Schur: $\underset{M_n(\mathbb{C})}{\overset{\in}{A}} = \underset{U_n}{\overset{\in}{U}} \begin{pmatrix} \lambda_1 & \cdots & * \\ 0 & & \lambda_n \end{pmatrix} U^*$

What about $A \in B(H)$? Issues: (1) Invariant subspace problem

— " — $A \in M \subseteq B(H)$ — " — (2) Spectral distribution?

(M,τ) tracial vNa $\hookrightarrow$ Brown measure

<u>Thm (Brown; 1986)</u>: Every $A \in M$ has a Borel prob measure ν_A on $\mathbb{C}$ (called the Brown measure) s.t.

$$\tau(\log|A-\lambda|) = \int \log|z-\lambda| \, d\nu_A(z) \qquad \lambda \in \mathbb{C}$$

(also $\tau(|A|^p) = \int |z|^p \, d\nu_A(z)$)

One issue: $\mathrm{supp}(\nu_A) \subseteq \sigma(A)$ but could be strict

<u>Prop (Brown)</u> If $p \in M$ is A-inv. $\left(A = \begin{pmatrix} A_p & * \\ 0 & A(1-p) \end{pmatrix}\right)$ then

$$\nu_A(p) = \tau(p)\, \nu_{A_p} + \tau(1-p)\, \nu_{A(1-p)}$$

<u>Thm [Haagerup + Schultz; 2009]</u> $\forall$ Borel $B \subseteq \mathbb{C}$, $\forall T \in M$, $\exists$ a T-inv. proj. $p \in M$ s.t.

- $\tau(p) = \nu_T(B)$
- $\nu_{Tp}^{(pMp)} = \nu_T|_B$
- $\nu_{(1-p)T}^{((1-p)M(1-p))} = \nu_T|_{B^c}$

Notation: $p(T,B) = p$, called the Haagerup + Schultz projection.

<u>Thm (H+S; 2009)</u> $T \in M$, then $\nu_T = \delta_0 \iff \underset{n\to\infty}{\text{SOT-lim}} |T^n|^{1/n} = 0$.
(say T is "sot-quasinilpotent")

<u>Thm (Dykema + Sukochev + Zanin; 2015)</u>
Given $T \in M$, $\exists$ increasing family $(q_t)_{0\le t\le 1}$ of T-inv proj. $q_0 = 0$, $q_1 = 1$, s.t. letting

$$\mathcal{D} = W^*(\{q_t : 0 \le t \le 1\})$$

$N = E_{\mathcal{D}}(T)$ then $\nu_N = \nu_T$ and $T = N + Q$ and Q is sot-quasinilpotent.

<u>Construction:</u> Choose a "cts spectral ordering" of T, namely $\rho: [0,1] \to \mathbb{C}$ cts s.t. $\operatorname{supp}(\nu_T) \subseteq \rho([0,1])$. Let $q_t = P(T, \rho([0,t]))$

<u>Properties of $P(T,B)$:</u> [Schultz; 2006]:
- $B_1 \subseteq B_2 \Rightarrow P(T, B_1) \leq P(T, B_2)$
- $P(T, B_1) \wedge P(T, B_2) = P(T, B_1 \cap B_2)$
  — " — $\vee$ ——— " ——— $\cup$ — " — and etc.

<u>Def</u> (Foias, 1963, 1968; Apostol, 1968)
$T \in B(\mathcal{H})$ is <u>decomposable</u> if it has a <u>spectral capacity</u>, namely $\{\text{closed subsets of } \mathbb{C}\} \ni K \mapsto E(K)$ closed, T-inv subspace s.t.
(1) $\sigma(T|_{E(K)}) \subseteq K$
(2) $\bigcap_{n=1}^{\infty} E(K_n) = E(\bigcap_{n=1}^{\infty} K_n)$
(3) If $U_1, \ldots, U_n$ is an open cover of $\mathbb{C}$, then
$E(\bar{U}_1) + \cdots + E(\bar{U}_n) = \mathcal{H}$.

<u>Thm</u> If T is decomposable, then E is unique and $E(K) = \mathcal{H}_T(K)$
("local spectral subspace")

<u>Def</u> $X \subseteq \mathbb{C}$ $\quad \mathcal{H}_T(X) := \{\xi \in \mathcal{H} : \sigma_T(\xi) \subseteq X\}$

$\sigma_T(\xi) := \bigcup \{U \subseteq \mathbb{C} \mid U \text{ open}, \exists f: U \to \mathcal{H} \text{ s.t. } (T-\lambda)f(\lambda) = \xi \quad \lambda \in U\}$

If T decomp, K closed $\Rightarrow \mathcal{H}_T(K)$ is closed.

<u>Thm</u> [HS; 2007] If $T \in M \subseteq B(\mathcal{H})$ and T is decomposable, then
$P(T, B)\mathcal{H} = \overline{\mathcal{H}_T(B)}$

<u>Thm</u> [HS; 2009] $\forall T \in M$, $\quad \text{sot-}\lim_{n\to\infty} |T^n|^{1/n} \overset{*}{=} A$ s.t.
$1_{[0,r]}(A) = P(T, r\bar{\mathbb{D}})$

<u>Def</u> We say T has the <u>norm convergence property</u> if $(*)$ converges in norm.
We say T has the <u>shifted norm convergence property</u> if $\forall \lambda \in \mathbb{C}$ $T - \lambda$ has the n.c.p.

<u>Def</u> T is <u>strongly decomposable</u> if T is decomposable, and $\forall$ closed $K \subseteq \mathbb{C}$ $T|_{\mathcal{H}_T(K)}$ is decomposable.

[Eschmeier 1988]: $\exists\ T \in B(\mathcal{H})$ decomposable but ~~not~~ strongly decomp.

Thm (Dykema + Noles + Zanin; 2018)
Let $T \in M$. Then:
(a) T is "Borel decomposable"
$\Downarrow$
(b) T is strongly decomp
$\Downarrow$
(c) $\forall$ cts spectral orderings ρ, in $T = N + Q$, Q is quasinilpotent
$\Downarrow$
(d) T is decomposable
$\Downarrow$
(e) T has the shifted n.c.p
$\Downarrow\!\!\!\!\!\!\times$
(f) T has full spectral dist (i.e. $\mathrm{supp}(\nu_T) = \sigma(T)$)

Def We say $T \in M$ is Borel decomposable if $\forall\ X \subseteq Y \subseteq \mathbb{C}$ Borel sets, letting $p = P(T, Y) - P(T, X)$, pT_p has full spectral dist.

Prop: $T \in M$
(1) T is Borel decomp. $\iff$ $\forall\ X \subseteq Y \subseteq \mathbb{C}$ Borel w/ p as above
$\sigma(pTp) \subseteq \overline{Y \setminus X}$
(2) T is strongly decomp $\iff$ The above holds for Y closed and X rel. open in X.

Ex. • [Dykema + Haagerup, 2004] The DT-operators (including circular op's) are Borel decomposable
• If $T \in M$, and $\sigma(T)$ is totally disconnected, then T is Borel decomp.

Thm Let $T \in M$, assume $\mathrm{supp}(\nu_T)$ is totally disconnected. Then (a) – (f) are equiv. and are equiv. to each of:
(g) $\sigma(T)$ is totally disc.
(h) T^* has shifted n.c.p.

Thm: $T \in M$ and $\mathrm{supp}(\nu_T)$ is finite, then (a) – (h) are equiv to each of
(i) $\sigma(T)$ is finite
(j) $\exists$ cts spectral ordering ρ, s.t. in $T = N + Q$ Q is quasinilpotent