- Conjecture (Peterson–Thom): If $N_1, N_2 < L(\mathbb{F}_r)$, $r > 1$, are diffuse, hyperfinite, and $N_1 \cap N_2$ diffuse $\Rightarrow$ $N_1 \vee N_2$ hyperfinite

- Motivation:
  - Group version is true for $G_1 * G_2$, $G_j$ amenable + inf.
  - Analytic proof using $L^2$-Betti numbers (Peterson–Thom)
  - True for any $G \underset{ME}{\sim} \mathbb{F}_2$.

Conj $\Leftrightarrow$ Given any diffuse, hyperfinite $N < M \Rightarrow \exists!$ diffuse $N < P < M$ s.t. $P$ is max'l amenable
Moreover, if $P_1, P_2 < M$ are max'l amen. $P_1 \cap P_2 \Rightarrow P_1 = P_2$.

Ex
- [Houdayer; 2015]: prove PT-conj. for $L(\mathbb{Z}) * 1 < L(\mathbb{Z}) * L(\mathbb{Z})$
- [Wen; 2015]: —— " —— radial masa
- [Brothier + Wen; 2015]: —— " —— cup subalg.
- [Parekh + Shimada + Wen; 2015]: — " — $q$-deformed free group factors

- 1-Bounded Entropy Hayes 2018 following Jung 2008:
  Given $N < M$ diffuse, hyperfinite: $h(N : M)$ = 1-bdd entropy of $N$ in the presence of $M$
  "count how many $N \hookrightarrow \prod_{n \to \omega} M_n(\mathbb{C})$
  which extend to $M \hookrightarrow \prod_{n \to \omega} M_n(\mathbb{C})$ "

Axioms:
- $h(N : M) = 0$ if $N$ is hyperfinite
- $h(M) = h(M : M) = \infty$ if $M = W^*(x)$ s.t. $\delta_0(x) > 1$, e.g. $M = L(\mathbb{F}_r)$
- $h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$ if $N_1 \cap N_2$ is diffuse
- $h(N : M) = h(N : M^\omega)$
- $h(W^*(N_M(N)) : M) = h(N : M)$
- $h(N_1 : M_1) \leq h(N_2 : M_2)$ if $N_1 \leq N_2 \leq M_2 \leq N_1$
- $h(\bigvee N_j : M) = \sup_j h(N_j : M)$ if $N_j \leq N_{j+1}$

Ex: $L(\mathbb{F}_2)$ does not have prop. $\Gamma$: If it did, then $\exists\, A \leq M' \cap M^\omega$ diffuse, abelian. $\Rightarrow$ $h(M) = h(M : M) = h(M : M^\omega) \leq h(W^*(N_{M^\omega}(A)) : M^\omega)$
$= h(A : M^\omega) \leq 0$.

- Conj:
  - If $N < L(\mathbb{F}_r)$ diffuse, $r > 1$, and $h(N : L(\mathbb{F}_r)) = 0 \Rightarrow N$ amenable
  - $L(\mathbb{F}_2) \cong L(\mathbb{Z})^{*\mathbb{Z}} \rtimes \mathbb{Z}$

**Def:** $X_1^{(N)}, \dots, X_r^{(N)}$ random $N\times N$-matrices s.t. $\sup_N \|X_j^{(N)}\| < \infty$

$$\mu_X : \mathbb{C}^*\langle T_j : 1 \le j \le r\rangle \to \mathbb{C}$$
$$p \mapsto \tfrac{1}{N}\mathrm{Tr}(p(X_1^{(N)}, \dots, X_r^{(N)}))$$

Given $x \in M^r$, $(M,\tau)$-tracial vNa, with law $\mu_x$, we say
$$(X_1^{(N)}, \dots, X_r^{(N)}) \xrightarrow[N\to\infty]{\mathcal{D}} (x_1, \dots, x_r)$$

if $\forall\, \mathcal{O}$ wk$^*$-nbhd of $\mu_x$, $\mathbb{P}(\{X : \mu_X \in \mathcal{O}\}) \to 1$

**Ex:** Voiculescu's Asy. Freeness: $U_1^{(N)}, \dots, U_r^{(N)}$ ind. Haar $N\times N$-unitaries
$$(U_1^{(N)}, \dots, U_r^{(N)}) \xrightarrow{\mathcal{D}} (\lambda(a_1), \dots, \lambda(a_r)) \qquad \mathbb{F}_r = \langle a_1, \dots, a_r\rangle$$

**Def** Say $X^{(N)} = (X_1^{(N)}, \dots, X_r^{(N)})$ converges to $X$ strongly if $X^{(N)} \xrightarrow{\mathcal{D}} X$ and $\forall P \in \mathbb{C}^*\langle T_j : 1 \le j \le r\rangle$
$$\mathbb{P}(\{X : |\|P(X^{(N)})\| - \|P(x)\|| > \varepsilon\}) \to 0$$

**Thm** (Haagerup–Thorbjørnsen; 2005) GUE converges strongly to free semicircular

**Conj ③** $\underbrace{U_1^{(N)}, \dots, U_r^{(N)}}_{U^{(N)}}, \underbrace{V_1^{(N)}, \dots, V_r^{(N)}}_{V^{(N)}}$ Haar unitaries

$$(U^{(N)} \otimes 1, 1 \otimes \overline{V^{(N)}}) \xrightarrow{\text{strongly}} (\lambda(a) \otimes 1, 1 \otimes \lambda(b)) \in C_r^*(\underbrace{\mathbb{F}_r}_{\langle a_1, \dots, a_r\rangle} \times \underbrace{\mathbb{F}_r}_{\langle b_1, \dots, b_r\rangle})^{2r}$$

**Thm:** Consider the following statements:
1. PT-conj
2. $r > 1$, $h(N : L(\mathbb{F}_r)) = 0 \Rightarrow N$ amenable
3. Conj ③ ($\otimes$-HT)

Then $3 \Rightarrow 2 \Rightarrow 1$

**Subtlety:** Take $(W_1^{(N)}, \dots, W_r^{(N)}) \xrightarrow{\text{strongly}} (\lambda(a_1), \dots, \lambda(a_r))$
$$(W^{(N)} \otimes 1, 1 \otimes \overline{W^{(N)}}) \xrightarrow{\cancel{\text{strongly}}} (\lambda(a_1), \dots, \lambda(a_r))$$

Since $\left\|\frac{1}{r}\sum_{j=1}^r W_j^{(N)} \otimes \overline{W_j^{(N)}}\right\| = 1$ but $\left\|\frac{1}{r}\sum_{j=1}^r \lambda(a_j) \otimes \lambda(b_j)\right\| < 1$

$\mathcal{U} = (\mathcal{U}^{(N)})_{N=1}^{\infty} \in \prod_{N=1}^{\infty} \mathcal{U}(N)^{r} = \Omega$

Choose $\mathcal{U}$ at random: $\exists!\ \Theta_{\mathcal{U}} : L(\mathbb{F}_r) \to \prod_{N \to \omega} M_n(\mathbb{C})$ $\quad \Theta_{\mathcal{U}}(\lambda(a_j)) = (U_j^{(N)})_{N \to \omega}$

Choose $\mathcal{U}, \mathcal{V} \in \Omega$ ind.

<u>Conj. 2.5</u>: $\forall Q < L(\mathbb{F}_r)$ non-amen., a.e. $(\mathcal{U}, \mathcal{V})$ $\Theta_{\mathcal{U}}|_Q \sim_{a.e.} \Theta_{\mathcal{V}}|_Q$

$3 \Rightarrow 2.5 \Rightarrow 2 \Rightarrow 1$