Spatial-Temporal Nonlocal Homogenization Model for Transient Anti-plane Shear Wave Propagation in Periodic Viscoelastic Composites

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Abstract

This manuscript presents a spatial-temporal nonlocal homogenization model for transient anti-plane shear wave propagation in viscoelastic composites. The proposed model is formulated through asymptotic homogenization with higher order corrections incorporated to extend the applicability of the homogenization theory to shorter wavelength regime. A nonlocal homogenization model in the form of a fourth order PDE is consistently derived with all model parameters directly computed from the microstructure equilibrium. A reduced order model in the form of a second order PDE is then proposed for efficient transient wave propagation analysis. The reduced model retains the dispersive character of the original nonlocal model through the effective stiffness tensor. Transient shear wave propagation in two-dimensional domain with periodic elastic and viscoelastic microstructures is investigated and the proposed models were verified against direct numerical simulations. The spatial-temporal nonlocal homogenization model is shown to accurately capture shear wave dispersion in the first pass band and attenuation within the first stop band.

Keywords: High order homogenization; Wave dispersion; Band gap; Viscoelastic composites; Gradient elasticity

1 Introduction

Wave propagation in manufactured composite materials has received increasing research interest over the past decade due to the opportunities for achieving favorable dynamic properties

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within targeted frequency ranges (e.g., acoustic band gaps [41, 39]). Phononic crystals [61, 52] and acoustic metamaterials [38, 18] demonstrate significant potential in many novel engineering applications, such as cloaking [59, 9], acoustic diode [36, 35] and blast mitigation [47]. These materials exhibit unique wave propagation and attenuation patterns possible through the design of the microstructure and constituent material properties. The majority of the literature in this area so far focused on elastic materials as composite constituents. It has been recently recognized that employing viscoelastic materials could significantly expand the possibilities for these materials by leveraging the interactions between material damping and heterogeneity induced dispersion, such as shifting the stop band to lower frequencies and enhancing wave attenuation [42, 43, 33]. This manuscript presents a new homogenization model that accounts for wave dispersion due to both material heterogeneity and damping.

Direct numerical modeling of transient wave propagation in composites is computationally prohibitive due to the necessity to resolve the fine scale features of the microstructure. Generalized continuum theories, pioneered by Mindlin [46], Suhubi and Eringen [60] and others were developed to describe the macroscopic phenomena accounting for the microstructural effects without the need to resolve them. Gradient elasticity modeling, a special class of the generalized theories, introduces higher order gradients in the governing equation in addition to the terms that pertain to the classical continuum theory. The resulting equations of motion capture the dispersive and attenuative behavior induced by the material heterogeneity. The fourth order gradient elasticity model (with fourth order spatial, fourth order temporal and mixed spatial-temporal derivative terms) has been investigated by Askes et al. [5], Metrikine [44], Pichugin et al. [55] and others. Dontsov et al. [16] demonstrated the capability of this model in predicting the dispersion relation of one-dimensional elastic layered composites up to the second pass band, provided that the length-scale parameters associated with the higher order gradients are appropriately calibrated. However, the identification of these parameters in multidimensional problems is an outstanding issue [4] and this approach has so far been applied to elastic composites only.

Homogenization models based on Willis' theory [45, 64, 50, 58, 49] have also been proposed in the past decade. The nonlocal nature of wave propagation in composites is modeled by the coupled constitutive relation that connects the stress to velocity, and momentum to strain. This construction allows an accurate description of dispersion band structure for elastic composites. To the best of the authors' knowledge, transient wave propagation in composites has not been previously simulated with this type of homogenization models. Computational homogenization models [53, 57, 37], on the other hand, pose nested and coupled initial-boundary value problems at the scale of the material microstructure and the scale of the structure. The problems at two disparate scales are evaluated numerically in a coupled fashion. The energetic consistency between the problems at the two scales are achieved through the Hill-Mandel condition. While these models have been successful in capturing wave dispersion in small-scale transient simulations, the computational cost is high for multi-dimensional structure-scale simulations.

