TEACHER'S CORNER

Fitting Nonlinear Latent Growth Curve Models With Individually Varying Time Points

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Individual growth trajectories of psychological phenomena are often theorized to be nonlinear. Additionally, individuals' measurement schedules might be unique. In a structural equation framework, latent growth curve model (LGM) applications typically have either (a) modeled nonlinearity assuming some degree of balance in measurement schedules, or (b) accommodated truly individually varying time points, assuming linear growth. This article describes how to fit 4 popular nonlinear LGMs (polynomial, shape-factor, piecewise, and structured latent curve) with truly individually varying time points, via a definition variable approach. The extension is straightforward for certain nonlinear LGMs (e.g., polynomial and structured latent curve) but in the case of shape-factor LGMs requires a reexpression of the model, and in the case of piecewise LGMs requires introduction of a general framework for imparting piecewise structure, along with tools for its automation. All 4 nonlinear LGMs with individually varying time scores are demonstrated using an empirical example on infant weight, and software syntax is provided. The discussion highlights some advantages of modeling nonlinear growth within structural equation versus multilevel frameworks, when time scores individually vary.

Keywords: definition variables, latent curve models, nonlinear growth

Many longitudinal data sets in the social sciences have truly unique, individually varying measurement schedules (Cook & Ware, 1983; Finkel, Reynolds, McArdle, Gatz, & Pedersen, 2003; Mehta & West, 2000; Xu, Styner, Gilmore, Piven, & Gerig, 2008). Such data structures can arise when time is measured very precisely (e.g., age in days or months rather than years) and data collection schedules vary over persons. Other possibilities include daily diary studies with signal- or participant-initiated response schedules (Walls & Schafer, 2006), or longitudinal studies beginning with a retrospective report of a prior event that occurred at different times for different persons (Blozis & Cho, 2008).

Researchers may be interested in using such longitudinal data to model growth trajectories in a social or behavioral construct. If followed for a long enough time span, such trajectories are likely to exhibit *nonlinear* change—in that the construct has a nonlinear relationship to time. Nonlinear change is posited by developmental theories in, for instance, cognitive aging (Finkel et al., 2003; Hertzog & Nesselroade, 2003), literacy (Skibbe, Grimm, Bowles, & Morrison, 2012), information seeking (De Vos & Freese, 2011), functional health (Haas, 2008), pediatric brain imaging (Xu et al., 2008), and math ability (Harring, 2009). Hence, important initial tasks for the researcher include determining the functional form of the mean trend over time, often by comparing alternative nonlinear models (by which we mean models that accommodate nonlinearity with respect to time[1]), and determining the extent to which individual growth trajectories vary around that mean trend. Growth curve

[1]Some models discussed later (namely, structured latent curve [SLC] models) also allow certain parameters to enter the model nonlinearly. This kind of nonlinearity is distinguished from whether a model accommodates a nonlinear relationship between the construct and time (our focus here).

TABLE 1
Case I, II, and III Unbalancedness and Their Respective Latent Growth Model Specification and Estimation

	Data	*Model Specification*	*Model Estimation*
Case I: Observed data are balanced	All persons are measured at the same occasions, and there are no missing data.	Factor loadings are fixed to the same set of time scores for all persons.	Estimation methods (e.g., conventional maximum likelihood [ML]) employ sufficient statistics as input.
Case II: Complete data are balanced	Persons may contribute data at different occasions and different numbers of occasions, so long as the maximum number of possible occasions across the sample as a whole remains limited.	Factor loadings are fixed to the same set of time scores for all persons.	Full information ML (FIML) is employed to allow individual contributions to the likelihood. In Person i's contribution, the dimensionality of data vectors and parameter matrices are individual-specific (due to missing data) to conform with available occasions for Person i.
Case III: Truly individually varying	No two people need be measured at exactly the same occasions and persons can have different numbers of occasions.	Factor loadings for Person i are fixed to the observed measurement times for Person i. Loadings are *definition variables*—fixed parameters assigned the values of a predictor (time) that varies over persons.	FIML is employed. Individual contributions to the likelihood can now depend on individual-specific values of time scores (due to definition variables) as well as individual-specific dimensionality of data vectors and parameter matrices (due to missing data).

Note. Specification and estimation methods are backward compatible (e.g., Case II methods accommodate Cases I or II; Case III methods accommodate Cases I–III).

models can be used to accomplish these tasks. Growth curves can be fit using a multilevel modeling (MLM; Raudenbush & Bryk, 2002) or structural equation modeling (SEM) framework (e.g., Bollen & Curran, 2006; McArdle & Epstein, 1987; Meredith & Tisak, 1990). This article concerns growth curve models in an SEM framework; when using this framework we call them latent growth curve models (LGMs).

Historically, one distinction between the SEM and MLM approaches was that the former placed certain restrictions on the nature and spacing of measurement occasions, whereas the latter did not. This distinction has eroded over the course of the last two decades, in the following manner. Table 1 defines and distinguishes among three types of (un)balancedness, labeled Case I (observed data are balanced), Case II (complete data are balanced), and Case III (truly unique, individually varying measurement occasions; e.g., Raudenbush & Bryk, 2002). Model specification and estimation methods needed to fit LGMs with Case I, II, or III data in an SEM framework are also described in Table 1. The LGM model specification and estimation methods accommodating Case I data were widely used through the early 1990s (e.g., Willett & Sayer, 1994, 1995).[2] Starting in the mid-1990s, estimation methods for fitting LGMs with Case II data became available (Arbuckle, 1995, 1996; Neale, 1994) and were widely employed (e.g., Bollen & Curran, 2006; T. E. Duncan, Duncan, & Strycker, 2006; Ferrer, Hamagami, & McArdle, 2004; McArdle & Bell,

2000; Raykov, 2005). For over a decade, LGM specifications that allow Case III data have been available in SEM software (e.g., M*plus,* Muthén & Muthén, 1998–2012; Mx, Neale, Boker, Xie, & Maes,1999–2003; see also Hamagami, 1997; OpenMx, Boker et al., 2011).

However, demonstrations of how to accommodate truly individually varying time scores with LGMs to date have concerned *linear* LGMs (i.e., LGMs that accommodate a constant relationship between the construct and time; e.g., Bauer, 2003; Burt, McGue, Carter, & Iacono, 2007; Hamagami, 1997; Mehta & West, 2000; Preacher, Wichman, MacCallum, & Briggs, 2008). Some methodological sources on LGM in an SEM framework have allowed for the possibility of individually varying time scores with certain parametric nonlinear LGMs (polynomials or SLC; e.g., Blozis, 2004, 2007; Blozis, Conger, & Harring, 2007) but suggested this was not possible for shape factor LGMs. Furthermore, no studies have demonstrated implementing semiparametric LGMs (piecewise or shape-factor) with truly individually varying time scores in an SEM framework, to our knowledge. More generally, a detailed treatment of how to accommodate truly individually varying time scores in a variety of nonlinear LGMs in an SEM framework is lacking, but this extension has been recommended (Mehta & West, 2000, p. 40).

Perhaps relatedly, the vast majority of LGM applications using SEM continue to assume data are Case I or II. In these applications, a more precise measure of time, such as age in days or months, might have first been rounded to age in years to facilitate assuming that observed or complete data are balanced. Some undesirable consequences of treating individually varying time scores as if they are balanced were discussed by Blozis and Cho (2008), Mehta and West (2000),

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[2]If there were a limited number of different measurement schedules, a modified specification was used that placed individuals with the same schedule in a group and fit the LGM simultaneously to all groups (e.g., Duncan & Duncan, 1994; McArdle & Hamagami, 1992; Muthén, Kaplan, & Hollis, 1987).

and Maes and Neale (2009). These consequences can include biased intercept variance, intercept–slope covariance, and residual variance, but can differ depending on the fitted model and nature of individual variation in time scores. There are advantages to staying within an SEM framework, rather than moving to MLM, when confronted with individually varying time scores. This is particularly true when further modeling goals include specifying complex structural relations (e.g., directional paths among parallel growth processes) and latent variable predictors or outcomes of growth. See the Discussion for other examples.

This article fills an existing gap by describing how to fit four popular nonlinear LGMs—polynomial, shape-factor, piecewise, and an SLC (exponential)—with truly individually varying time scores in an SEM framework. Whereas the extension from linear LGMs with individually varying time scores is more intuitive for some models (polynomials, SLC) it is more involved for others (piecewise, shape-factor). The remainder of this article is organized as follows. First we briefly review specification of the balanced (Case I or II) linear LGM and contrast it with the Case III specification. Next, we present the specification of polynomial, shape-factor, piecewise, and exponential models, in turn, for a balanced (Case I or II) LGM versus an individually varying time score (Case III) LGM. Third, we briefly discuss two topics handled somewhat differently in Case I and II versus Case III LGMs (model fit and missing data). Fourth, we illustrate these models using a longitudinal study of infant weight in the Philippines and supply accompanying software syntax. We conclude with implications of these extensions for practice when time scores individually vary, including considerations for choosing among nonlinear SEMs, and rationales for choosing between SEM and MLM for modeling growth.

