Replica Symmetry Breaking for mean field spin glass models

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September 15, 2018

Outline

1. Models and background
2. Replica Symmetry Breaking for mixed p-spin models
3. Replica Symmetry Breaking for spherical mixed p-spin models

Outline

What are spin glasses?

- In the 1960s, physicists observed a low temperature state of certain magnetic alloys, such as CuMn, which is distinct from conventional ferromagnetic materials. This new state is called spin glass.

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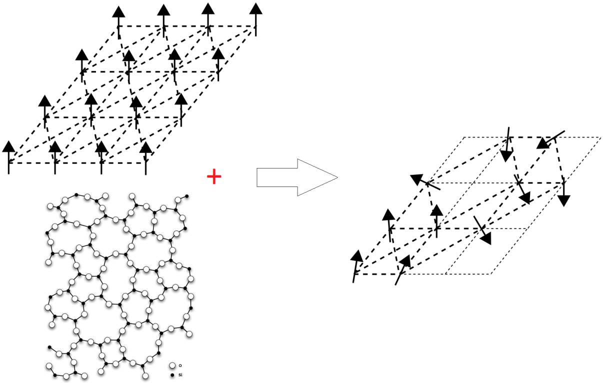

Figure: Wikipedia

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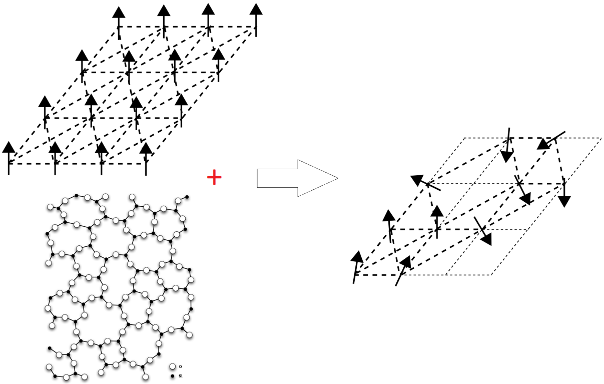

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- Models: RKKY $\Rightarrow$ Edward–Anderson $\Rightarrow$ Sherrington–Kirkpatrick, etc.

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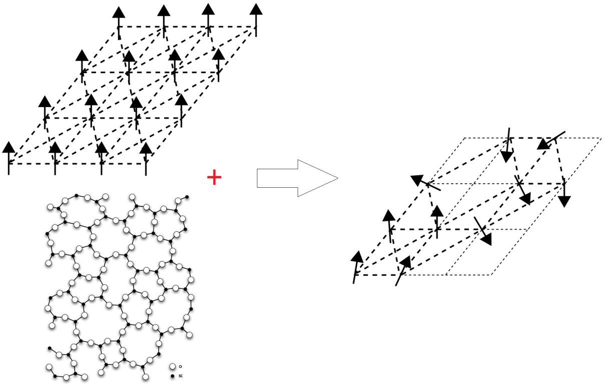

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- Models: RKKY ⇒ Edward–Anderson ⇒ Sherrington–Kirkpatrick, etc.
- Application to combinatorial optimization problems

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- Goal: maximize the "happiness" function!

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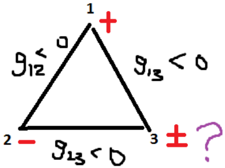
- Frustration:



The Sherrington–Kirkpatrick (SK) model (1975)

- Spins $\sigma = (\sigma_1, ..., \sigma_N) \in \{\pm 1\}^N$; Hamiltonian (or energy)

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \le i,j \le N} g_{ij} \sigma_i \sigma_j.$$

- g_{ij}: independent normal (=Gaussian) $N(0,1)$: $\mathbb{E} g_{ij} = 0$, $\mathbb{E} g_{ij}^2 = 1$.

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  Given by inner product (thus distance) of two spins!
- Define the overlap for $\sigma^1, \sigma^2 \in \{\pm 1\}^N$,

$$R_{1,2} = R(\sigma^1, \sigma^2) = \frac{1}{N} \sum_{i=1}^N \sigma_i^1 \sigma_i^2 .$$

Fundamental problem and quantities

- The problem: compute the ground state energy

$$\max_{\sigma\in\{\pm 1\}^N} H_N(\sigma).$$

- Statistical mechanics: smooth approximation through the partition function

$$Z_{N,\beta} = \sum_{\sigma\in\{\pm 1\}^N} e^{\beta H_N(\sigma)}, \quad \beta = 1/T,$$

and the free energy

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- High temperature: T large and β small;
  low temperature: T small and β large;
  zero temperature: $T = 0$ and $\beta = \infty$.