Asymptotic homogenization has been frequently employed to compute the effective properties of composite materials. In the context of dynamics, Boutin and Auriault [7] first investigated the role of higher order expansions in capturing wave dispersion. Fish et al. [20, 19] proposed a spatial nonlocal homogenization model that incorporates a fourth order spatial derivative term in addition to the terms that pertain to classical homogenization model. The spatial nonlocal model was shown to capture wave dispersion in the long-wavelength regime [3]. By deriving a homogenization model that employed a mixed spatial-temporal nonlocal term, Hui and Oskay [29] studied the dispersion and attenuation of transient waves in elastic composites. More recently, Wautier and Guzina [62] extended the spatial nonlocal homogenization model by incorporating the temporal and mixed spatial-temporal nonlocal terms using the argument of asymptotic equivalence [55]. With appropriately calibrated parameters associated with the nonlocal terms, this model was shown to capture the onset and size of the first stop band in the context of dispersion analysis.

While the approaches mentioned above account for wave dispersion due to material heterogeneity in elastic composites, the effects of material damping were typically not considered. In contrast, among many homogenization models developed to capture the wave attenuation due to material damping in viscoelastic composites [22, 65, 66], few investigated the dispersion due to material heterogeneity. Hui and Oskay [28, 30] investigated transient wave propagation in viscoelastic composites using the spatial nonlocal homogenization model which accurately captures wave dispersion induced by material heterogeneity in the long wavelength regime. Hu and Oskay [23] developed a spatial-temporal nonlocal homogenization model in one-dimension by incorporating higher order asymptotic expansions. It was shown that dispersion and attenuation in viscoelastic composite were well captured for shorter waves.

In this manuscript, we propose a spatial-temporal nonlocal homogenization model for transient anti-plane shear wave propagation in viscoelastic composites accounting for dispersion due to material heterogeneity and material damping. The proposed model is derived based on the asymptotic expansions of up to the eighth order in order to extend the applicability of asymptotic homogenization to shorter wavelength scenarios within the limit of separation of scales. Compared with the previous work by the authors in nonlocal homogenization modeling of transient wave propagation, the current manuscript provides the following novel contributions: (1) the current formulation consistently derives the temporal nonlocal term, which is demonstrated to be important in accurately capturing the initiation of the first stop band and beyond. The previous multi-dimensional models [28, 29, 30] did not include this effect; and (2) the current work generalizes the one-dimensional, semi-analytical homogenization approach presented in Ref. [23] to the two-dimensional anti-plane shear wave propagation case, which is evaluated using a numerical solution strategy. Noting that the spatial-temporal nonlocal equation for anti-plane shear wave propagation in elastic composites has been analyzed by Wautier and Guzina [62] with calibrated length-scale parameters, a novelty of the proposed model in this regard is that it provides a consistent recipe for computing all model parameters including the high order ones, and extends the approach to viscoelastic composites. In addition, we propose a reduced order model for transient wave propagation, which enables the characterization of wave dispersion and attenuation in the presence of structural effects, such as geometry and boundary conditions.

The remainder of this manuscript is organized as follows: Section 2 provides the problem description of anti-plane shear wave propagation in viscoelastic composites and the multiscale setting. Section 3 describes the asymptotic analysis and homogenization procedure. Section 4 derives the spatial-temporal nonlocal homogenization model and the reduced order model appropriate for transient simulations. Section 5 provides procedures to numerically implement the proposed model. Section 6 verifies the proposed model in two examples, i.e., elastic bimaterial layered and viscoelastic matrix-fiber composites. The conclusions and future research directions are presented in Section 7.

In what follows, scalars are denoted by italic Roman or Greek characters; vectors by boldface lowercase Roman characters; tensors by boldface italic Roman characters. Boldface uppercase Roman characters are reserved for matrices. Boldface Greek characters denote both vectors or tensors with their meaning explicitly specified when used. Indicial notation is used when necessary and Einstein summation convention applies to repeated indices.