Subsequent sections assume time scores are already in the desired metric of time. There are often alternative metrics from which to choose within a given study; some might entail more individual variation in time scores than others. Although researchers might be more familiar with thinking about individual variation in time scores when the metric is age, when the desired metric is wave, there is often still variation in actual data collection times around a planned occasion. For instance, Coffey, Schumacher, Brady, and Cotton (2007) attempted data collection at 2, 5, 10, 14, 21, and 28 days after last substance abuse, and considered these waves as their metric of time. But there was doubtless variation in the timing of individual data collection around each attempted data collection occasion. Finally, examples considered throughout subsequent sections assume time scores have already been recoded to have the desired *origin* (or 0-point) and *spacing* of time (Biesanz, Deeb-Sossa, Papadakis, Bollen, & Curran, 2004; Hancock & Choi, 2006). Also, for all persons time is coded with respect to the same origin (see Blozis & Cho, 2008, or Mehta & West, 2000, for other options).

REVIEW OF LINEAR LGM

Balanced (Case I or II)

The linear LGM is given as

$$y_i = \Lambda \eta_i + \varepsilon_i \tag{1}$$

$$\eta_i = \mu_\eta + \zeta_i \tag{2}$$

y_i is a $T \times 1$ vector of repeated measures for person i, where $i = 1 \ldots N$. For Case I and II, T is defined in Table 2. η_i is a $q \times 1$ vector containing person i's scores on latent growth factors—here, intercept (α_i) and linear slope (β_i). μ_η is a $q \times 1$ vector of growth factor means—here μ_α and μ_β. ε_i is a $T \times 1$ vector of normally distributed time-specific errors for person i, where $\varepsilon_i \sim N(\mathbf{0}, \Theta_\varepsilon)$ and usually $\Theta_\varepsilon = \theta_\varepsilon \mathbf{I}$. ζ_i is a $q \times 1$ vector of normally distributed person-specific deviations from growth factor means where $\zeta_i \sim N(\mathbf{0}, \Psi)$. Denote the intercept variance $\psi_{\alpha\alpha}$, linear variance $\psi_{\beta\beta}$, and covariance $\psi_{\alpha\beta}$. Λ is a $T \times q$ matrix of fixed time scores. The first column, of 1s, defines level. The second column, with values proportional to measurement occasion, represents linear change. Supposing $T = 7$,

$$\Lambda = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \\ 1 & \lambda_3 \\ 1 & \lambda_4 \\ 1 & \lambda_5 \\ 1 & \lambda_6 \\ 1 & \lambda_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \tag{3}$$

In this Case I and II model specification, all persons are assumed to be measured at the same occasions because they are all assigned the same values in the second column of Equation 3. Persons are also assumed to be measured at the same number of occasions because they are assigned the same dimension of Λ in Equation 3. In estimation, allowing for missing data using full information maximum likelihood (FIML), persons could be present for different numbers of occasions (Arbuckle, 1996).

Individually Varying Time Scores (Case III)

For Case III, T is defined in Table 2. Note that two alternative definitions for T are provided, and the choice between them depends on the study design and chosen model. This choice also has relevance for the construction of the analysis data set. For the linear LGM with individually varying time scores and $T = 7$, person i's second column of loadings is populated with values of the T observed time variables for person i.

TABLE 2
Number of Rows, *T*, in the Factor Loading Matrix Under Case I, II, and III Specifications: Definitions and Example Designs

	T (No. of Rows in Factor Loading Matrix)	Example Design(s)
Case I	T = number of measurement occasions per person	T = 7 for a panel study where all persons are repeatedly measured at ages 15, 16, 17, 18, 19, 20, and 21 years, with no missingness.
Case II	T = maximum number of possible measurement occasions across the entire sample as a whole (no longer equivalent to the number of measurement occasions per person). A person might not be present at all occasions.	T = 7 for a panel study where measurements for all persons were attempted at ages 15, 16, 17, 18, 19, 20, and 21 years, but Person *i* contributes between 1 and 7 time points due to subject-initiated absences. T = 7 for a cohort-sequential study where Cohort A is measured at 15, 16, 17, and 18; Cohort B at 17, 18, and 19; and Cohort C at 19, 20, and 21 years. Person *i* contributes a maximum of 4 time points, but 7 time points are observed across the sample as a whole.
Case III	Two alternative definitions of *T* (a) T = the maximum number of possible measurement "windows" across the entire sample as a whole. (Here, a window is a time span of actual data collection around a common attempted occasion; an individual can provide up to one measurement per window but need not provide data at every window. Windows could overlap in some applications.) The shape-factor model requires this definition. (b) T = the maximum number of occasions for which the design allows a single person to provide data.	For some designs, *T* definitions (a) and (b) imply the same loading matrix. Under (a), T = 7 if in a panel study everyone has the opportunity to be measured 7 times, but individual data collection times vary within measurement windows. So long as at least 1 person has complete data, under (b), T = 7 also. For such designs, all 4 nonlinear LGMs are applicable. For other designs, only *T* definition (b) is used. Under (b), T = 20 for a daily diary study yielding up to 20 measurements per person on a purely participant-initiated response schedule. Definition (a) is not used as there are no common occasion windows. For such designs, the shape-factor LGM is not applicable (see text). For other designs, *T* definitions (a) and (b) imply different loading matrices but can yield the same results. Consider a cohort sequential design in which Person *i* has an opportunity to contribute data at up to 4 windows, although the sample as a whole spans 7 windows. Under definition (a), T = 7. But under definition (b), T = 4.[3] For such designs, fitting a shape factor LGM requires definition (a). Polynomial, piecewise, and SLC LGMs also require definition (a) if they allow time-varying residual variances. (With time-invariant residual variances the loading matrix under (a) collapses to the loading matrix under (b) in full information maximum likelihood estimation).

Note. LGM = latent growth model; SLC = structured latent curve.

$$\boldsymbol{\Lambda}_i = \begin{pmatrix} 1 & time_{i1} \\ 1 & time_{i2} \\ 1 & time_{i3} \\ 1 & time_{i4} \\ 1 & time_{i5} \\ 1 & time_{i6} \\ 1 & time_{i7} \end{pmatrix} \tag{4}$$

When an observed variable (time) is placed into a parameter matrix (lambda) it is called a *definition variable* (Mehta & Neale, 2005; Neale, 1998, 2000). A path diagram for this

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[3]For an example person in Cohort 2, definition (a) implies the loading matrix on the left for a fitted quadratic LGM, and definition (b) implies the matrix on the right.

$$\boldsymbol{\Lambda}_i = \begin{pmatrix} 1 & \times & \times \\ 1 & time_{i2} & time_{i2}^2 \\ 1 & time_{i3} & time_{i3}^2 \\ 1 & time_{i4} & time_{i4}^2 \\ 1 & time_{i5} & time_{i5}^2 \\ 1 & \times & \times \\ 1 & \times & \times \end{pmatrix} \quad \boldsymbol{\Lambda}_i = \begin{pmatrix} 1 & time_{i2} & time_{i2}^2 \\ 1 & time_{i3} & time_{i3}^2 \\ 1 & time_{i4} & time_{i4}^2 \\ 1 & time_{i5} & time_{i5}^2 \end{pmatrix}.$$

Case III linear LGM is given in Figure 1, where definition variables are denoted with diamonds (Mehta & Neale, 2005). The raw data input for a linear LGM under Case III now contains 2*T* variables—*T* repeated measures as well as *T* measurement occasion times.

In all the balanced Case I or II LGMs presented in this article, the estimated LGM parameters could be used to construct a single model-implied mean vector ($\hat{\boldsymbol{\mu}}_y = \boldsymbol{\Lambda}\hat{\boldsymbol{\mu}}_\eta$) and covariance matrix ($\hat{\boldsymbol{\Sigma}}_y = \boldsymbol{\Lambda}\hat{\boldsymbol{\Psi}}\boldsymbol{\Lambda}' + \hat{\boldsymbol{\Theta}}_\varepsilon$). In all the Case III LGMs presented here, there is no longer a single model-implied mean vector and covariance matrix; rather, there are *N* of each ($\hat{\boldsymbol{\mu}}_{y_i} = \boldsymbol{\Lambda}_i\hat{\boldsymbol{\mu}}_\eta$ and $\hat{\boldsymbol{\Sigma}}_{y_i} = \boldsymbol{\Lambda}_i\hat{\boldsymbol{\Psi}}\boldsymbol{\Lambda}_i' + \hat{\boldsymbol{\Theta}}_\varepsilon$). Throughout, interpretation of growth factor means, (co-)variances, and Level-1 residual variances remains unchanged when moving from the balanced case (I or II) to Case III.

POLYNOMIAL LGMS

Balanced (Case I or II)

In the balanced setting, a commonly employed nonlinear model is a polynomial LGM (see also Cudeck & du Toit,
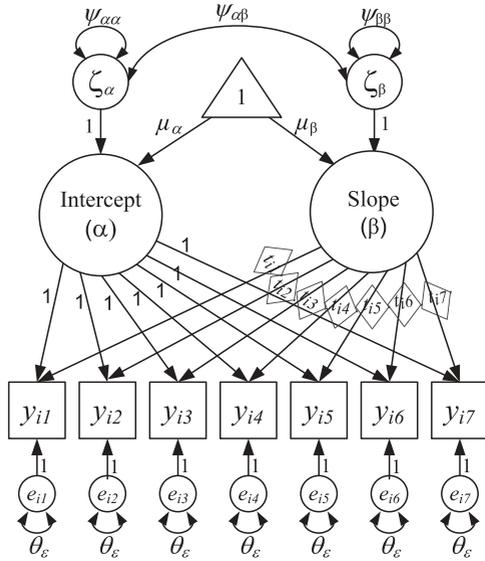
FIGURE 1   Linear latent growth curve model with truly individually vary-ing (Case III) time scores. *Note.* Boxes = measured variables; circles = latent factors (growth factors or residuals); straight arrow = regression path; curved arrow = (co)variance; triangle = constant. Diamonds repre-sent definition variables, here corresponding to observed individual-specific measurement times. See text for other notation.

