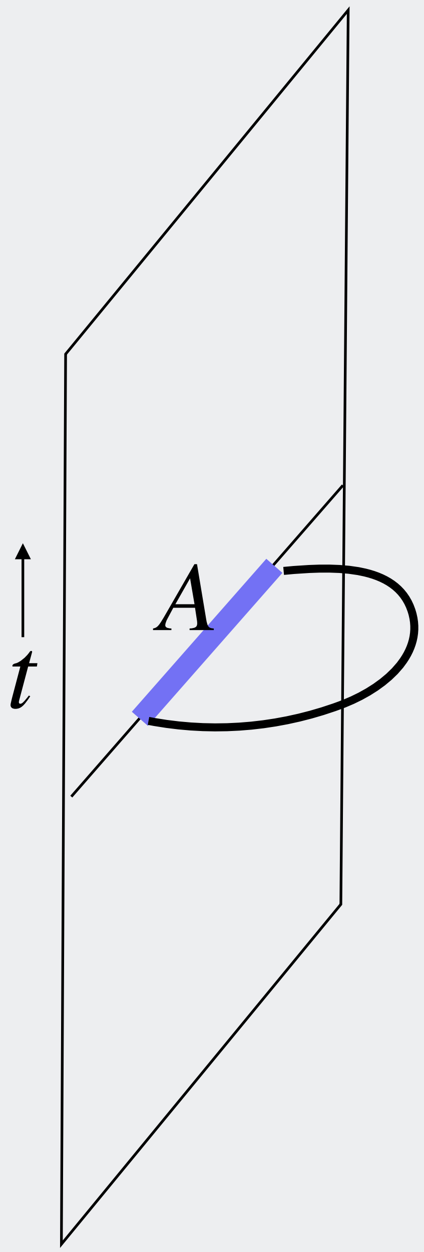
$$S_{AB} + S_{AC} + S_{BC} \geq S_A + S_B + S_C + S_{ABC}$$

$$I_{A:BC} \geq I_{A:B} + I_{A:C}$$

# Holographic Entanglement Entropy

So far, we've ignored time

In space*time*, find minimal-area codimension-2 spacelike surface homologous to $A$
(Hubeny-Rangamani-Takayanagi '07)




Bulk spacetime geometry (Einstein equation) directly related to quantum information properties in field theory:

- Strong subadditivity (Wall '12)
- Causality (no faster-than-light signalling) (MH-Hubeny-Lawrence-Rangamani '14)
- First law of entanglement:

$$\Delta S = \langle H_{\mathrm{mod}} \rangle$$

To some extent, Einstein equation can be *derived* from properties of entanglement (Lashkari-McDermott-Van Raamsdonk '13)

# Holographic Entanglement Entropy

Many other developments:

- Relation between bulk & boundary modular Hamiltonians and relative entropies
  (Jafferis-Lewkowycz-Maldacena-Suh '15)
- Derivation of RT formula (MH '10, Lewkowycz-Maldacena '13)
- Tensor networks for modelling holography (Swingle '08)
- Bit threads & entanglement structures (Cui et al '18)
- Holography as quantum error-correcting code (Almheiri-Dong-Harlow '14)
- ...

# Holographic Entanglement Entropy

Many other developments:

- Relation between bulk & boundary modular Hamiltonians and relative entropies
  (Jafferis-Lewkowycz-Maldacena-Suh '15)
- Derivation of RT formula (MH '10, Lewkowycz-Maldacena '13)
- Tensor networks for modelling holography (Swingle '08)
- Bit threads & entanglement structures (Cui et al '18)
- Holography as quantum error-correcting code (Almheiri-Dong-Harlow '14)
- ...