joint w/ Zhi hong Li, Kyle Mayor, Drew Nguyen, Jennifer Pi, Anna Raichev

$$\begin{tikzcd}
 & \text{Free} \arrow[dl] \arrow[dr] & \\
\text{Inf. Free / Type B free} \arrow[dr] & & \text{Bi-free} \arrow[dl] \\
 & ? &
\end{tikzcd}$$

<u>Def</u>: A coxeter system is a pair (G, S) where G is a grp, S is a generating set ($e \notin S$) s.t.

$$G = \langle S \mid R \rangle$$

where R is a set of rel's of the form $(s_i s_j)^{m_{ij}} = e$ s.t. $m_{ij} = 1 \iff i = j$, and $m_{ij} = m_{ji}$.

<u>Thm</u> (Coxeter)

The finite indecomposible Coxeter systems are classified by Coxeter diagrams:

$n \geq 1$ A_n $\underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{n}$ $\qquad A_n \cong S_{n+1}$ $\overset{(1,2)}{\bullet}\!-\!\overset{(2,3)}{\bullet}\!- \cdots \to \overset{(n,n+1)}{\bullet}$

$n \geq 2$ B_n $\underbrace{\bullet\overset{4}{=}\bullet\!-\cdots-\!\bullet}_{n}$ $\qquad B_n \cong S_{\pm n} = \{\sigma \in S_{\{-n,\dots,n\}}$ s.t. $\sigma(-i) = -\sigma(i)\}$

$\underset{(-1,1)}{\bullet} = \underset{(1,2)(-1,-2)}{\bullet} - \underset{(2,3)(-2,-3)}{\bullet} - \cdots \bullet$ $(n-1,n)(-(n-1),-n)$

<u>Def</u>: The absolute order on G (coming from (G,S)) is the following

let $T = \{gsg^{-1} \mid g \in G, s \in S\}$

Say $g \leq h$ if g is on a geodesic from e to h in the Cayley graph for G w/ generating set T.

A coxeter element in (G,S) is an elt of the form $\prod_{s \in S} s$ (in any order)

$NC(G,S) :\cong [e, c]$ for any coxeter elt. c (same for any $c \in C$)

For A_n or B_n, by sending cycles to blocks, we get a subset of $P(n)$ or of $P(\pm n)$.

An inf. n.c.p.s is a triple (A, φ, φ') where A is unital $*$-alg, $\varphi: A \to \mathbb{C}$ unital, and $\varphi': A \to \mathbb{C}$ s.t. $\varphi'(1) = 0$

Say $\varphi_t : A \to \mathbb{C}$ all states s.t. $t \mapsto \varphi_t(a)$ is diff'ble $\forall a \in \mathbb{C}$, take $\varphi' = \frac{d}{dt}\Big|_{t=0} \varphi_t$

<u>Def/Thm</u>: • $A_1, \ldots, A_n \subseteq (A, \varphi, \varphi')$ are inf. free

• If $a_i \in A_{j_i}$, $\varphi(a_i) = 0$, $j_i \neq j_{i+1}$

$$\varphi(a_1 \cdots a_n) = 0$$

$$\varphi'(a_1 \cdots a_n) = \begin{cases} \varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{\frac{n-1}{2}} a_{\frac{n+3}{2}})\,\varphi'(a_{\frac{n+1}{2}}) & \text{if } n \text{ odd: } j_1 = j_n, j_2 = j_{n-1}, \ldots \\ 0 & \text{otherwise} \end{cases}$$

• The mixed type-B and type A cumulants vanish

• If $a_i \in A_{j_i}$, $j_i \neq j_{i+1}$, and $\varphi_t = \varphi + t\varphi'$

$$\varphi_t\big((a_1 - \varphi_t(a_1)) \cdot (a_2 - \varphi_t(a_2)) \cdots (a_3 - \varphi_t(a_3))\big) = o(t)$$

Using relation $BNC(\chi) = S_\chi^{-1} \cdot NC(n)$, can obtain analogous version of above for type-B-bifree probability.