2002). A polynomial LGM includes additional column(s) (i.e., latent curves or basis functions) in $\mathbf{\Lambda}$ containing powered versions of time scores (often squared, $q = 3$ or cubed, $q = 4$). For instance, extending Equation 3 to a quadratic LGM yields

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{pmatrix} \tag{5}$$

Means for these intercept, linear, and quadratic growth fac-tors can be estimated ($\mu_\alpha, \mu_{\beta_1}, \mu_{\beta_2}$), as can their variances ($\psi_{\alpha\alpha}, \psi_{\beta_1\beta_1}, \psi_{\beta_2\beta_2}$) and covariances ($\psi_{\alpha\beta_1}, \psi_{\alpha\beta_2}, \psi_{\beta_1\beta_2}$), to allow individual differences among correlated aspects of change.

Individually Varying Time Scores (Case III)

In the individually varying measurement setting, a similar manipulation is done either via data management or a non-linear constraint. Using data management, for a quadratic LGM, T new variables—squares of measurement occasion times—are created to serve as T new definition variables in $\mathbf{\Lambda}_i$ (bringing the data set variables to $3T$). Or, more con-veniently, a nonlinear constraint can be employed to set a

second set of definition variables equal to the squared values of the original set. Either approach yields:

$$\mathbf{\Lambda}_i = \begin{pmatrix} 1 & time_{i1} & time_{i1}^2 \\ 1 & time_{i2} & time_{i2}^2 \\ 1 & time_{i3} & time_{i3}^2 \\ 1 & time_{i4} & time_{i4}^2 \\ 1 & time_{i5} & time_{i5}^2 \\ 1 & time_{i6} & time_{i6}^2 \\ 1 & time_{i7} & time_{i7}^2 \end{pmatrix} \tag{6}$$

SHAPE-FACTOR LGMS

An advantage of the polynomial LGM is that it allows spec-ification and testing of known parametric functional forms of change over time. However, as the degree of the polyno-mial increases, the number of estimated parameters increases sharply, potentially resulting in a nonparsimonious model. An alternative is to abandon the fully parametric mindset to estimate some loadings on an optimal "shape" curve. This kind of model has been presented for the balanced (Case I or II) setting and is called a shape-factor or freed-loading (Bollen & Curran, 2006; McArdle, 1986), fully latent (e.g., Aber & McArdle, 1991), unspecified (Duncan et al., 2006), or latent basis LGM (Grimm, Ram, & Hamagami, 2011). The extension of the shape-factor LGM to individually varying time scores is less straightforward than for the polynomial LGM.

Balanced (Case I or II)

In this setting, a convention is to fix all loadings to unity on the intercept factor and fix just two anchor loadings on a "shape" factor to two different values. This convention is sufficient to set the scale of the estimated loadings and overi-dentify the LGM covariance structure when $T > 3$ (Bollen & Curran, 2006). A popular choice is to fix $\lambda_1 = 0$ to establish interpretation of the intercept factor as initial status and fix λ_T to the final time score to allow interpretation of the pat-tern of loadings on the second factor—the shape factor—in comparison to a linear LGM. When time scores are 0, 1, 2, 3, 4, 5, 6, this yields:

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 1 & \lambda_2 \\ 1 & \lambda_3 \\ 1 & \lambda_4 \\ 1 & \lambda_5 \\ 1 & \lambda_6 \\ 1 & 6 \end{pmatrix} \tag{7}$$

Means (μ_α, μ_β) and (co)variances ($\psi_{\alpha\alpha}, \psi_{\beta\beta}, \psi_{\alpha\beta}$) of the $q = 2$ growth factors are also estimated.

At first, the very definition of the shape-factor LGM seems to prohibit individually varying time scores. There

are not enough degrees of freedom to estimate every person's freed loadings at different values. To illustrate how this extension can be accomplished, we begin by reexpressing the shape factor loadings' capability for capturing departures from linearity, in the balanced setting. Note that other related parameterizations of shape factor models have been previously considered for the balanced case (e.g., McArdle, 1988). An equivalent way of conveying that the shape-factor model's estimated loadings λ_2 to λ_6 capture departures from linearity is to represent a loading at time t as the sum of (a) the time score at t, and (b) the amount by which the loading departs from linearity at time t—denoted δ_t. The fixed (first and last) anchor loadings are also sums of (a) and (b), however, in their case (b) is 0. For the setting considered earlier, this reexpression is shown in Equation 8:

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 1 & 1 + \delta_2 \\ 1 & 2 + \delta_3 \\ 1 & 3 + \delta_4 \\ 1 & 4 + \delta_5 \\ 1 & 5 + \delta_6 \\ 1 & 6 \end{pmatrix} \tag{8}$$

where δ_2, δ_3, δ_4, δ_5, and δ_6 are the amounts by which the second, third, fourth, fifth, and sixth loadings would have to move to correspond with linearity. Because $\delta_1 = \delta_7 = 0$, they are not shown. $\delta_2 \ldots \delta_{T-1}$ are estimated and used to

solve for $\lambda_2 \ldots \lambda_{T-1}$. In the reexpression, $\hat{\mu}_\alpha$ is still the average intercept and $\hat{\mu}_\beta$ is still the average net gain or loss per unit in time. The reexpressed model tells us about occasion-specific deviations from linearity either in y-units ($\hat{\mu}_\beta \times \hat{\delta}_t$) or in time-units ($\delta_t$). For example, in a balanced-case single sample simulated example in Figure 2, the predicted y at the third occasion is farthest from linearity (($\hat{\mu}_\beta \times \hat{\delta}_t$) = $-.47 \times 3.21 = -1.51$ y-units lower). Further, linearity implies the y score of 3.33 is reached when time is 2.74, but the shape-factor model implies it is reached when time is 1.0—a difference of $\hat{\delta}_2$ time units. (In Figure 2 note the model-implied mean trajectory is gray and a superimposed linear reference line is black.)

Individually Varying Time Scores (Case III)

We can extend the reexpressed shape-factor LGM to accommodate individually varying time scores. Fixed linear time scores in Equation 8 are replaced with T individual-specific time scores (definition variables):

$$\mathbf{\Lambda}_i = \begin{pmatrix} 1 & time_{i1} \\ 1 & time_{i2} + \delta_2 \\ 1 & time_{i3} + \delta_3 \\ 1 & time_{i4} + \delta_4 \\ 1 & time_{i5} + \delta_5 \\ 1 & time_{i6} + \delta_6 \\ 1 & time_{i7} \end{pmatrix} \tag{9}$$
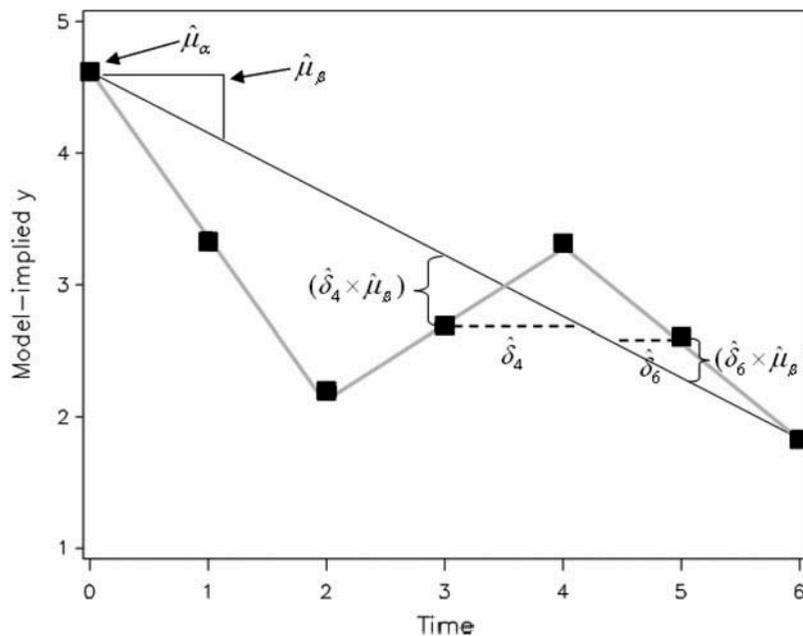
FIGURE 2   Model-implied y scores from a reexpressed shape factor latent growth model (LGM) fitted to an $N = 1,000$ simulated sample with balanced time scores. *Note.* The seven boxes connected with a gray line define the model-implied mean trajectory for the sample. The solid black diagonal line is a reference line superimposed to demarcate net change between anchor time points. The intercept and slope of this line correspond with the means of the corresponding growth factors. Dashed horizontal lines represent offset parameter values—occasion-specific departures from linearity, in time units (shown only for two occasions). Products of offset parameter and mean slopes, denoted by curly brackets, represent occasion-specific departures from linearity, in y units (shown only for two occasions). Symbols discussed in the text. The sample was generated from a conventional shape-factor LGM: $\mu_\alpha = 4.65$; $\mu_\beta = -.47$; $\psi_{\alpha\alpha} = 1.4$; $\psi_{\beta\beta} = .35$; $\psi_{\alpha\beta} = -.13$; $\lambda_1 = 0$; $\lambda_2 = 2.80$; $\lambda_3 = 5.25$; $\lambda_4 = 4.2$; $\lambda_5 = 2.75$; $\lambda_6 = 4.4$; $\lambda_7 = 6$; $\sigma_1^2 - \sigma_7^2 = .5$.