2 Problem setting

Let $\Omega \in \mathbb{R}^2$ denote the domain of a body constructed by periodic unit cells composed of two or more constituents. Ω is described using the Cartesian coordinate, **x**. The momentum balance equation that governs wave propagation in this heterogeneous body is expressed as:

$$\boldsymbol{\nabla}_{x} \cdot \boldsymbol{\sigma}^{\zeta}(\mathbf{x}, t) = \rho^{\zeta}(\mathbf{x}) \ddot{\mathbf{u}}^{\zeta}(\mathbf{x}, t) \tag{1}$$

where, σ^{ζ} denotes the stress tensor; ρ^{ζ} the density; and \mathbf{u}^{ζ} the displacement vector. ∇_x . is the divergence operator and superimposed dot denotes derivative with respect to time. The superscript, ζ , indicates that the response fields oscillate spatially due to the microstructural heterogeneity.

Considering anti-plane shear deformation, the displacement field is expressed over the two-

dimensional problem domain, whereas the displacement vector is normal to the plane:

$$\mathbf{u}^{\zeta}(\mathbf{x},t) = u^{\zeta}(\mathbf{x},t)\hat{\mathbf{e}}_3 \tag{2}$$

where, $\mathbf{x} = \{x_1, x_2\}$ denotes the position vector of the material point and $\hat{\mathbf{e}}_j$ is the unit vector in the x_j direction. $\hat{\mathbf{e}}_3$ is taken to be the normal to the plane of the problem domain. Under the assumption of small deformation, the engineering shear strain is expressed in vector form after degeneration by contraction with $\hat{\mathbf{e}}_3$:

$$\boldsymbol{\gamma}^{\zeta}(\mathbf{x},t) = \left[\boldsymbol{\nabla}_{x}\mathbf{u}^{\zeta} + (\boldsymbol{\nabla}_{x}\mathbf{u}^{\zeta})^{T}\right] \cdot \hat{\mathbf{e}}_{3} = u_{,x_{j}}^{\zeta}(\mathbf{x},t)\hat{\mathbf{e}}_{j}$$
(3)

in which, the subscript dot denotes single contraction and subscript comma the spatial derivative. $\nabla_x = \partial(\cdot)/\partial x_j \hat{\mathbf{e}}_j$ is the gradient operator.

The constitutive behavior at a material point is taken to be linear viscoelastic, expressed using the hereditary integral:

$$\boldsymbol{\tau}^{\zeta}(\mathbf{x},t) = \int_0^t G^{\zeta}(\mathbf{x},t-t') \dot{\boldsymbol{\gamma}}^{\zeta}(\mathbf{x},t') dt'$$
(4)

where, $\tau^{\zeta}(\mathbf{x}, t)$ is the shear stress vector and $G^{\zeta}(\mathbf{x}, t')$ is the time-varying relaxation moduli. Elastic behavior is recovered by taking G^{ζ} as constant in time. The densities and relaxation modulus of all constituents forming the composite are assumed of the same order of magnitude.

Under the anti-plane shear wave propagation condition, the momentum balance equation (Eq. 1) reduces to:

$$\boldsymbol{\nabla}_{x} \cdot \boldsymbol{\tau}^{\zeta}(\mathbf{x}, t) = \rho^{\zeta}(\mathbf{x}) \ddot{u}^{\zeta}(\mathbf{x}, t) \tag{5}$$

The boundary conditions are:

$$u^{\zeta}(\mathbf{x},t) = \tilde{u}(\mathbf{x},t); \quad \mathbf{x} \in \Gamma^u \tag{6a}$$

$$\boldsymbol{\tau}^{\zeta}(\mathbf{x},t).\mathbf{n} = \tilde{t}(\mathbf{x},t); \quad \mathbf{x} \in \Gamma^t$$
(6b)

where, **n** denotes the outward unit normal vector along the traction boundaries; $\tilde{u}(\mathbf{x}, t)$ and $\tilde{t}(\mathbf{x}, t)$ are the displacement and traction data prescribed on Γ^u and Γ^t , respectively, and $\partial \Omega = \Gamma^u \cup \Gamma^t$; $\Gamma^u \cap \Gamma^t = \emptyset$. Homogeneous initial displacement and velocity conditions are assumed:

$$u^{\zeta}(\mathbf{x},0) = 0; \quad \mathbf{x} \in \Omega \tag{7a}$$

$$\dot{u}^{\zeta}(\mathbf{x},0) = 0; \quad \mathbf{x} \in \Omega \tag{7b}$$

Equations 4-7 define the initial-boundary value problem in the time domain. The convolutional form of the hereditary integral that describe the viscoelastic behavior allows a simpler formulation of the problem when posed in the Laplace domain [29, 23]. The momentum balance equation, Eq. 5, expressed in the Laplace domain is:

$$\boldsymbol{\nabla}_{x}.\boldsymbol{\tau}^{\zeta}(\mathbf{x},s) = \rho^{\zeta}(\mathbf{x})s^{2}u^{\zeta}(\mathbf{x},s) \tag{8}$$

where, $s = \sigma + iw$ is the Laplace variable, $s \in \mathbb{C}$. In the Laplace domain, the response variable u^{ζ} is complex valued and the viscoelastic constitutive relation is written in the proportional form:

$$\boldsymbol{\tau}^{\zeta}(\mathbf{x},s) = G^{\zeta}(\mathbf{x},s)\boldsymbol{\gamma}^{\zeta}(\mathbf{x},s)$$
(9)

where $G^{\zeta}(\mathbf{x}, s) = s \mathscr{L}(G^{\zeta}(\mathbf{x}, t))$ is the shear modulus function in the Laplace domain. The boundary conditions in the Laplace domain are written as:

$$u^{\zeta}(\mathbf{x},s) = \tilde{u}(\mathbf{x},s); \quad \mathbf{x} \in \Gamma^u$$
(10a)

$$\boldsymbol{\tau}^{\zeta}(\mathbf{x}, s) \cdot \mathbf{n} = \tilde{t}(\mathbf{x}, s); \quad \mathbf{x} \in \Gamma^t$$
(10b)

2.1 Multiscale setting

The domain of the canonical unit cell is denoted as $\Theta \in \mathbb{R}^2$ expressed using the Cartesian coordinate, \mathbf{y} , which is related to the macroscale coordinate by $\mathbf{y} = \mathbf{x}/\zeta$, where $0 < \zeta \ll 1$ is the small scaling parameter. In the context of wave propagation, the scaling parameter is defined as the ratio between the size of microstructure and the characteristic length of deformation wave (i.e., $\zeta = l/\lambda$, where λ is the characteristic deformation wavelength).

Consider an arbitrary response function, $f^{\zeta}(\mathbf{x}, s)$, which oscillates in space due to fluctuations induced by material heterogeneity. The response field is assumed to allow a two-scale separation in terms of macroscale and microscale coordinates:

$$f^{\zeta}(\mathbf{x},s) = f(\mathbf{x},\mathbf{y}(\mathbf{x}),s) \tag{11}$$

The shear modulus and material density are taken to depend on the microscale coordinate only, i.e., $G^{\zeta}(\mathbf{x}, s) = G(\mathbf{y}, s)$ and $\rho^{\zeta}(\mathbf{x}) = \rho(\mathbf{y})$. The spatial derivative of f^{ζ} is obtained by applying the chain rule:

$$f_{,\mathbf{x}}^{\zeta}(\mathbf{x},s) = f_{,\mathbf{x}}(\mathbf{x},\mathbf{y},s) + \frac{1}{\zeta}f_{,\mathbf{y}}(\mathbf{x},\mathbf{y},s)$$
(12)

where, subscript comma followed by \mathbf{x} and \mathbf{y} denote the spatial derivative with respect to the macroscale and microscale coordinates, respectively. All response fields are assumed to be locally periodic:

$$f(\mathbf{x}, \mathbf{y}, s) = f(\mathbf{x}, \mathbf{y} + \mathbf{N}\hat{\mathbf{l}}, s)$$
(13)

where, $\hat{\mathbf{l}} = [\hat{l}_1, \hat{l}_2]^T$ denotes the period of the microstructure in the microscale coordinates, i.e., $\hat{\mathbf{l}} = \mathbf{l}/\zeta$, \mathbf{l} is the period of the microstructure in the physical (macroscale) coordinates, and \mathbf{N} is a 2 × 2 diagonal matrix with integer components.