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- Free energy vs. ground state energy

$$\frac{1}{N}\max_{\sigma\in\{\pm1\}^N} H_N(\sigma) \le \frac{1}{\beta N}\log Z_{N,\beta} \le \frac{\log 2}{\beta} + \frac{1}{N}\max_{\sigma\in\{\pm1\}^N} H_N(\sigma).$$

Parisi predictions (1979–1983) about the SK model

The problem:

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2. $\mathbb{E}{G_{N,\beta}}^{\otimes 2}(|R_{1,2}|\in B)\to \mu_\beta(B)$ as $N\to\infty$. Here
   - $G_{N,\beta}(\sigma) := \frac{1}{Z_{N,\beta}}e^{\beta H_N(\sigma)}$ is the Gibbs measure,
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4. The Parisi Full-step Replica Symmetry Breaking (FRSB) prediction: at low temperature, the symmetry of replicas is broken infinitely many times.

Remarks about the Parisi predictions

- The Parisi functional: for $\mu \in \mathrm{Prob}[0,1]$

$$\mathcal{P}_\beta(\mu) := \frac{\log 2}{\beta} - \int_0^1 \beta\mu[0,s]s\mathrm{d}s + \Psi_{\mu,\beta}(0,0),$$

where $\Psi_{\mu,\beta}(t,x)$ is the solution to

$$\partial_t \Psi_{\mu,\beta}(t,x) = -\partial_{xx}\Psi_{\mu,\beta}(t,x) - \beta\mu[0,t](\partial_x \Psi_{\mu,\beta}(t,x))^2$$

for $(t,x) \in [0,1) \times \mathbb{R}$ with boundary condition $\Psi_{\mu,\beta}(1,x) = \frac{\log\cosh\beta x}{\beta}$.

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- The Gibbs measures:

$$G_{N,\beta}(\sigma) := \frac{1}{Z_{N,\beta}} e^{\beta H_N(\sigma)}.$$

Typically, low temperature problems are harder!

The pure p-spin model (Derrida, 1981)

- For $p \geq 3$, $\sigma = (\sigma_1, ..., \sigma_N) \in \{\pm 1\}^N$

$$H_N(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{1 \leq i_1, \dots, i_p \leq N} g_{i_1,\dots,i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

$$g_{i_1,\dots,i_p} \quad \text{independent } N(0,1).$$

- Covariance

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N R_{1,2}^p.$$

The mixed p-spin model

- $\sigma = (\sigma_1, ..., \sigma_N) \in \{\pm 1\}^N$,

$$H_N(\sigma) = \sum_{p \geq 2} \frac{c_p}{N^{(p-1)/2}} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

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- Covariance

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N \xi(R_{1,2}),$$

  where $\xi(x) = \sum_{p=2}^{\infty} c_p^2 x^p$.

- The SK model: $\xi(x) = x^2$.
- The pure p-spin model: $\xi(x) = x^p, p \geq 3$.

The mixed p-spin model

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- The SK model: $\xi(x) = x^2$.
- The pure p-spin model: $\xi(x) = x^p, p \geq 3$.
- ξ is generic if

$$\mathrm{span}\{1, x, x^p : c_p \neq 0\} \text{ is dense in } C[-1,1].$$

SK and pure p-spin are not generic!

Parisi predictions for mixed p-spin model: progress

The Parisi predictions have driven the field for the last 40 years.

1. The Parisi formula: $\lim_{N\to\infty} \frac{1}{\beta N} \log Z_{N,\beta} = \inf_{\mu\in\mathrm{Prob}[0,1]} \mathcal{P}_\beta(\mu)$.
   Talagrand (Ann. Math. 2006): true for convex ξ.
2. $|R_{1,2}| \Rightarrow \mu_\beta$ as $N \to \infty$
   Talagrand (JFA 2006): true for generic ξ.
3. The Parisi ultrametricity conjecture
   Panchenko (Ann. Math. 2013): true for generic ξ.
4. The Parisi FRSB prediction
   Auffinger–Chen–Z. (2017): true at $T = 0$ for any ξ.

Parisi FRSB prediction: significance

- FRSB is indispensable in Parisi's original deduction of his formula.