As in the balanced case, we are still estimating only $T - 2$ parameters in lambda (here, $\delta_2 \ldots \delta_{T-1}$). Note that the first and last (anchor) loadings on the shape factor are no longer fixed to universal values for identification purposes, as in the balanced case. Rather, the anchor loadings for person i are constrained to the observed values of person i's own first and last time scores, which serves to identify the model. Thus, person j can have different anchor loading values than person i.

In this model, persons within the tth measurement window (see Table 2 for definition of measurement window) are now allowed different measurement times, but, as in the original shape-factor model, still have the same horizontal offset from linearity (δ_t). Thus, persons within the tth window take on different model-implied mean y values due to their individually varying time scores within that window (as will be visually depicted later in the context of the empirical example).

Some additional properties of this model can be noted. First, if linearity were to hold, all offset parameters would be zero: $\delta_2 = \ldots = \delta_{T-1} = 0$. Second, although this model is not necessarily limited in the degree of individual variation in time that can be accommodated,[4] it is limited in the kind of individual variation in time that can be accommodated. Regarding the kind of individual variation accommodated, observations need to be classifiable by the researcher into measurement windows. The timing of data collection is allowed to individually vary within such windows. Examples of designs that would qualify are given in Table 2; in such contexts there may be compelling reasons to consider this extension.

PIECEWISE LGMS

We have discussed a fully parametric approach to modeling nonlinearity (polynomial LGM) and one semiparametric approach (shape-factor LGM). A third alternative— a piecewise LGM—provides a compromise between the potential overrestrictiveness of the former and the flexibility of the latter. Like polynomials, piecewise LGMs capture nonlinearity through the use of additional latent growth factors. Like shape-factors, piecewise LGMs are not fully parametric.

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[4]Regarding *degree* of individual variation, a simulation was conducted (generating parameters $\mu_\alpha = 4.65$; $\mu_\beta = -.47$; $\psi_{\alpha\alpha} = 1.4$; $\psi_{\beta\beta} = .35$; $\psi_{\alpha\beta} = -.13$; $\lambda_1 = 0$; $\lambda_2 = 2.80$; $\lambda_3 = 5.25$; $\lambda_4 = 4.2$; $\lambda_5 = 2.75$; $\lambda_6 = 4.4$; $\lambda_7 = 6$; $\sigma_1^2 - \sigma_7^2 = .5$) with normally distributed time scores generated to have some variation at Levels 1 and 2: $time_{it} = lev1\_time_{it} + lev2\_time_i$. Two alternative amounts of individual variation in time were considered; either ($lev1\_time_{it} \sim N(t, .10^2)$ and $lev2\_time_i \sim N(0, .05^2)$), or ($lev1\_time_{it} \sim N(t, .50^2)$ and $lev2\_time_i \sim N(0, .25^2)$, the latter implying considerably overlapping windows. $t = 0, 1, 2, 3, 4, 5,$ or 6. There was trivial to no bias ($\leq 1.2\%$ relative bias) in any model parameter across 100 samples of $N = 1,000$.

Piecewise LGM's "piece" together at least two shorter, low-order polynomial (often linear or quadratic) segments to approximate a more complex underlying functional form. In addition to a shared intercept factor, one growth factor is added per basis function, per piece. For example, a two-piece linear LGM has intercept and linear factors for each piece 1 and 2 ($q = 3$). A three-piece quadratic LGM has intercept, linear, and quadratic slope factors for pieces 1, 2, and 3 ($q = 7$). Means and (co)variances of the q growth factors can be estimated to determine average aspects of change per piece and (co)variation in aspects of change. We focus on piecewise linear LGMs, but explain how principles can be extended to higher order pieces.

Piecewise LGMs attach curve fragments together at *knot points*. Knot points may be chosen in a data-driven way (Kwok, Luo, & West, 2010) or based on theory (Flora, 2008; Hancock & Lawrence, 2006). Here we assume the knot point is at the same predetermined place for all persons, but some recent work has addressed relaxing this assumption (Preacher & Hancock, 2010). Knot point location(s) in piecewise LGMs are specified in Λ. As usual, each of the q growth factors is associated with its own column of loadings in Λ.

Balanced (Case I or II)

In an SEM framework, piecewise LGMs have been presented for balanced (Case I or II) data (e.g., Bollen & Curran, 2006, Duncan et al., 2006, Flora, 2008). Flora (2008, p. 523) provided a set of rules used to generate Λ elements for any two-piece linear LGM for balanced Cases I or II. However, no fully general algorithm for specifying elements in Λ has been provided that applies to any number of pieces (phases), and any coding of the origin or spacing of time. In the balanced case (I or II), it might be possible to infer Λ elements for more complex designs by extrapolating from available rules. However, in the individually varying time scores case, fully general guidelines are necessary. Furthermore, their application needs to be automated (for reasons discussed later). Such general guidelines are first presented in the balanced case (I or II), and later applied to Case III.

Preliminarily, a researcher with balanced (Case I or II) data needs to choose the number of phases (i.e., pieces), denoted M. There are $M - 1$ knots. We index phases $m = 1 \ldots M$ and knots $m = 1 \ldots M - 1$. Next, the locations of the knot(s) need to be chosen. An algorithm for generating linear slope loadings for each phase (piece) in a piecewise LGM are given in Table 3. In Table 3, the phase containing the origin of time is termed the *intercept phase*. When the intercept falls on a boundary between two phases (i.e., at a knot), we always assign it to the previous phase.[5] The general guidelines in Table 3 are applied, in turn, to each of the T time scores

---

[5]This is an arbitrary decision. If the alternative decision had been made (i.e., when an intercept falls at the boundary between two phases, assign it to

TABLE 3
Algorithm for Transforming Each of the T Time Scores to Have Desired Piecewise Structure for Each $m = 1 \ldots M$

If phase $(m) <$ *intercept phase:*
- If knot (m) is $\geq$ *time* and [either *time* > knot $(m-1)$ or knot $(m-1)$ absent] then loading $= (time - \text{knot}(m))$.
- If knot $(m-1) \geq$ *time,* then loading $= (\text{knot }(m-1) - \text{knot }(m))$.
- Otherwise, loading $= 0$.

If phase$(m) =$ *intercept phase:*
- If [either knot $(m) \geq$ *time* or knot (m) absent] and [either *time* > knot $(m-1)$ or knot $(m-1)$ absent] then loading $= time$.
- If knot $(m-1) \geq$ *time* then loading $=$ knot $(m-1)$.
- If knot $(m) <$ *time* then loading $=$ knot (m)

If phase $(m) >$ *intercept phase:*
- If knot $(m-1) <$ *time* and [either age $\leq$ knot (m) or knot (m) absent] then loading $= (time - \text{knot }(m-1))$.
- If knot $(m) <$ *time* then loading $= (\text{knot }(m) - \text{knot }(m-1))$.
- Otherwise, loading $= 0$.

Note. Time = already-recoded time score; number of phases is $m = 1 \ldots M$; number of knots is $m = 1 \ldots M - 1$; *intercept phase* = phase containing the origin (in the already-recoded time metric). Note that knot $(m-1)$ will be "absent" if $m = 1$, and knot (m) will be "absent" if $m = M$. When the intercept falls at a knot, it is assigned to the previous phase. The Table 3 algorithm is automated in SAS code in the online Appendix.

(denoted *time* in Table 3), for $m = 1 \ldots M$. Slope loading for phases with higher order polynomial (quadratic or cubic, etc.) factors do not need to be separately generated; they can be obtained through nonlinear constraints on the generated linear slope loadings (see empirical example and code). These general guidelines are implemented in a SAS macro available in the online Appendix at www.vanderbilt.edu/peabody/sterba/appxs.htm. One example is provided here; two other examples, of greater and lesser complexity, are provided in the online Appendix (with code).

For this Case I or II data illustration, suppose time scores are *time* = –6.5 –5, –1, 0, 2, 3.5, 6, 8. Suppose there are $M = 3$ phases (pieces), and $M - 1 = 2$ knots at *time* = –1, and 3.5. The intercept phase is Phase 2 because *time* = 0 falls between the two knots. Applying general guidelines in Table 3 yields

$$\Lambda = \begin{pmatrix} 1 & -5.5 & -1 & 0 \\ 1 & -4 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 3.5 & 0 \\ 1 & 0 & 3.5 & 2.5 \\ 1 & 0 & 3.5 & 4.5 \end{pmatrix} \qquad (10)$$

This balanced case example affords intuitive understanding. The aspect of change described by the first piece goes offline at the first knot. There the second piece picks up until the second knot, where the final piece takes over, ensuring linear slope loadings sum across rows to time scores.