3 Two-scale asymptotic analysis

In this section, the asymptotic analysis procedure [7, 19, 29, 30], in which, asymptotic expansions of first few orders are employed to derive the momentum balance equations, is generalized for the development of momentum balance equations considering higher order expansions. Based on the two-scale setting described above, the displacement is approximated by the asymptotic expansion of up to the eighth order:

$$u^{\zeta}(\mathbf{x},s) \equiv u(\mathbf{x},\mathbf{y},s) = u^{(0)}(\mathbf{x},s) + \sum_{i=1}^{8} \zeta^{i} u^{(i)}(\mathbf{x},\mathbf{y},s) + O(\zeta^{9})$$
(14)

where, $u^{(0)}$ denotes the macroscopic displacement field and is dependent on the macroscale coordinate only [20]; and $u^{(i)}$ are high order displacement fields which depend on both macroscale and microscale coordinates. We note that $u^{(0)}$ depends only on **x** if the assumption of moderate material property contrast between the composite constituents is satisfied. For composites with highly contrasted constituents, $u^{(0)}$ depends on both micro- and macroscale coordinates [56, 6]. The shear strain field is obtained as:

$$\boldsymbol{\gamma}^{\zeta}(\mathbf{x},s) \equiv \boldsymbol{\gamma}(\mathbf{x},\mathbf{y},s) = \sum_{\alpha=0}^{7} \zeta^{\alpha} \boldsymbol{\gamma}^{(\alpha)}(\mathbf{x},\mathbf{y},s)$$
(15)

where,

$$\boldsymbol{\gamma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, s) = \left(u_{,x_j}^{(\alpha)} + u_{,y_j}^{(\alpha+1)}\right)\hat{\mathbf{e}}_j \tag{16}$$

 $(\cdot)_{,x_j} = \partial(\cdot)/\partial x_j$ and $(\cdot)_{,y_j} = \partial(\cdot)/\partial y_j$.

Employing Eq. 16 along with the constitutive relation (Eq. 9), the shear stress field at order $O(\zeta^{\alpha})$ is obtained as:

$$\boldsymbol{\tau}^{(\alpha)}(\mathbf{x}, \mathbf{y}, s) = G(\mathbf{y}, s)\boldsymbol{\gamma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, s)$$
(17)

Substituting Eqs. 9, 14-17 into the momentum balance equation (Eq. 8), and collecting terms with equal orders yield the balance equations at each order of ζ :

$$O(\zeta^{-1}): \quad \nabla_{y} \cdot \boldsymbol{\tau}^{(0)}(\mathbf{x}, \mathbf{y}, s) = 0 \tag{18a}$$

$$O(\zeta^{\alpha}): \quad \boldsymbol{\nabla}_{x}.\boldsymbol{\tau}^{(\alpha)}(\mathbf{x},\mathbf{y},s) + \boldsymbol{\nabla}_{y}.\boldsymbol{\tau}^{(\alpha+1)}(\mathbf{x},\mathbf{y},s) = \rho(\mathbf{y})s^{2}u^{(\alpha)}(\mathbf{x},\mathbf{y},s)$$
(18b)

We start by additively decomposing the displacement field at each order of the asymptotic expansion. Following the procedure proposed in [7], the displacement field at each order is expressed in terms of a macroscopically constant displacement field and series of locally varying fields which have zero average over the unit cell:

$$u^{(i)}(\mathbf{x}, \mathbf{y}, s) = U^{(i)}(\mathbf{x}, s) + \sum_{k=0}^{i-1} \tilde{U}^{(i,k)}(\mathbf{x}, \mathbf{y}, s)$$
(19)

where, $\tilde{U}^{(i,k)}(\mathbf{x}, \mathbf{y}, s)$ is the k^{th} locally varying field of $u^{(i)}(\mathbf{x}, \mathbf{y}, s)$ and it is associated with the successive gradients of macroscopic strain of an inferior order by a locally periodic influence function that is defined over the unit cell:

$$\tilde{U}^{(i,k)}(\mathbf{x},\mathbf{y},s) = \boldsymbol{H}^{(k+1)}(\mathbf{y},s)(.\boldsymbol{\nabla}_x)^{k+1}U^{(i-k-1)}(\mathbf{x},s)$$
(20)

where, $\boldsymbol{H}^{(k+1)}$ is the microstructural influence function at order (k+1), and $(\boldsymbol{\nabla}_x)^{k+1}$ is the $(k+1)^{th}$ gradient with respect to the macroscale coordinate, \mathbf{x} , with k+1 contractions to the microstructural influence function (e.g., $\tilde{U}^{(3,2)} = \boldsymbol{H}^{(3)}(\boldsymbol{\nabla}_x)^3 U^{(0)}$ is written as $H^{(3)}_{ijk} U^{(0)}_{,x_{ijk}}$ in indicial notation). The microstructural influence function, $\boldsymbol{H}^{(k+1)}$, is an order (k+1) tensor and has (k+2) independent components.

Remark 1. For a k^{th} order tensor \boldsymbol{H} contracting k times with $\nabla^k U$, only the symmetric part of \boldsymbol{H} affects the result of contraction, i.e., $\boldsymbol{H}(.\nabla)^k U = \operatorname{Sym}(\boldsymbol{H})(.\nabla)^k U$.

This can be shown by defining the symmetric part of H as [11]:

$$\operatorname{Sym}(\boldsymbol{H}) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}} \sigma(\boldsymbol{H})$$
(21)

for all permutations $\sigma \in \mathfrak{S}$, and \mathfrak{S} denotes the symmetric group of permutations on $\{1, ..., k\}$. The linear operator σ is defined as: $\sigma(\boldsymbol{v}_{i_1} \otimes \cdots \otimes \boldsymbol{v}_{i_k}) := \boldsymbol{v}_{i_{\sigma(1)}} \otimes \cdots \otimes \boldsymbol{v}_{i_{\sigma(k)}}$ for $\boldsymbol{v}_{i_1} \otimes \cdots \otimes \boldsymbol{v}_{i_k} \in \boldsymbol{H}^k(\mathbb{C}^2)$, where $[i_1, \cdots, i_k] \in \{1, 2\}$. $\boldsymbol{H}^k(\mathbb{C}^2)$ is the set of all k^{th} order tensors. Because the sequence of differentiation is interchangable, $\boldsymbol{\nabla}^k U$ is invariant under the permutation operation. Since the symmetric part of a scalar is the scalar itself, we have:

$$\boldsymbol{H}(.\boldsymbol{\nabla})^{k}U = \operatorname{Sym}\left(\boldsymbol{H}(.\boldsymbol{\nabla})^{k}U\right) = \operatorname{Sym}\left(\boldsymbol{H}\right)(.\boldsymbol{\nabla})^{k}U$$
(22)

Substitution of Eqs. 16, 19 and 20 into 17 leads to the expression for the stress field at $O(\zeta^{\alpha})$:

$$\boldsymbol{\tau}^{(\alpha)}(\mathbf{x}, \mathbf{y}, s) = \sum_{k=0}^{\alpha} \boldsymbol{G}^{(k)}(\mathbf{y}, s) . (.\boldsymbol{\nabla}_x)^k \boldsymbol{\nabla}_x U^{(\alpha-k)}(\mathbf{x}, s)$$
(23)

in which, $G^{(k)}(\mathbf{y}, s)$ is an order (k+2) tensor and written as:

$$\boldsymbol{G}^{(k)}(\mathbf{y},s) = \boldsymbol{G}(\mathbf{y},s) \left[\boldsymbol{H}^{(k)}(\mathbf{y},s) \otimes \boldsymbol{I} + \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{H}^{(k+1)}(\mathbf{y},s) \right]$$
(24)

