joint w/ Ian Charlesworth

- Fix a tracial vNa (M,τ) with self-adjoint gen $x_1, \ldots, x_n$.
- Denote $X := (x_1, \ldots, x_n)$ and $\mathbb{C}\langle X\rangle = \mathbb{C}\langle x_1, \ldots, x_n\rangle$.

- Recall the jth - free difference quotient

$$\partial_j : \mathbb{C}\langle X\rangle \longrightarrow \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$$

$$\partial_j (x_{i_1} \cdots x_{i_d}) = \sum_{k=1}^{d} \delta_{i_k = j}\, x_{i_1} \cdots x_{i_{j-1}} \otimes x_{i_{j+1}} \cdots x_{i_d}$$

- Define the non-commu Jacobian as:

$$\mathcal{J} : \mathbb{C}\langle X\rangle^n \longrightarrow M_n(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle)$$

$$\mathcal{J}(p_1, \ldots, p_n) = \begin{bmatrix} \partial_1 p_1 & \cdots & \partial_n p_1 \\ \vdots & \ddots & \vdots \\ \partial_1 p_n & \cdots & \partial_n p_n \end{bmatrix}$$

- View $\mathcal{J}$ as a densely defined (unbdd op.):

$$\mathcal{J} : L^2(M)^n \longrightarrow M_n(L^2(M \bar{\otimes} M))$$

with adjoint

$$\mathcal{J}^* : M_n(L^2(M \bar{\otimes} M)) \longrightarrow L^2(M)^n$$

- Notation: $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n) \in L^2(M)^n$ $\qquad \langle P, Q\rangle_2 := \sum_{j=1}^{n} \langle p_j, q_j\rangle_2$

$A, B \in M_n(L^2(M \bar{\otimes} M))$ $\qquad \langle A, B\rangle_{HS} := \sum_{j,k=1}^{n} \underbrace{\langle [A]_{jk}, [B]_{jk}\rangle_{HS}}_{\langle \cdot, \cdot \rangle_{\tau \otimes \tau}}$

Def Let

$$\mathbb{1} := \begin{bmatrix} 1\otimes 1 & & 0 \\ & \ddots & \\ 0 & & 1\otimes 1 \end{bmatrix} \in M_n(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle)$$

If $\mathbb{1} \in \mathrm{dom}(\mathcal{J}^*)$, then $\Xi := (\xi_1, \ldots, \xi_n) := \mathrm{dom}(\mathcal{J}^*)$ are called the conjugate variables to $X = (x_1, \ldots, x_n)$.

The free Fisher information of X is the qty:

$$\Phi^*(x_1, \ldots, x_n) := \|\Xi\|_2^2 = \sum_{j=1}^{n} \|\xi_j\|_2^2$$

if the conj. var. Ξ exist, and is $+\infty$ otherwise.

Rem.
- $\Phi^*(X) < \infty \iff \mathbb{1} \in \mathrm{dom}(\mathcal{J}^*)$
- $\langle \xi_j, p\rangle_2 = \langle 1\otimes 1, \partial_j p\rangle_{HS}$ $\quad \forall p \in \mathbb{C}\langle X\rangle$, $j = 1, \ldots, n$.

Thm: Assume $\mathbb{1} \in \mathrm{dom}(\partial^*)$

[Voiculescu; 1993]: $\frac{d\mu_1}{dm}, \ldots, \frac{d\mu_n}{dm} \in L^3(\mathbb{R}, m)$

[Dabrowski; 2010]: M does not have Property Γ.

[Mai + Speicher + Weber; 2014]: every $p \in \mathbb{C}\langle X\rangle_{s.a} \setminus \mathbb{R}$ is diffuse

Question. What if $\mathbb{1} \in \overline{\mathrm{dom}(\partial^*)} \setminus \mathrm{dom}(\partial^*)$?