Individually Varying Time Scores (Case III)

In Case II or III some, no, or all persons may be measured exactly at a knot point, or exactly at the origin. Also, some persons may not be measured in the span of an entire piece. In Case II these complications were not burdensome because we were obliged to specify Λ only once; variations due to missing data were handled exclusively in estimation, not specification. In Case III, we need to specify N different Λ_i to impart the piecewise structure on N different sets of transformed time scores. The variety of possibilities and scope of this task make it more difficult to verify via visual inspection that Λ_i conforms with the desired piecewise structure. To illustrate such complications, consider a situation in which there are $M = 3$ linear phases, and $M - 1 = 2$ knots, desired to be at time = –1 and 2 for the sample, such that the intercept phase is Phase 2. Consider the measurement schedules for four persons from this sample.[6]

Person 1 time scores: –3.9, –1.9, –1.0, 0, .90, 2.0, 4.8, 5.9
Person 2 time scores: –4.3, –2.1, –1.0, .60, 1.7, 3.1, 4.2, 5.8
Person 3 time scores: –4.5, –2.5, –.50, 0, 1.9, 2.8, 3.9, 5.2
Person 4 time scores: –5.0, –3.3, –2.1, –1.8, 3.0, 3.2, 4.5, 5.6

Here Person 1 is measured exactly at both knots as well as at the origin. Person 2 is measured exactly at the first knot, but at neither the second knot nor origin. Person 3 is measured exactly at the origin, but at neither knot. Person 4 is measured at neither at a knot, nor the origin, and furthermore has no measurements in Piece 2 at all.

The Table 3 guidelines can be used, personwise, to generate Λ_i. But because of the preceding complexities and the need to specify N different Λ_i, automated implementation of

the later—not earlier—phase), the exact same lambda matrix would result, so long as the decision was made consistently.

[6]These persons could have missing data; missing data are not immediately relevant to this illustration.

these guidelines for generating loadings is no longer just a convenience, but a necessity. The software tools described earlier can be used to generate all N $\mathbf{\Lambda}_i$ simultaneously; instead of inputting one set of time scores that apply to all persons in the sample, an entire data set of N persons' time scores is inputted. For instance, the Table 3 guidelines generate $\mathbf{\Lambda}_i$ for Persons 1, 2, 3, and 4 as:

$$\mathbf{\Lambda}_1 = \begin{pmatrix} 1 & -2.9 & -1 & 0 \\ 1 & -.9 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & .9 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2.8 \\ 1 & 0 & 2 & 3.9 \end{pmatrix} \quad \mathbf{\Lambda}_2 = \begin{pmatrix} 1 & -3.3 & -1 & 0 \\ 1 & -1.1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & .6 & 0 \\ 1 & 0 & 1.7 & 0 \\ 1 & 0 & 2 & 1.1 \\ 1 & 0 & 2 & 2.2 \\ 1 & 0 & 2 & 3.8 \end{pmatrix}$$

$$\mathbf{\Lambda}_3 = \begin{pmatrix} 1 & -3.5 & -1 & 0 \\ 1 & -1.5 & -1 & 0 \\ 1 & 0 & -.5 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1.9 & 0 \\ 1 & 0 & 2 & .8 \\ 1 & 0 & 2 & 1.9 \\ 1 & 0 & 2 & 3.2 \end{pmatrix} \quad \mathbf{\Lambda}_4 = \begin{pmatrix} 1 & -4 & -1 & 0 \\ 1 & -2.3 & -1 & 0 \\ 1 & -1.1 & -1 & 0 \\ 1 & -.8 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1.2 \\ 1 & 0 & 2 & 2.5 \\ 1 & 0 & 2 & 3.6 \end{pmatrix}$$

Importantly, Person i's linear slope loadings still sum across columns of their $\mathbf{\Lambda}_i$ to produce their own individual-specific time scores (given earlier). Further, because Person 3 was not measured exactly at the second knot, the second knot occurs between their fifth and sixth loadings. Also, because Person 2 was not measured exactly at the origin, the intercept occurs between their fourth and fifth loadings. Finally, as the Table 3 guidelines accommodate any number of pieces, very complex piecewise LGMs (see online Appendix) could have individually varying loadings using the same procedure.

We have thus far described automatically generating $\mathbf{\Lambda}_i$ matrices for piecewise LGMs. When specifying the model in SEM software, transformed individually varying time scores constituting slope loadings in $\mathbf{\Lambda}_i$ are definition variables, as illustrated in Equation 11 for a two-piece linear LGM:

$$\mathbf{\Lambda}_i = \begin{pmatrix} 1 & time_{i1}^{piece1} & time_{i1}^{piece2} \\ 1 & time_{i2}^{piece1} & time_{i2}^{piece2} \\ 1 & time_{i3}^{piece1} & time_{i3}^{piece2} \\ 1 & time_{i4}^{piece1} & time_{i4}^{piece2} \\ 1 & time_{i5}^{piece1} & time_{i5}^{piece2} \\ 1 & time_{i6}^{piece1} & time_{i6}^{piece2} \\ 1 & time_{i7}^{piece1} & time_{i7}^{piece2} \\ 1 & time_{i8}^{piece1} & time_{i8}^{piece2} \end{pmatrix} \tag{11}$$

SEM software expects transformed, individually varying time scores serving as definition variables to be in wide

format.[7] In wide format, our piecewise LGM data set has T + TM columns (T outcomes + TM transformed linear time scores), even if some pieces have higher order slope factors.[8]

STRUCTURED LATENT GROWTH CURVE MODELS

Earlier we discussed that one advantage of polynomial LGMs is their known functional form. With complex growth patterns, however, high-degree polynomial LGMs can become difficult to interpret. Other known parametric functional forms for a mean trajectory, or target functions (e.g., Gompertz, monomolecular, cosine, hyperbolic, exponential) could yield parameters that are easier to interpret. However, such functional forms are often more difficult to specify in SEM software because this software assumes growth coefficients enter the model linearly.[9] The target function can be linearized to be compatible with SEM software—as done previously for a variety of functions (e.g., Blozis, 2004, 2007; Browne, 1993; Browne & Du Toit, 1991; Grimm & Ram, 2009; Grimm et al., 2011). Such models are called structured latent curves (SLCs). Our goal is to highlight how specification of an SLC differs between the balanced cases (I or II) and Case III; hence, we provide this contrast for one popular exemplar SLC—the exponential LGM.

Balanced (Case I or II)

The (here, negative) exponential target function at time t can be given as follows (e.g., Bollen & Curran, 2006):

$$y_t = \alpha + \beta(1 - e^{(-\gamma\lambda_t)}) \tag{12}$$

where α is the intercept, β is the total change in the outcome as time approaches infinity, and γ is a rate parameter governing the rate at which the outcome approaches an asymptotic level. The corresponding SLC is specified here with balanced Case I or II time scores of 0, 1, 2, 3, 4, 5, 6.

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[7]The %PIECEWISE macro outputs the transformed time scores in long format. The M*plus* syntax provided in the online Appendix transforms them to wide format before fitting piecewise LGMs.

[8]Loadings for any pieces with higher order (e.g., quadratic, cubic) slopes can be specified as squares or cubes of linear time scores in the data, for instance using nonlinear constraints (see example code).

[9]That is, repeated measures are assumed by SEM software to be simple weighted sums of growth coefficients × predictors, plus error. In more complex functional forms, some [growth coefficients × predictor] terms might be in an exponent or logarithm, under a radical, or inside a trigonometric term.

$$
\begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \\ y_{i5} \\ y_{i6} \\ y_{i7} \end{pmatrix} = \begin{pmatrix} 1 & 1 - e^{(-\mu_\gamma 0)} & \mu_\beta 0 \times e^{(-\mu_\gamma 0)} \\ 1 & 1 - e^{(-\mu_\gamma 1)} & \mu_\beta 1 \times e^{(-\mu_\gamma 1)} \\ 1 & 1 - e^{(-\mu_\gamma 2)} & \mu_\beta 2 \times e^{(-\mu_\gamma 2)} \\ 1 & 1 - e^{(-\mu_\gamma 3)} & \mu_\beta 3 \times e^{(-\mu_\gamma 3)} \\ 1 & 1 - e^{(-\mu_\gamma 4)} & \mu_\beta 4 \times e^{(-\mu_\gamma 4)} \\ 1 & 1 - e^{(-\mu_\gamma 5)} & \mu_\beta 5 \times e^{(-\mu_\gamma 5)} \\ 1 & 1 - e^{(-\mu_\gamma 6)} & \mu_\beta 6 \times e^{(-\mu_\gamma 6)} \end{pmatrix}
$$

$$
\times \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \varepsilon_{i4} \\ \varepsilon_{i5} \\ \varepsilon_{i6} \\ \varepsilon_{i7} \end{pmatrix} \tag{13}
$$

$$
\begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} = \begin{pmatrix} \mu_\alpha \\ \mu_\beta \\ 0 \end{pmatrix} + \begin{pmatrix} \zeta_{\alpha i} \\ \zeta_{\beta i} \\ \zeta_{\gamma i} \end{pmatrix} \tag{14}
$$

Here, the linearization involved taking the partial derivative of the target function with respect to each of its growth parameters—evaluated at the parameters of the target function. For instance, the partial derivatives of the target function with respect to the qth growth parameter were placed in the qth column of Λ in Equation 13. Also, to ensure μ_y equals the target function, parameters (α, β) entering the target function linearly have corresponding coefficient means in μ_η, in Equation 14, whereas the parameter entering nonlinearly (γ) has a corresponding mean estimated within Λ. (Although the population mean vector follows the target function, individual observations do not necessarily; Browne, 1993). To fit Equation 13, nonlinear constraints are placed on loadings during model specification—for example if $t = 0$, set $\lambda_{12} = 1 - e^{(-\mu_\gamma 0)}$.