where, \otimes and I are the dyadic product and second order identity tensor, respectively, and $H^{(0)} = 1$. Substituting Eqs. 23 and 24 into Eq. 18a, we obtain the equilibrium equation for $H^{(1)}(\mathbf{y}, s)$:

$$\boldsymbol{\nabla}_{y} \cdot \left\{ G(\mathbf{y}, s) \left[\boldsymbol{I} + \boldsymbol{\nabla}_{y} \boldsymbol{H}^{(1)}(\mathbf{y}, s) \right] \right\} = 0$$
(25)

which is defined over the unit cell domain. $\mathbf{H}^{(1)}(\mathbf{y}, s)$ is unique when the local periodicity condition is applied and average is set to vanish, i.e., $\langle \mathbf{H}^{(1)}(\mathbf{y}, s) \rangle = \mathbf{0}$, where the averaging operator $\langle \cdot \rangle$ is defined as:

$$\langle \cdot \rangle = \frac{1}{|\Theta|} \int_{\Theta} (\cdot) \, d\mathbf{y} \tag{26}$$

where $|\Theta|$ is the area of the unit cell. The O(1) macroscale balance equation is obtained by applying the averaging operator to Eq. 18b (with $\alpha = 0$) and considering Eqs. 19 and 23:

$$\bar{\boldsymbol{G}}^{(0)}..\boldsymbol{\nabla}_{x}^{2}U^{(0)}(\mathbf{x},s) = \rho_{0}s^{2}U^{(0)}(\mathbf{x},s)$$
(27)

where, $\bar{\mathbf{G}}^{(0)}(s) = \left\langle \mathbf{G}^{(0)}(\mathbf{y}, s) \right\rangle$ is the O(1) homogenized shear moduli, and $\rho_0 = \left\langle \rho(\mathbf{y}) \right\rangle$ is the homogenized density. Overbar denotes the homogenized value throughout this manuscript.

The procedure to evaluate influence functions and derive momentum balance equations presented above can be generalized for higher orders, i.e., $\alpha \geq 0$. At order $O(\zeta^{\alpha})$, the equilibrium equation for the influence function, $\mathbf{H}^{(\alpha+2)}$, is derived by substituting Eqs. 19, 20, 23, 24 and lower order equilibrium equations for the influence functions (e.g., Eq. 25) and macroscale balance equations up to order $O(\zeta^{\alpha})$ (e.g., Eq. 27) into Eq. 18b. The macroscale balance equation at order $O(\zeta^{\alpha+1})$ is then obtained by substituting Eqs. 19, 20, 23, 24 and lower order equilibrium equations for influence functions and macroscale balance equations into Eq. 18b at $O(\zeta^{\alpha+1})$ and applying the averaging operator. The equilibrium equation for the influence function, $\mathbf{H}^{(\alpha+2)}$, is derived as:

$$\boldsymbol{G}^{(\alpha)}(\mathbf{y},s) + \boldsymbol{\nabla}_{y}.\boldsymbol{G}^{(\alpha+1)}(\mathbf{y},s) = \theta(\mathbf{y})\sum_{j=0}^{\alpha} \boldsymbol{H}^{(j)}(\mathbf{y},s) \otimes \boldsymbol{D}^{(\alpha-j)}(s)$$
(28)

where, $D^{(\alpha-j)}(s)$ is the $O(\zeta^{\alpha-j})$ homogenized shear moduli expressed as:

$$\boldsymbol{D}^{(\alpha-j)}(s) = \bar{\boldsymbol{G}}^{(\alpha-j)}(s) - \sum_{m=1}^{\alpha-j} \overline{\theta(\mathbf{y})} \boldsymbol{H}^{(m)}(\mathbf{y},s) \otimes \boldsymbol{D}^{(\alpha-j-m)}(s)$$
(29)

for $\alpha - j \ge 1$ and $D^{(0)}(s) = \overline{G}^{(0)}(s)$. The macroscale balance equation at $O(\zeta^{\alpha})$ is obtained as:

$$\sum_{n=0}^{\alpha} D^{(n)}(.\nabla)_x^{n+2} U^{(\alpha-n)}(\mathbf{x},s) = \rho_0 s^2 U^{(\alpha)}(\mathbf{x},s)$$
(30)

The boundary value problems to evaluate the influence functions and compute the homogenized shear moduli are summarized in Box 1.

