(Γ, V, E, op, μ) $\mu: E \to \mathbb{R}^+$ s.t. $\mu(e)\mu(e^{op}) = 1$.

involution on E

directed graph

Remark: If $\bar{\mu}: V \to \mathbb{R}^+$ $\rightsquigarrow$ $\mu: E \to \mathbb{R}^+$ by $\mu(e) = \frac{\bar{\mu}(t(e))}{\bar{\mu}(s(e))}$

$A = \ell^\infty(V) = \text{span}\{p_v : v \in V\}$

Path Space: $\text{span}\{e_1 \cdots e_n : t(e_j) = s(e_{j+1})\}$
equipped with A-valued inner product:

$$\langle \sigma | \sigma' \rangle = \delta_{\sigma, \sigma'} p_{t(\sigma)}$$

Def: $Y_e = \ell(e) + \sqrt{\mu(e)}\, \ell(e^{op})^*$

$$S(\Gamma, \mu) = C^*(A, \{Y_e\}_{e \in E})$$

Expectation
$$\mathbb{E} : S(\Gamma, \mu) \to A$$
$$x \mapsto \sum_{v \in V} \langle p_v | x p_v \rangle$$

States on $S(\Gamma, \mu)$: Pick $\Gamma_{Tr} < \Gamma$ max'l subject to $\mu(e_1) \cdots \mu(e_n) = 1$ for all loops $e_1 \cdots e_n$ in Γ_{Tr}

Pick $* \in V$, set $\varphi(p_*) = 1$, then for $v \in V$
$$\varphi(p_v) = \mu(\sigma)$$

where σ is a path in Γ_{Tr} s.t. $s(\sigma) = *$ $t(\sigma) = v$.
Extend φ to $S(\Gamma, \mu)$ by $\varphi \circ \mathbb{E}$.
Let
$$(M(\Gamma, \mu), \varphi) := (S(\Gamma, \mu), \varphi)''$$

Fact: 1 $\Delta_\varphi Y_e = \mu(e)\mu(\sigma) Y_e$ where σ path in Γ_{Tr} with $s(\sigma) = t(e)$, $t(\sigma) = s(e)$

Thm (Hartglass + Nelson, 2018) Assume $\Gamma_{Tr} \neq \Gamma$.
$$(M(\Gamma, \mu), \varphi) \cong (T_H, \varphi_H) \oplus \bigoplus_{v \in V} \overset{q_v}{\underset{s_v}{\mathbb{C}}}$$

where
$$s_v = \max\{0, \varphi(p_v) \cdot (1 - \sum_{e : s(e) = v} \mu(e))\}$$

and
$$H = \langle \mu(\sigma) : \sigma \text{ a loop in } \Gamma \rangle$$

Applications to free products of fin. dim'l vNa's

Thm [Dykema; 1993] Let (A, ϕ_1), (B, ϕ_2) be fin. dim'l, tracial vNa's, with $\dim(A), \dim(B) \geq 2$ and $\dim(A) + \dim(B) \geq 5$

$$(A, \phi_1) * (B, \phi_2) \cong L(\mathbb{F}_t) \oplus D$$

where D is fin. dim, possibly 0.

Thm [Dykema; 1997] Same as above, but assume at least one of ϕ_1, ϕ_2 is non-tracial. Then

$$(A, \phi_1) * (B, \phi_2) \cong \underbrace{(M, \varphi)}_{\text{type III}} \oplus \overbrace{D}^{\text{as above}}$$

Then $M^{\varphi} \cong L(\mathbb{F}_\infty)$, φ is almost periodic, pt spectrum of Δ_φ is group gen. by pt. spectrum of $\Delta \phi_i$ $i = 1, 2$.

Q: (Dykema + Shlyakhtenko)
If $\langle \sigma(\Delta \phi_1), \sigma(\Delta \phi_2) \rangle = \langle \sigma(\Delta \psi_1), \sigma(\Delta \psi_2) \rangle$, are the M's isomorphic? Are they free Araki–Woods factors?

Thm (Hartglass + Nelson; 2018)
(A, ϕ_1), (B, ϕ_2) as before, then

$$(A, \phi_1) * (B, \phi_2) \cong (T_H, \varphi_H) \oplus D$$

Ex
$$\underset{\alpha_1, \dots, \alpha_n}{M_n(\mathbb{C})} * \underset{\beta_1, \dots, \beta_m}{M_m(\mathbb{C})} \cong (T_H, \varphi_H)$$

where $H = \left\langle \frac{\alpha_i}{\alpha_j}, \frac{\beta_i}{\beta_j} \right\rangle$

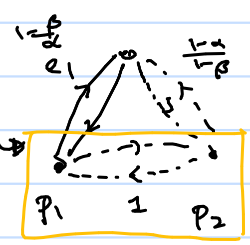
Ex (Houdayer; 2007) $\alpha \geq \beta \geq \frac{1}{2}$, $\alpha \neq \frac{1}{2}$
$$\underset{\alpha, 1-\alpha}{\overset{p_1, p_2}{M_2(\mathbb{C})}} * \Big(\underset{\beta}{\mathbb{C}} \oplus \underset{1-\beta}{\mathbb{C}} \Big) \cong (T_\lambda, \varphi_\lambda)$$

$$\lambda = \frac{1-\alpha}{\alpha}$$

If $\beta > \alpha$, set $D = W^*(p_1, p_2)$
Then

$$\underset{\alpha, 1-\alpha}{M_2(\mathbb{C})} * \Big(\underset{\beta}{\mathbb{C}} \oplus \underset{1-\beta}{\mathbb{C}} \Big) \cong \underset{\alpha, 1-\alpha}{M_2(\mathbb{C})} *_D \Big[\Big(\underset{\beta}{\mathbb{C}} \oplus \underset{1-\beta}{\mathbb{C}} \Big) * D \Big]$$

$$\cong M_2(\mathbb{C}) *_D \Big[\underset{\alpha-(1-\beta)}{\mathbb{C}} \oplus \underset{1-\beta, 1-\beta}{M_2(L(\mathbb{Z}))} \oplus \underset{\beta-\alpha}{\mathbb{C}} \Big] \rightsquigarrow$$



Consider compression instead by $p_1 + p_2$.