Individually Varying Time Scores (Case III)

Similar to the polynomial LGM, extending the SLC from the balanced case (I or II) to Case III presents no complications. Simply replace time scores—previously fixed to the same values for all persons in Λ —with T definition variables representing individual-specific measurement occasions in Λ_i.

$$
\Lambda_i = \begin{pmatrix} 1 & 1 - e^{(-\mu_\gamma time_{i1})} & \mu_\beta time_{i1} \times e^{(-\mu_\gamma time_{i1})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i2})} & \mu_\beta time_{i2} \times e^{(-\mu_\gamma time_{i2})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i3})} & \mu_\beta time_{i3} \times e^{(-\mu_\gamma time_{i3})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i4})} & \mu_\beta time_{i4} \times e^{(-\mu_\gamma time_{i4})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i5})} & \mu_\beta time_{i5} \times e^{(-\mu_\gamma time_{i5})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i6})} & \mu_\beta time_{i6} \times e^{(-\mu_\gamma time_{i6})} \\ 1 & 1 - e^{(-\mu_\gamma time_{i7})} & \mu_\beta time_{i7} \times e^{(-\mu_\gamma time_{i7})} \end{pmatrix} \tag{15}
$$

Note that the same T time scores are incorporated $3T$ times, inside nonlinear constraints, in this Λ_i.

MODEL SELECTION IN CASE I AND II VERSUS CASE III LGMs

For LGMs, conventional model fit indices are not available under Case III—similar to MLM—whereas they are available always under Case I and sometimes under Case II (depending on the extent and pattern of "missing" data). However, model selection indices (e.g., information criteria or likelihood ratio difference tests [LRTs]) are available under Cases I, II, and III for LGMs. As in the balanced Case (I–II), for Case III, the random linear LGM is nested within each of the (a) random quadratic LGM, (b) reexpressed random shape-factor LGM, and (c) random piecewise linear LGM. In the empirical example, LRTs are used to compare each pair of nested models.

MISSING DATA IN CASE I-II VS. CASE III LGMs

In Cases I and II, unconditional LGMs (where the only predictor is time) are only subject to missingness on y scores; time scores are treated as known for all persons regardless of whether their y scores were missing. In Case III, such LGMs are subject to missingness on (a) y scores only, (b) both time scores and y scores, or (c) time scores only. Regarding (a), if a person is missing some y score(s) *only*, all available observations for that person are retained under FIML. Regarding (b), if a person is missing *pairs* of y score(s) and time score(s) for some occasions, SEM software using a conditional likelihood specification (e.g., M*plus* 6.1 or later) will by default listwise delete that person. SEM software using a joint likelihood specification will not. That is, a conditional likelihood is conditional on predictors, and in the Case III definition variable approach, time scores are considered observed predictors.[10] When fitting an LGM with a conditional likelihood, the following strategy can be used to retain persons that have some missing pairs of y scores and time scores. Arbitrary time data are inserted as placeholders for time scores at occasions where y scores are missing. This serves to retain the entire person (data set row). Only that person's *observed* y scores—along with their paired, true time scores—contribute to model estimation and resulting parameter estimates. This strategy was employed in the empirical example (also see online Appendix). Regarding (c), it is not common in practice for a person to have missing time score(s) for occasion(s) at which they have present y score(s). If this circumstance arises, time scores could, for instance, be multiply imputed to retain that person when using a conditional likelihood.

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[10]Note that MLM also uses a conditional likelihood, and will listwise delete observations, not cases, with missing time scores (but will do so for Case II *or* Case III data).
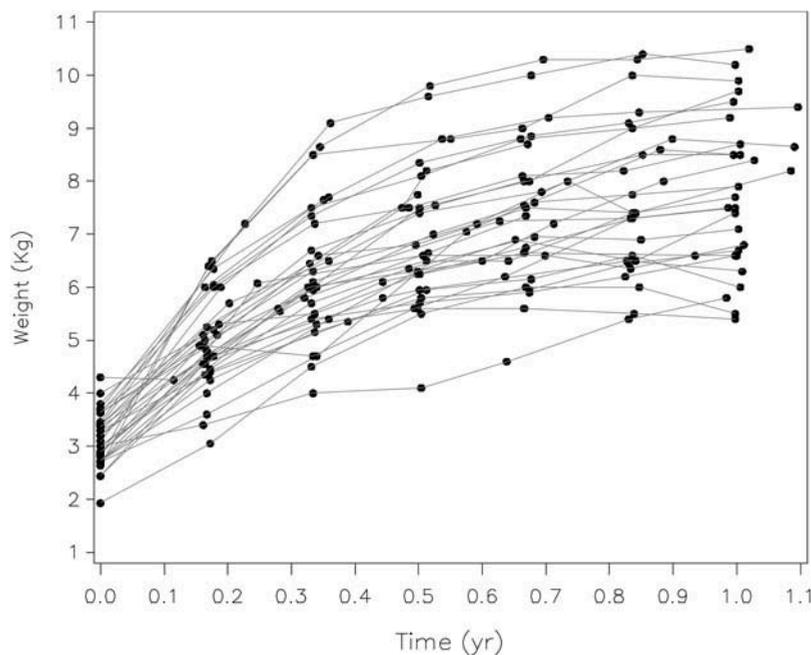
FIGURE 3 Observed *y* scores for a subset of infants from the empirical example data set. *Note.* Although study researchers intended to collect infant weight every 2 months, horizontal variation of dots around each 2-month mark indicates that individuals vary in measurement times.

EMPIRICAL EXAMPLE

In this section we illustrate fitting all four types of nonlinear LGMs with individually varying loadings for an empirical example on growth in infant weight. This example uses panel data from the Longitudinal Health and Nutrition Survey ($N = 2,631$)[11] (Adair, 1989; Adair & Popkin, 2001), which was concerned with tracking the functional form of infant growth in the Philippines, identifying heterogeneity in growth, and explaining this heterogeneity with social and health behaviors. Even today, 20% of Filipino children have restricted growth (United Nations Children's Fund, 2009). Pregnant mothers in one province in the Philippines were enrolled in 1983 and 1984. These data contain individually varying time scores. Although researchers attempted to collect measurements of weight (in kg) from all infants every 2 months from birth to 1 year ($T = 7$ using Case III definition (a) in Table 2), age, recorded in days, varied at all measurement windows except the first (birth). For example, at the second measurement window infants ranged from 42 to 102 days old; at the third infants ranged from 90 to 173 days old. A subset of observed trajectories is plotted in Figure 3.

As a starting point, a linear LGM is fit, followed by quadratic, shape-factor, piecewise linear, piecewise quadratic, and exponential LGMs, each with individually varying time scores. Variances were estimated for all growth coefficients. All models had heteroscedastic Level 1 residual

variances across time. M*plus* input code for these five nonlinear LGMs with individually varying loadings is given in the online Appendix. A total of 2,061 out of 2,631 infants had complete data. When an outcome was missing, its companion time score was missing. Using the strategy described previously in the missing data section, all 2,631 infants were retained. The intercept was coded at birth (0 years) and the unit change in time was coded to be 1 year.

Model Fitting and Interpretation

For the linear, quadratic, shape-factor, and exponential LGMs our (wide format) analysis data set contains N rows and $2T$ columns: $T = 7$ columns of repeated measures and $T = 7$ columns of time scores used as definition variables. The quadratic LGM also uses squares of these T time scores (via nonlinear constraint) as definition variables. Following automatic generation of TM transformed linear time scores (see online Appendix), the fitted piecewise linear LGM uses these as definition variables. The piecewise quadratic LGM also uses their squares (via nonlinear constraint) as definition variables.

Estimates and standard errors for all parameters except residual variances are given in Table 4 for all six fitted LGMs with individually varying loadings. Interpretation of parameters in Table 4 for each nonlinear model is briefly considered. The quadratic LGM implies that, on average, infants weigh 3.22 kg at birth, with an instantaneous velocity of 9.92 kg per year, that decelerates by $2\times$ –5.38 kg/year; there is significant individual variability in all aspects of change. The

---

[11]One extreme case (18 kg at 1 year) who had small to moderate influence on estimates was deleted.