Def: The free Stein information of X is the qty

$$\Sigma^*(X) := \mathrm{dist}_{HS}(\mathbb{1}, \mathrm{dom}(\partial^*)) = \inf\{\|\mathbb{1} - A\|_{HS} : A \in \mathrm{dom}(\partial^*)\}$$

For $R > 0$ the R-bounded free Stein information of X is the qty

$$\Sigma^*_R(X) := \inf\{\|\mathbb{1} - A\|_{HS} : A \in \mathrm{dom}(\partial^*), \|\partial^*(A)\|_2 \leq R\}$$

Observations: • $R \mapsto \Sigma^*_R(X)$ is decreasing:

$$\Sigma^*(X) = \inf_{R>0} \Sigma^*_R(X) = \lim_{R\to\infty} \Sigma^*_R(X)$$

• $R \mapsto \Sigma^*_R(X)$ is convex

Thm (Charlesworth + N.) For $n=1$, if $X = x_1$ has spectral measure μ then

$$\Sigma^*(x)^2 = \sum_{t\in\mathbb{R}} \mu(\{t\})^2$$

Ex: Let $\omega_1, \ldots, \omega_n \in (0,1)$ s.t. $\sum \omega_j = 1$, and let $a_1 < a_2 < \cdots < a_n$. Suppose x has spectral measure

$$\mu := \sum_{j=1}^n \delta_{a_j} \omega_j$$

Then $\Sigma^*(X) = \sqrt{\omega_1^2 + \cdots + \omega_n^2}$. Claim $\Sigma^*_R(x) = \Sigma^*(x)$ for finite R.

Consider

$$g(t) = 2 \cdot p.v. \int \frac{1}{t-s}\, d\mu(s) = \sum_{j : a_j \neq t} \frac{2}{t - a_j} \omega_j$$

Then

$$\int p(t) g(t)\, d\mu(t) = \sum_{i=1}^n p(a_i) \cdot \sum_{j\neq i} \frac{2}{a_i - a_j} \omega_j \cdot \omega_i$$

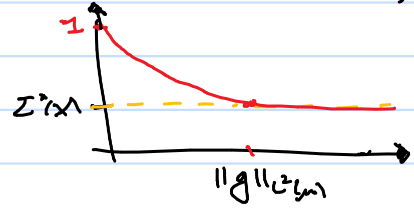
$$= \sum_{i=1}^n \sum_{j\neq i} \frac{p(a_i) - p(a_j)}{a_i - a_j} \omega_j \omega_i$$

$$= \iint \chi_{s\neq t} (\partial p)(s,t)\, d\mu(s)\, d\mu(t)$$

Hence $X_{s\neq t} \in \mathrm{dom}(\partial^*)$ with $\partial^*(X_{s\neq t}) = g$. Let $R \geq \|g\|_{L^2(\mu)}$, then

$$\Sigma_R^*(X)^2 \leq \|X_{s\neq t} - 1\|_{HS}^2 = \iint X_{s=t}\, d\mu(s)\, d\mu(t) = \sum_{j=1}^{n} w_j^2 = \Sigma^*(X)$$

$$\Rightarrow \quad \Sigma_R^*(X) = \Sigma^*(X)$$



- Formula due to Voiculescu: $\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle \curvearrowright \mathrm{dom}(\partial^*) \Rightarrow \overline{\mathrm{dom}(\partial^*)}$ is $M\bar{\otimes} M^\circ$-module

$\boxed{\text{Thm (Charlesworth + N.)}}$

$$n - \Sigma^*(X)^2 = \frac{1}{n} \dim_{M\bar{\otimes} M^\circ}\left(\overline{\mathrm{dom}(\partial^*)}\right)$$

$\boxed{\text{Cor}}$ $\Sigma^*(X) = 0 \iff \partial$ is closable

$\boxed{\text{Cor}}$ Let $Y = (y_1, \ldots, y_m) \in M_{sa}^m$. Then

$$\Sigma^*(X)^2 + \Sigma^*(Y)^2 \leq \Sigma^*(X,Y)^2$$

with equality if X and Y are free.