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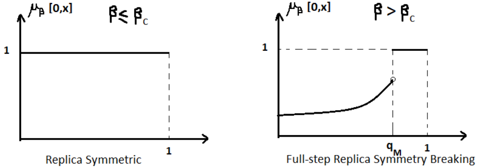
- It indicates phase transition: for the SK model $\beta_c = \frac{1}{\sqrt{2}}$,



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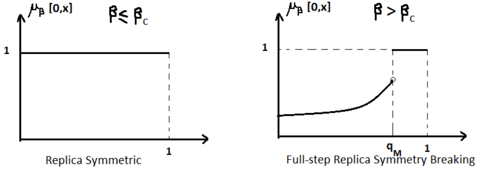
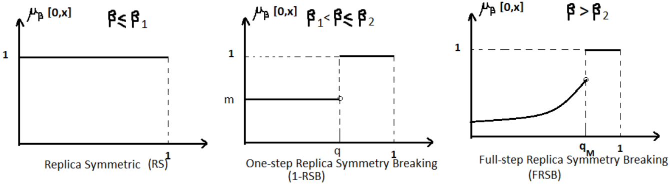


- For pure p-spin model (i.e. $\xi(x) = x^p$): Gardner transition (1985)



Replica Symmetry Breaking: meaning in physics

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- Replica Symmetric (RS): all the ℓ replicas are the same (symmetric).

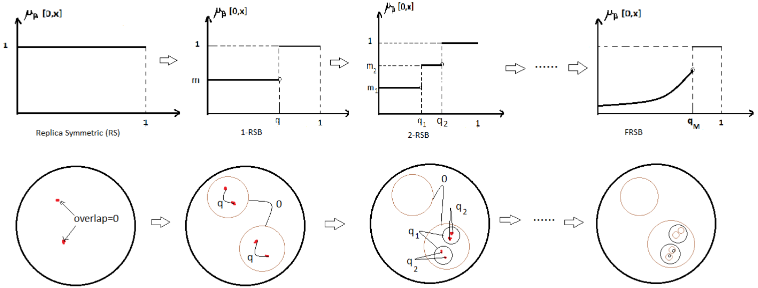
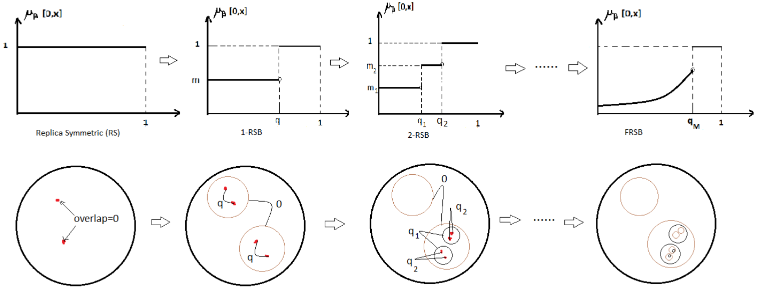


Figure: The first row is the functional order parameter. The second row is the corresponding geometric picture in the large N limit.

Replica Symmetry Breaking: meaning in physics

$$\log Z_N = \lim_{\ell \to 0} \frac{Z_N^\ell - 1}{\ell}.$$

- 1-RSB: ℓ replicas should be divided into two distinct groups (Break Replica Symmetry).



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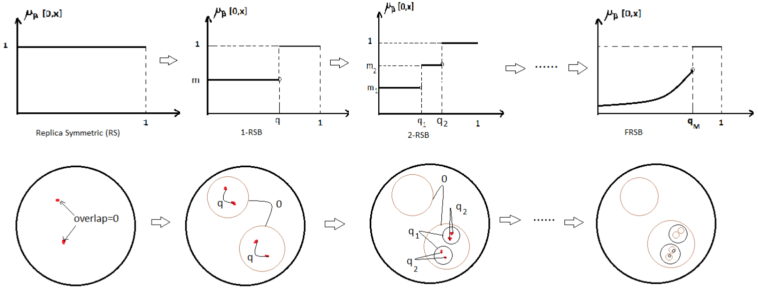
Replica Symmetry Breaking: meaning in physics

$$\log Z_N = \lim_{\ell \to 0} \frac{Z_N^\ell - 1}{\ell}.$$

- 2-RSB: each group should be divided into two distinct subgroups.