TABLE 4
Fitted Latent Growth Curves (LGMs) With Individually Varying Time Scores for Empirical Example

	Linear LGM			Quadratic LGM			Shape-Factor LGM			Piecewise Linear LGM			Piecewise Quadratic LGM						Exponential LGM		
Parm	Est	SE	Parm	Est	SE	Parm	Est	SE	Parm	Est	SE	Parm	Est	SE	Parm	Est	SE	Parm	Est	SE
μ_α	5.031	.018	μ_α	3.216	.013	μ_α	3.026	.010	μ_α	3.382	.019	μ_α	3.024	.009	$\psi_{\alpha\beta_1}$	−0.224	.072	μ_α	3.021	.009
μ_β	3.135	.025	μ_{β_1}	9.923	.062	μ_β	4.878	.022	μ_{β_1}	7.166	.049	μ_{β_1}	12.435	.082	$\psi_{\alpha\beta_2}$	0.298	.105	μ_β	5.221	.026
$\psi_{\alpha\alpha}$	.581	.024	μ_{β_2}	−5.384	.057	$\psi_{\alpha\alpha}$	.152	.007	μ_{β_2}	1.961	.028	μ_{β_2}	−9.602	.129	$\psi_{\beta_1\beta_2}$	−16.175	.899	μ_γ	2.625	.022
$\psi_{\beta\beta}$	.937	.042	$\psi_{\alpha\alpha}$	.107	.007	$\psi_{\beta\beta}$	.900	.030	μ_{β_2}	.038	.008	μ_{β_3}	2.821	.058	$\psi_{\alpha\beta_3}$	0.037	.029	$\psi_{\alpha\alpha}$	0.189	.010
$\psi_{\alpha\beta}$	−.155	.026	$\psi_{\beta_1\beta_1}$	4.221	.202	$\psi_{\alpha\beta}$	.004	.010	$\psi_{\beta_1\beta_1}$	1.693	.089	μ_{β_4}	−1.275	.103	$\psi_{\beta_1\beta_3}$	−0.873	.264	$\psi_{\beta\beta}$	1.433	.049
			$\psi_{\beta_2\beta_2}$	2.288	.140	δ_2	.208	.002	$\psi_{\beta_2\beta_2}$	.947	.046	$\psi_{\alpha\alpha}$	.186	.012	$\psi_{\beta_2\beta_3}$	2.327	.508	$\psi_{\gamma\gamma}$	0.863	.038
			$\psi_{\alpha\beta_1}$	.258	.030	δ_3	.297	.002	$\psi_{\alpha\beta_1}$	.347	.021	$\psi_{\beta_1\beta_1}$	12.563	.607	$\psi_{\alpha\beta_4}$	−0.025	.050	$\psi_{\alpha\beta}$	−0.007	.013
			$\psi_{\alpha\beta_2}$	−.221	.025	δ_4	.280	.002	$\psi_{\alpha\beta_2}$	−.060	.014	$\psi_{\beta_2\beta_2}$	25.196	1.455	$\psi_{\beta_1\beta_4}$	0.764	.452	$\psi_{\alpha\gamma}$	−0.055	.014
			$\psi_{\beta_1\beta_2}$	−2.821	.162	δ_5	.206	.002	$\psi_{\beta_1\beta_2}$	.255	.046	$\psi_{\beta_3\beta_3}$	3.231	.313	$\psi_{\beta_2\beta_4}$	−2.436	.818	$\psi_{\beta\gamma}$	−0.541	.033
						δ_6	.106	.002				$\psi_{\beta_4\beta_4}$	9.960	.851	$\psi_{\beta_3\beta_4}$	−4.502	.494			

Model Assessment

	Linear LGM			Quadratic LGM			Shape-Factor LGM			Piecewise Linear LGM			Piecewise Quadratic LGM						Exponential LGM		
Loglik	−17,577			−11,708			−10,752			−12,393			−9,534						−9,880		
AIC	35,177			23,449			21,538			24,818			19,121						19,793		
BIC	35,248			23,543			21,638			24,912			19,274						19,887		
#parm	12			16			17			16			26[a]						16		

Note: AIC = Akaike's information criterion; BIC = Bayesian information criterion. Growth factor mean (μ) and (co)variance (ψ) notation was defined in the text, except for piecewise quadratic subscripts: 1 refers to linear component of the first piece, 2 refers to quadratic component of the first piece, 3 to linear component of the second piece, and 4 to quadratic component of the second piece.

[a]One residual variance in the piecewise quadratic was near 0. It was constrained to 0 to prevent a nonpositive definite solution, resulting in 26 rather than 27 free parameters.

FIGURE 4   Contrasting model-implied trajectories from five nonlinear latent growth models (LGMs) with individually varying time scores against a Loess-smoothed observed data trajectory for the empirical example.

shape factor LGM implies an average infant birthweight of 3.03 kg, and an average net gain in weight of 4.88 kg per year of life. Both aspects of change vary randomly across infants. Infants' model-implied ys are farthest from linearity at the third window ($4.88 \times .30 = 1.46$ kg higher than linearity). Stated differently, linear change between anchor points would imply that infants would reach 6 kg at approximately 6 months of age, whereas the shape-factor model implies that they can reach this weight by 3 months of age (a difference of $\hat{\delta}_3$). The piecewise linear LGM implies that average infant birthweight is 3.38 kg. For the first 6 months infants grow at a faster rate of 7.17 kg per year and for the second 6 months infants grow at a slower rate of 1.96 kg per year. All three aspects of change vary randomly across infants. The piecewise quadratic LGM implies that average infant birthweight is 3.02 kg, and the first piece is defined by an instantaneous velocity of 12.43 kg per year weight gain at birth, which decelerates by 2×-9.60 kg per year over the first half of the year. During the second half of the year the deceleration is less (2×-1.28 kg/year). All aspects of change show significant random variation across infants. Finally, the exponential LGM implies that average infant birthweight is 3.02 kg, and the total change in weight as time goes to infinity is 5.22 kg, approaching an asymptote with a rate parameter of 2.63. (Note the implausibility of extrapolating beyond the observed data here.)

Model Comparison

Information criteria (Bayesian information criterion and Akaike's information criterion) in Table 4, which are penalized for model complexity, indicate the piecewise quadratic

LGM is better fitting than the other models, followed by exponential and shape-factor. We can reject linearity with all nested model pairs. The quadratic LGM fits significantly better than the linear LGM, $\chi^2 (4) = 11738, p < .001$, as do the shape-factor LGM, $\chi^2 (5) = 13,650, p < .001$, and the piecewise-linear LGM, $\chi^2 (4) = 10,368, p < .001$.[12]

When complete data are balanced (Case I or II), researchers often assess and compare mean structure fit by plotting the T observed means (denoted $\bar{y}$) versus model-implied means ($\hat{\mu}_y$). In the case of individually varying time scores, instead of this $\bar{y}$ we can use a smoothed (e.g., Loess) observed data trajectory across the observed time range (here, 0–1.1 years). In the case of individually varying time scores, model-implied means, $\Lambda_i \hat{\mu}_\eta$ are also specific to the time score. For polynomial, piecewise, and exponential LGMs, regardless of a person's own time scores, their model-implied mean values fall along the same trajectory. Hence, for these models, we generated model-implied mean values of y across a range of hypothetical time scores from 0 to 1.1 for comparison to the observed Loess trajectory, in Figure 4.

For the shape-factor LGM, Person i's model-implied mean at each window is specific to his or her own time score; hence, model-implied means fall along a trajectory unique to Person i (as depicted for a subset of infants in Figure 5a). That is, Person i's model-implied mean at window t is:

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[12]Normally, we could also conduct an LRT between a piecewise linear and piecewise quadratic LGM, but a very small Time 7 residual variance in the latter model had to be constrained to 0 to prevent a nonpositive definite solution. This additional constraint made the models nonnested.
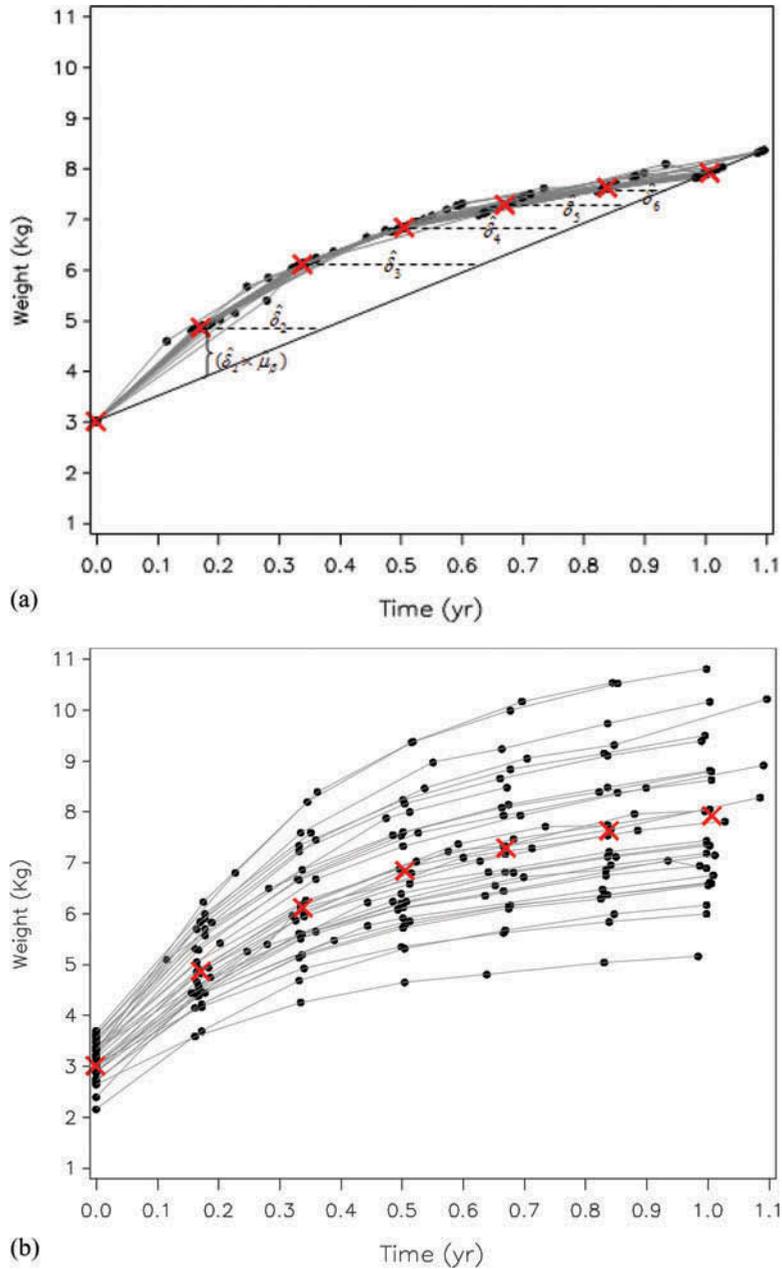
FIGURE 5   Model implied *y* scores from the shape factor latent growth model (LGM) with individually varying time scores, fit to the empirical example data set, for a subset of infants (same subset as shown in Figure 3). (a) Model-implied *y* scores are conditional on individual time scores (as in Equation 16). (b) Model-implied *y* scores are conditional on individual time scores and factor scores (as in Equation 17). *Note for (a)*. Solid black line = superimposed linear reference line; delta parameters (dashed lines) and a vertical offset (bracket) are noted for an example person; 7 *X*s = model-implied *y*s at average time scores per window. Person *i*'s model-implied trajectory is denoted by their (up to) 7 dots connected by an individual gray line.