$\boxed{\text{Cor}}$ If $Y = (y_1, \ldots, y_n) \in \mathbb{C}\langle X\rangle_{s.a.}^n$ satisfy $\mathbb{C}\langle Y\rangle = \mathbb{C}\langle X\rangle$ then

$$\Sigma^*(Y) = \Sigma^*(X)$$

Pf: $Y = F(X)$ for $F = (f_1, \ldots, f_n)$ poly. If $A \in \mathrm{dom}(\partial_X^*)$ with $\Xi = \partial_X^*(A)$ then

$$\langle \Xi, P(Y)\rangle_2 = \langle \Xi, P(F(X))\rangle_2 = \langle A, \partial_X(P(F(X)))\rangle_{HS}$$

$$= \langle A, \partial_Y(P(Y)) \cdot \partial_X(F(X))\rangle_{HS}$$

$$= \langle A \cdot \partial_X(F(X))^*, \partial_Y(P(Y))\rangle_{HS}$$

Thus $A \cdot \partial_X(F(X))^* \in \mathrm{dom}(\partial_Y^*)$.

$$\Rightarrow \quad n - \Sigma^*(Y)^2 = \frac{1}{n}\dim_{M\bar{\otimes} M^\circ}\left(\overline{\mathrm{dom}(\partial_Y^*)}\right) \geq \frac{1}{n}\dim_{M\bar{\otimes} M^\circ}\Big(\overline{\mathrm{dom}(\partial_X^*)} \cdot \underbrace{\partial_X(F(X))^*}_{\text{invertible with inverse } \partial_Y(F^{-1}(Y))^*}\Big)$$

$$= \frac{1}{n}\dim_{M\bar{\otimes} M^\circ}\left(\overline{\mathrm{dom}(\partial_X^*)}\right) = n - \Sigma^*(X)^2$$

Thus $\Sigma^*(Y) \leq \Sigma^*(X)$. By symmetry, we have equality. $\square$

$\boxed{\text{Remark}}$ $\Phi^*(X) < \infty \iff 1 \in \mathrm{dom}(\partial^*) \Rightarrow 1 \in \overline{\mathrm{dom}(\partial^*)} \iff \Sigma^*(X) = 0$

Thm (Charlesworth + N.)

$$\Phi^*(X) \leq R^2 \iff \Sigma^*_R(X) = 0$$

Pf: ($\Rightarrow$) Thus $1 \in \mathrm{dom}(\partial^*)$ with $\Xi = \partial^*(1)$ satisfying $\|\Xi\|_2 \leq R$.
Hence
$$\Sigma^*_R(X) = \inf\{\|A - 1\|_{HS} : A \in \mathrm{dom}(\partial^*), \|\partial^*(A)\|_2 \leq R\} = 0.$$

($\Leftarrow$) Let $A_n \in \mathrm{dom}(\partial^*)$ be s.t. $\|\partial^*(A_n)\|_2 \leq R$ and
$$\|A_n - 1\|_{HS} \to 0.$$

Then $\forall P \in \mathbb{C}\langle x \rangle^n$
$$\langle \partial^*(A_n), P\rangle_2 = \langle A_n, \partial P\rangle_{HS} \longrightarrow \langle 1, \partial P\rangle_{HS}$$
Since $\{\partial^*(A_n)\}$ unif. bdd, $\exists$ weak limit Ξ, which are conj. variables by the above. Moreover
$$\|\Xi\|_2 \leq \limsup_{n\to\infty} \|\partial^*(A_n)\|_2 \leq R \qquad \square$$

Def $\delta^*(X) = \ldots$ Not so important. But $\Phi^*(X) < \infty \Rightarrow \delta^*(X) = n$

Thm Assume $\delta^*(X) = n$

[Voiculescu; 1994]: $x_1, \ldots, x_n$ are diffuse
[Dabrowski; 2010]: $W^*(X)$ is a factor
[Charlesworth + Shlyakhtenko; 2014]: $p \in \mathbb{C}\langle X\rangle_{s.a.} \setminus \mathbb{R}$ is diffuse

Using an estimate from [Shlyakhtenko; 2004] obtain:

