Quantum probabilities, synchronous games and C*-algebras

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joint work with many people references at end

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where $\lambda(x, y, a, b) = 1$ means that if Alice and Bob receive inputs x, y, respectively and produce respective outputs a, b, then they win. If $\lambda(x, y, a, b) = 0$, they lose.

They both know the rule function and can jointly create a strategy for winning, but once the game starts Alice and Bob must produce their outputs without knowing what input the other received and without knowing what output the other produced. This is what is meant by *non-communicating*.

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A *deterministic strategy* is a pair of functions, $f_A : I_A \to O_A$, $f_B : I_B \to O_B$ so that when Alice and Bob receive inputs x, y then they give outputs, $f_A(x), f_B(y)$.

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A deterministic strategy is called *perfect* if it always wins, i.e., $\lambda(x, y, f_A(x), f_B(y)) = 1, \forall x, y$.

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A *random strategy* just means that each time they receive the input pair (x, y) they do not necessarily produce the same output. In this case there is a conditional probability density $p(a, b|x, y)$ that represents the probabiity that they output the pair (a, b) given that they received input (x, y).

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The goal of this talk is to show that for a certain family of such games, called *synchronous games*, we can construct a *-algebra whose representation theory completely characterizes these behaviours.

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such densities are called *synchronous*.

Graph Homomorphisms

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Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ a **homomorphism** from G to H is a function $f : V(G) \to V(H)$ such that

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- $\omega(G) = max\{c : K_c \to G\}$(clique number)
- $\alpha(G) = max\{c : K_c \to \overline{G}\}$(independence number), where $\overline{G}$ is the graph with the same vertex set but the opposite edges.

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Also given a pair of graphs, there is a *graph isomorphism game*, which has a perfect deterministic strategy iff the two graphs are isomorphic.

The syncBCS game

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Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}/2$; that is, $A = (a_{i,j}) \in \mathbb{M}_{m,n}(\mathbb{Z}/2)$ and $b \in (\mathbb{Z}/2)^n$.

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- $a_{i,k} = 0 \implies v_k = 0$ and $a_{j,k} = 0 \implies w_k = 0$,
- whenever $a_{i,k} = a_{j,k} = 1$, then $v_k = w_k$.

Conditional Quantum Probabilities: Tsirelson and Connes

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One model is that Alice and Bob have finite dimensional state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For each experiment x, Alice has projections $\{E_{x,a}, 1 \le a \le m\}$ such that $\sum_a E_{x,a} = I_A$.

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$$p(a, b|x, y) = \langle \psi | E_{x,a} \otimes F_{y,b} | \psi \rangle.$$

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We let $C_{qc}(n,m)$ denote the possibly larger set that we could obtain if instead of requiring the common state space to be a tensor product, we just required one common state space, and demanded that $E_{a,x}F_{y,b} = F_{y,b}E_{x,a}$ for all a,b,x,y, this is called the *commuting model.*

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Here are some of the things that we know/don't know about these sets.

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Our work characterizes the existence of perfect t-strategies in terms of a *-algebra constructed from the game.

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The second relation can also be written as $A_G \circ (e_{g,h}) = (e_{g,h}) \circ A_H$, where A_X is the adjacency matrix of the graph X.

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Finally, using Slofstra's construction of a BCS game with a perfect qa-strategy but no perfect q-strategy, we are able to prove that there exists a $m \times n$ matrix with $m \sim 100$ and a vector b such that $syncBCS(A, b)$ has a perfect qa-strategy but no perfect q-strategy.

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Explicitly: for a fixed N, does there exist $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$, independent of n, such that if $E_{g,h} \in M_n, 1 \le g, h \le N$ are projections with $\| \sum_h E_{g,h} - I_n \|_2 < \epsilon$ and $\| \sum_g E_{g,h} - I_n \|_2 < \epsilon$ then there exist projections $F_{g,h}$ with $\sum_h F_{g,h} = \sum_g F_{g,h} = I_n$ and $\| E_{g,h} - F_{g,h} \|_2 < \delta$?

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- Fritz proves that if ZFC is consistent, then there is a synchronous game $\mathcal{G}$ such that whether or not $\mathcal{A}(\mathcal{G})$ has a representation on a Hilbert space is undecidable.

Thanks!

References

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