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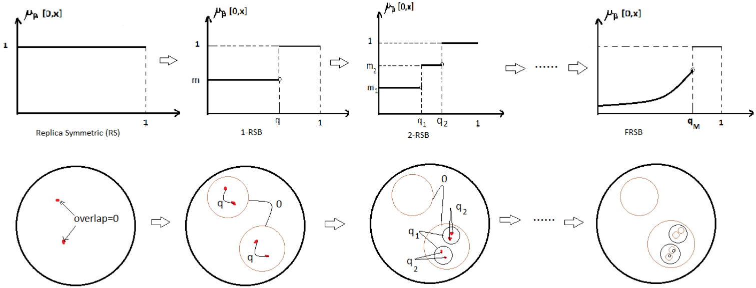
Replica Symmetry Breaking: meaning in physics

$$\log Z_N = \lim_{\ell \to 0} \frac{Z_N^\ell - 1}{\ell}.$$

- FRSB: the procedure should proceed infinitely many times.



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Outline

Main result I: FRSB at zero temperature

Recall $\mu_\beta := \operatorname{argmin} \mathcal{P}_\beta(\mu)$ is the functional order parameter.
A model ξ is

- Replica Symmetric (RS) if $\# \operatorname{supp} \mu_\beta = 1$,
- k-step Replica Symmetry Breaking (k-RSB) if $\# \operatorname{supp} \mu_\beta = k+1$,
- Full-step Replica Symmetry Breaking (FRSB) if $\# \operatorname{supp} \mu_\beta = \infty$.

Main result I: FRSB at zero temperature

- Parisi formula for ground state energy (Auffinger–Chen, AoP 2017): for any ξ,

$$\lim_{N\to\infty} \max_{\sigma\in\{\pm 1\}^N} \frac{H_N(\sigma)}{N} = \inf\{\mathcal{P}(\gamma) : \gamma \in \mathrm{Meas}[0,1)\}.$$

Here $\mathcal{P}(\gamma)$ is the Parisi functional at zero temperature:

$$\begin{aligned}
\mathcal{P}(\gamma) &= \Psi_\gamma(0,h) - \frac{1}{2}\int_0^1 t\xi''(t)\gamma([0,t])\mathrm{d}t,\\
\partial_t \Psi_\gamma(t,x) &= -\frac{\xi''(t)}{2}\left(\partial_{xx}\Psi_\gamma(t,x) + \gamma([0,t])(\partial_x \Psi_\gamma(t,x))^2\right),\\
\Psi_\gamma(1,x) &= |x|.
\end{aligned}$$

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Theorem (Auffinger–Chen–Z. arXiv:1703.06872)

Let $\gamma^* = \operatorname{argmin} \mathcal{P}(\gamma)$ be the functional order parameter (or the Parisi measure) at zero temperature. Then for any ξ,

$$\# \operatorname{supp} \gamma^* = \infty.$$

In other words, FRSB holds at zero temperature for any mixed p-spin model.

Consequences

Theorem

For any ξ, $\#\operatorname{supp}\mu_\beta \to \infty$ *as* $\beta \to \infty$.

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Corollary

For the pure p*-spin model* $\xi(x) = x^p$*, the Gardner* 1*-RSB phase cannot exist for arbitrarily low temperature.*

FRSB: progress and history

- The SK model is RS in the high temperature regime $\beta < \frac{1}{\sqrt{2}}$ (Aizenman–Lebowitz–Ruelle, CMP 1987).
- The SK model is not RS in the low temperature regime $\beta > \frac{1}{\sqrt{2}}$ (Toninelli, Europhys. Lett. 2002).
- For sufficiently low temperature the mixed p-spin model is at least 2-RSB (Auffinger–Chen, PTRF 2015).

FRSB: open problems

- Parisi prediction for mixed p-spin model at low but positive temperature: for any $\xi(x) = \sum_{p=2}^{\infty} c_p^2 x^p$, $\exists \beta_c > 0$ such that the model is FRSB for $\beta > \beta_c$.

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- Gardner transition for pure p-spin model.

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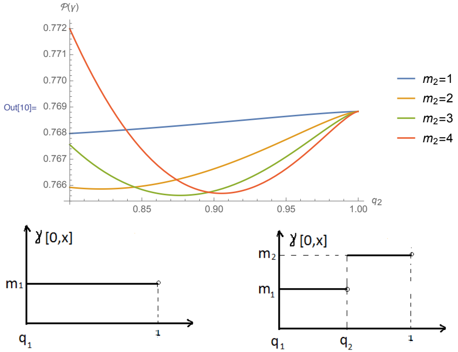
- Gardner transition for pure p-spin model.
- Conjecture: in the FRSB phase, μ_β has an absolutely continuous part.
- Conjecture (Oppermann–Sherrington, Phys. Rev. Lett. 2005): for the SK model at zero temperature, the inverse function of $\gamma^*(q)$ is

$$q(x) = \frac{\sqrt{\pi}x}{2\Xi}\mathrm{erf}\left(\frac{\Xi}{x}\right)$$

  for some constant $\Xi \approx 1.13$ and $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t$.