Thm (Charlesworth + N.)
$$n - \Sigma^*(X)^2 \leq \delta^*(X)$$

In particular,
$$\Sigma^*(X) = 0 \Rightarrow \delta^*(X) = n.$$

$$\Phi^*(X) = 0 \Longrightarrow \Sigma^*(X) = 0 \Longrightarrow \delta^*(X) = n$$
$$\Phi^*(X) = 0 \Longrightarrow \chi^*(X) > -\infty \Longrightarrow \delta^*(X) = n$$
$$\Sigma^*(X) = 0 \Downarrow \quad \Uparrow ? \quad \chi^*(X) > -\infty$$

For $n = 1$, $\Sigma^*(X) = 0 \iff \mu$ has no atoms $\iff \delta^*(X) = 1$
so in this case $\chi^*(X) > -\infty \Rightarrow \Sigma^*(X) = 0$.

Conjecture $\chi^*(X) > -\infty \Rightarrow \Sigma^*(X) = 0$

Let γ be the Gaussian measure on $\mathbb{R}^n$. Then $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \vec{x} \cdot \vec{\nabla}\varphi(x) \, d\gamma(\vec{x}) = \int_{\mathbb{R}^n} \langle 1, \mathrm{Hess}(\varphi) \rangle_{HS} \, d\gamma$$

$\boxed{\text{Def:}}$ Let μ be a centered prob measure on $\mathbb{R}^n$. $A : \mathbb{R}^n \to M_n(\mathbb{C})$ is called a Stein kernel for μ with respect to γ if $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \vec{x} \cdot \vec{\nabla}\varphi(x) \, d\mu(\vec{x}) = \int_{\mathbb{R}^n} \langle A, \mathrm{Hess}(\varphi) \rangle_{HS} \, d\gamma$$

The Stein discrepancy of μ with respect to γ is:

$$S_\gamma(\mu) := \| A - 1 \|_{HS}$$

- $S_\gamma(\mu)$ measures how badly μ violates the integration by parts formula.

- If $X = (x_1, \ldots, x_n)$ is a free semicircular family then $\forall p \in \mathbb{C}\langle x \rangle$

$$\langle X, Dp \rangle_2 = \langle 1, \mathcal{J} Dp \rangle_{HS}$$

In fact, $\forall P \in \mathbb{C}\langle x \rangle^n$

$$\langle X, P \rangle_2 = \langle 1, \mathcal{J} P \rangle_{HS}$$

$\boxed{\text{Def [Fathi + N.; 2017]}}$ $A \in M_n(L^2(M \bar{\otimes} M))$ is a free Stein kernel for X if

$$\langle X, P \rangle_2 = \langle A, \mathcal{J} P \rangle_{HS} \qquad \forall P \in \mathbb{C}\langle x \rangle^n$$

More generally, A is a free Stein kernel for X relative to $\Xi \in L^2(M)^n$ if

$$\langle \Xi, P \rangle_2 = \langle A, \mathcal{J} P \rangle_{HS} \qquad \forall P \in \mathbb{C}\langle x \rangle^n$$

The free Stein discrepancy of X relative to Ξ is the qty:

$$\Sigma^*(X \mid \Xi) = \inf_A \| A - 1 \|_{HS} = \| \Pi(A) - 1 \|_{HS}$$

$\boxed{\text{Rem}}$ A is a free Stein kernel rel to $\Xi \iff A \in \mathrm{dom}(\mathcal{J}^*)$ with $\mathcal{J}^*(A) = \Xi$

Thus

$$\Sigma^*(X) = \inf \{ \Sigma^*(X \mid \Xi) : \Xi \in L^2(M)^n \}$$

$$\Sigma^*_R(X) = \inf \{ \Sigma^*(X \mid \Xi) : \Xi \in L^2(M)^n, \ \|\Xi\|_2 \le R \}$$

$\boxed{\text{Prop [Mai 2016]}}$ If $\Xi \perp \mathbb{C}^n$ then

$$B_\Xi = \left(\tfrac{1}{2} \, (\xi_i \otimes 1 - 1 \otimes \xi_i) \# (x_j \otimes 1 - 1 \otimes x_j) \right)_{i,j=1}^n$$

is a free Stein kernel rel. to Ξ.