$$\hat{y}_{it}|\, time_{it} = \hat{\mu}_\alpha + \hat{\mu}_\beta(\lambda_{it}) = \hat{\mu}_\alpha + \hat{\mu}_\beta(time_{it}) + \hat{\mu}_\beta(\hat{\delta}_t)$$
(16)

One summary plotting option is to plot the mean trajectory for the average time scores per window $\bar{\boldsymbol{\Lambda}}\hat{\boldsymbol{\mu}}_\eta$ —depicted as *X*s in Figure 5a—against the observed-Loess. This is done in Figure 4; good correspondence is noted. Another option is to plot Person *i*'s model-implied *y* at window *t* also given their estimated individual intercept and slope factor scores $(\hat{\alpha}_i, \hat{\beta}_i)$:

$$\hat{y}_{it}|\, time_{it}, \hat{\alpha}_i, \hat{\beta}_i = \hat{\alpha}_i + \hat{\beta}_i(\lambda_{it}) = \hat{\alpha}_i + \hat{\beta}_i(time_{it}) + \hat{\beta}_i(\hat{\delta}_t)$$
(17)

as done in Figure 5b, for comparison to the observed trajectories in Figure 3. Good correspondence is also noted between Figure 3 and Figure 5b. (Similar plots to Figure 5b could be constructed for the other models, but are not shown here.)

DISCUSSION

Longitudinal data may often include individually varying measurement times. Social science theories also often include strong theoretical motivations for nonlinear trends over time. Whereas previous LGM demonstrations have focused on including individually varying measurement times or nonlinear growth, their combination has not been thoroughly addressed. LGM applications still rarely allow for individually varying time scores, perhaps in part because definition variable methods need further extension to popular nonlinear functional forms and further dissemination.

This article described how to fit four popular kinds of nonlinear LGMs with truly individually varying time scores. Two kinds—polynomial and SLC—represented straightforward extensions of the linear LGM with individually varying time scores. Two other kinds—piecewise and shape-factor—posed some complexities, with the latter requiring a reexpression of the model, and the former requiring a general framework and accompanying software tools to facilitate individual-specific transformations of time scores prior to model fitting. These software tools can be useful when data are balanced (Case I or II) as well. The ability to fit nonlinear LGMs with individually varying time scores can avert bias that could be incurred when such time scores are treated as balanced (Case I or II; Blozis & Cho, 2008; Mehta & West, 2000). Additionally, although LGMs already accommodate individual differences in growth factor scores, allowing individual specificity in fixed time scores is an additional way to tailor the model to the individual—a goal broadly advocated by many theories (e.g., person-oriented theory; see Sterba & Bauer, 2010a, 2010b).

Choosing Among Nonlinear LGMs With Individually Varying Time Scores

One distinction made previously among nonlinear LGMs is semiparametric (piecewise, shape) versus parametric (polynomial, SLC). All these LGMs are useful for explaining individual variation in level and shape using person-level predictors (e.g., gender). Also, parametric methods may better map onto theory about true data generating processes, and are useful for forecasting. For semiparametric methods the presence of model error stemming from the use of an approximation is transparent (Bauer, 2005; Fan & Li, 2004), and they may often do an equal or better job of flexibly describing a functional form. For instance, in Figure 4 the piecewise quadratic and shape-factor LGMs best correspond with the Loess observed trajectory in the empirical example. However, these models can be less useful for forecasting (see also Grimm et al., 2011). For instance, the shape-factor LGM is less applicable for forecasting *y* scores for new, hypothetical persons, unless the researcher is willing to specify both a time score and a window for the intended

predicted value.[13] This is not overly restrictive because the focus of balanced shape-factor model applications to date has mainly been on description and explanation rather than forecasting (e.g., Coffey et al., 2007; De Vos & Freese, 2011; Duncan, Duncan, & Hops, 1996; Ferrer et al., 2004; Gao, Raine, Dawson, Venables, & Mednick, 2007; Grimm, 2008; Haas, 2008; Hopwood & Zanarini, 2010; McArdle, 2005; Prinzie, Onghena, & Hellincks, 2006). Design considerations can also be more salient for semiparametric models. Shape-factor models are applicable to designs that define measurement windows within which time scores individually vary (see Table 2). Additionally, piecewise models might be more substantively interpretable when design-based or naturally occurring transitions (e.g., high school entry or treatment cessation; Flora, 2008) correspond with knots. In sum, when choosing among semiparametric and parametric nonlinear LGMs for individually varying time scores, researchers should take into consideration their own inferential goals and the nature of their design.

Choosing Between Nonlinear LGMs and MLMs With Individually Varying Time Scores

Researchers wishing to account for both individually varying time points and nonlinearity might have previously considered MLM their only option, given the limited guidance on combining the two within LGM. In light of this article, other reasons for deciding between LGM and MLM for Case III data can be considered. One remaining reason involves feasibility or practical challenges in representing nonlinear change. For instance, MLMs have not been specified to accommodate shape-factor growth, to our knowledge. As another example, our piecewise LGM for Case III data is arguably easier to specify than existing MLM implementations. For instance, in piecewise MLMs (e.g., Cudeck & Klebe, 2002; Fitzmaurice, Laird, & Ware, 2004, pp. 147–150; Raudenbush & Bryk, 2002, pp. 178–179; Snijders & Bosker, 1999, pp. 186–188), imparting piecewise structure has been part of the model specification itself (e.g., model includes a max() function per knot, with additional functions required for higher order terms). Whereas MLM specifications for selected example piecewise structures have been provided in several sources, specification of different or more complicated piecewise structures have been considered challenging in practice (Snijders & Berkhof, 2008, p. 163). In this article, knot point location and transformation of time scores are imparted in a precursor data management step that generates Λ_i elements, which is fully automated. Hence, a wide variety of piecewise structures can be accommodated with little effort.

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[13]Relately, it is useful to visualize that making additional observations of existing subjects necessitates adding rows to lambda (as in the balanced case shape-factor model also), corresponding with new measurement windows.

There are also general features of SEM versus MLM frameworks that bear on choosing between LGMs with individually varying time points and MLMs (e.g., MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997; Mehta & West, 2000). For instance, the SEM framework readily allows expanding the structural model so that growth factors predict distal outcomes. This framework also allows expanding the measurement model so that the repeated construct is a latent (error-free) variable with multiple indicators per occasion. Such expansions would be difficult or impossible with standard MLM software. On the other hand, MLM has historically more readily accommodated many levels of hierarchical nesting above (or below) the growth process (e.g., child within class within school; but see Asparouhov & Muthén, 2011).

Limitations and Extensions

Another approach for handling nonlinearity not addressed here involves transforming the outcomes and fitting alternative reparameterized LGMs (Choi, Harring, & Hancock, 2009). Nonlinear LGM modeling possibilities not discussed here could be explicated with individually varying time scores. It is also possible to combine nonlinear approaches, such as a two-piece LGM with a shape factor as the first piece and a known polynomial as the second piece.

Conclusions

Nonlinear patterns of change are theorized for many behavior, health, and learning outcomes. This article provided pedagogical explanations, new methodological developments, software tools, and examples to show how to accommodate truly individually varying time points with nonlinear LGMs. It is hoped that this article encourages more researchers to allow for individually varying time points in complex nonlinear LGMs.

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