Proof idea for $\# \operatorname{supp} \gamma^* = \infty$

Assume the SK model is 1-RSB. Numerical calculation yields $m_1 \approx 0.817264$. Plot for $m_2 = 1, 2, 3, 4$, $q_2 \in [0.8, 0.99)$.



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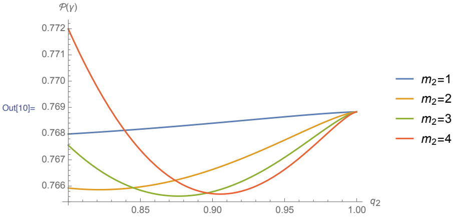

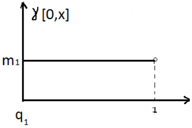

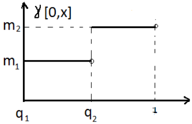

- Proof by contradiction: assume γ has n atoms, then show that the Parisi functional $\mathcal{P}(\gamma)$ is lowered if we add one more jump in a small neighborhood of 1.

Proof Sketch I

- Assume $\gamma_P = \gamma$ has n atoms and work with SK model ($\xi(s) = s^2/2, \xi'(s) = s, \xi''(s) = 1$).
- $\gamma(t) = \sum_{i=0}^{n-1} m_i \mathbf{1}_{[q_i, q_{i+1})}(t) + m_n \mathbf{1}_{[q_n, 1)}(t)$ where

$$\begin{aligned} q_0 &= 0 \leq q_1 < q_2 < \cdots < q_n < 1, \\ m_0 &= 0 < m_1 < m_2 < \cdots < m_n < \infty. \end{aligned}$$

- Perturb $\gamma(t)$: $q = q_{n+1}, 1 = q_{n+2}$,

$$\gamma_q(t) = \sum_{i=1}^{n-1} m_i \mathbf{1}_{[q_i, q_{i+1})}(t) + m_n \mathbf{1}_{[q_n, q)}(t) + m_{n+1} \mathbf{1}_{[q, 1)}(t).$$

- Compute $\partial_q \mathcal{P}(\gamma_q)$ as $q \to 1-$.

Proof Sketch II

- $z_0, \ldots, z_{n+1}$ i.i.d. $N(0,1)$ r.v.'s. Define

$$Y_{n+2} = \Big|\sum_{j=0}^{n+1} z_j \sqrt{q_{j+1} - q_j}\Big|,$$

  and iteratively for $1 \le i \le n+1$,

$$Y_i = \frac{1}{m_i} \log \mathbb{E}_{z_i} \exp m_i Y_{i+1},$$

  and $Y_0 = \mathbb{E}_{z_0} Y_1$.

- Using Cole–Hopf representation,

$$\mathcal{P}(\gamma_q) = Y_0 - \frac{1}{2}\sum_{i=1}^{n} m_i \int_{q_i}^{q_{i+1}} t\xi''(t)\mathrm{d}t - \frac{m_{n+1}}{2}\int_q^1 t\xi''(t)\mathrm{d}t.$$

Proof Sketch III

- It turns out

$$\lim_{q\to 1-} \partial_q \mathcal{P}(\gamma_q) = 0.$$

- After some hard work, one can show

$$\limsup_{q\to 1-} \partial^2_{qq} \mathcal{P}(\gamma_q) < 0.$$

- Main tool: Gaussian integration by parts

$$\mathbb{E}[zf(z)] = \mathbb{E}[f'(z)], \quad z \sim N(0,1).$$

Outline

The spherical mixed p-spin model

- $\sigma = (\sigma_1, ..., \sigma_N) \in S_N = \{(\sigma_1, ..., \sigma_N) \in \mathbb{R}^N : \sum_{i=1}^N \sigma_i^2 = N\}$,

$$H_N(\sigma) = \sum_{p\geq 2} \frac{c_p}{N^{(p-1)/2}} \sum_{1\leq i_1,...,i_p\leq N} g_{i_1,...,i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

$$g_{i_1,...,i_p} \ \text{ independent } N(0,1).$$

- Covariance

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N\xi(R_{1,2}),$$

where $\xi(x) = \sum_{p=2}^{\infty} c_p^2 x^p$.

- The spherical SK model (Kosterlitz–Thouless–Jones, 1976):

$$\xi(x) = x^2.$$

- The spherical pure p-spin model for $p \geq 3$:

$$\xi(x) = x^p.$$

Limiting free energy and ground state energy

Partition function:
$$Z_{N,\beta} = \int_{S_N} e^{\beta H_N(\sigma)} \lambda_N(\mathrm{d}\sigma).$$

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- For $T = 0$ (Chen–Sen, CMP 2017; Jagannath–Tobasco, CMP 2017)

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- Let

$$\mu_\beta := \operatorname{argmin} \mathcal{Q}_\beta(\mu),$$
$$\gamma^* := \operatorname{argmin} \mathcal{Q}(\gamma).$$

- The level of RSB is classified by $\#\,\mathrm{supp}\,\mu_\beta$ and $\#\,\mathrm{supp}\,\gamma^*$.

Replica Symmetry Breaking for spherical models

- Panchenko–Talagrand (AoP 2007): at positive temperature,
  - the spherical SK model is RS;
  - the spherical pure p-spin model is RS for $\beta \leq \beta_c$, and 1-RSB for $\beta > \beta_c$.
- Auffinger–Chen (PTRF 2015): at positive temperature, some $2+p$ spin models (i.e. $\xi(x) = (1-\lambda)x^2 + \lambda x^p$) are FRSB.
- Chen–Sen (CMP 2017): the zero temperature $\beta = \infty$ case is the same as the low temperature case.

RSB for spherical models: conjecture and prediction

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- Crisanti–Leuzzi (Phys. Rev. B 2007) predicted 2-RSB model exists at certain positive temperature for some $3 + 16$ model ($\xi(x) = (1 - \lambda)x^3 + \lambda x^{16}$).
- Auffinger–Ben Arous (AoP 2013) conjectured at zero temperature excluding SK, there is a classification of spherical mixed p-spin models: either 1-RSB (pure-like) or FRSB (full mixture).

Main result II: 2-RSB exists for spherical models

Theorem (Auffinger–Z. CMP 2019+)

At zero temperature, there exist 2-RSB spherical $s+p$ models ($\xi(x) = (1-\lambda)x^s + \lambda x^p$) in both pure-like and full mixture regimes.

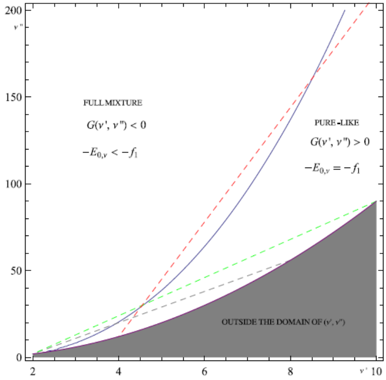

Figure: Auffinger–Ben Arous (AoP 2013)

Consequence for energy landscape of $H_N(\sigma)$

Corollary

If ξ is a 2-RSB model (with some mild extra properties), we have for any $\varepsilon > 0$ there exist $\eta, K > 0$ such that for all $N \geq 1$,

$$\mathbb{P}_N(\eta, [-1+\varepsilon, -q-\varepsilon] \cup [-q+\varepsilon, -\varepsilon] \cup [\varepsilon, q-\varepsilon] \cup [q+\varepsilon, 1-\varepsilon]) \leq Ke^{-N/K},$$

where for any Borel set $A \subset [-1,1]$,

$$\mathbb{P}_N(\eta, A) := \mathbb{P}(\exists\, \sigma^1, \sigma^2 \in \mathcal{L}(\eta),\ \textit{with } R_{1,2} \in A).$$

and

$$\mathcal{L}(\eta) := \{\sigma \in S_N : H_N(\sigma) \geq N(\max_{\sigma' \in S_N} H_N(\sigma') - \eta)\}.$$

In words, with overwhelming probability, near maxima have distance within $\sqrt{2\varepsilon}$, or $\sqrt{2(1-q\pm\varepsilon)}$ apart, or $\sqrt{2(1\pm\varepsilon)}$ apart (orthogonal).

Some open questions

- Easy criterion for 2-RSB at zero temperature?
- k-RSB ($2 \leq k < \infty$) for positive temperature?
- In general, classify the RSB phase diagram?

Thank you very much for your attention!