Given: Viscoelastic shear modulus, $G(\mathbf{y}, s)$, and density, $\rho(\mathbf{y})$.

Find: Microscale influence functions, $\boldsymbol{H}^{(n+1)}(\mathbf{y}, s)$, and

the homogenized shear moduli, $D^{(n)}(s)$, for n = 0, 1, ..., 6, such that:

• Equilibrium:

$$n = 0, \quad \nabla_{y}.\boldsymbol{G}^{(0)}(\mathbf{y}, s) = 0$$

$$n = 1, ..., 6, \quad \nabla_{y}.\boldsymbol{G}^{(n)}(\mathbf{y}, s) = \theta(\mathbf{y}) \sum_{j=0}^{n-1} \boldsymbol{H}^{(j)}(\mathbf{y}, s) \otimes \boldsymbol{D}^{(n-1-j)}(s) - \boldsymbol{G}^{(n-1)}(\mathbf{y}, s)$$

• Periodic boundary condition:

 $\langle \alpha \rangle$

$$\boldsymbol{H}^{(n+1)}(\mathbf{y},s) = \boldsymbol{H}^{(n+1)}(\mathbf{y}+\hat{\mathbf{l}},s), \quad \mathbf{y} \in \partial \Theta$$

• Normalization condition:

$$\left\langle \boldsymbol{H}^{(n+1)}(\mathbf{y},s) \right\rangle = 0$$

• The homogenized shear moduli:

$$\begin{aligned} \boldsymbol{D}^{(0)}(s) &= \bar{\boldsymbol{G}}^{(0)}(s) \\ \boldsymbol{D}^{(n)}(s) &= \bar{\boldsymbol{G}}^{(n)}(s) - \sum_{m=1}^{n} \overline{\boldsymbol{\theta}(\mathbf{y})} \boldsymbol{H}^{(m)}(\mathbf{y},s) \otimes \boldsymbol{D}^{(n-m)}(s) \end{aligned}$$

Box 1: Boundary value problems for the evaluation of the influence functions and computation of the homogenized shear moduli.

4 Spatial-temporal nonlocal homogenization model

4.1 Higher order gradient formulation

In this section, we propose a novel spatial-temporal nonlocal homogenization model for antiplane shear wave propagation in viscoelastic composites. The nonlocal model is derived based on the macroscale momentum balance equations and all the model parameters are computed from the homogenized shear moduli and density which are determined through the asymptotic analysis. The key novelty of the proposed model is the introduction of the temporal nonlocal term that is derived from high order macroscale momentum balance equations.

Remark 2. For macroscopically orthotropic composites, the homogenized shear moduli at odd and even orders have the following characteristics: (1) $D^{(i)} = 0$, *i* is odd; (2) The components of $D^{(0)}$, $D^{(2)}$, $D^{(4)}$, $D^{(6)}$ with odd number of repeated indices are zero (e.g., $D_{12}^{(0)} = D_{1112}^{(2)} = D_{11112}^{(4)} = D_{111112}^{(6)} = 0$).

This can be shown by considering the two planes of material symmetry that have unit normals, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. The corresponding transformation matrix for reflection with respect to the two planes are:

$$\bar{\mathbf{R}} = \begin{bmatrix} -1 & 0\\ 0 & 1 \end{bmatrix}, \quad \bar{\bar{\mathbf{R}}} = \begin{bmatrix} 1 & 0\\ 0 & -1 \end{bmatrix}$$

The orthotropy condition requires the shear moduli is invariant under reflection [34], i.e., $D^{(i)} = \bar{\mathbf{R}}^T D^{(i)} \bar{\mathbf{R}} = \bar{\mathbf{R}}^T D^{(i)} \bar{\mathbf{R}}$, which implies that the components of the moduli tensor with odd number of repeated index have to be 0. For example, $D_{111}^{(1)} = \bar{R}_{11} \bar{R}_{11} \bar{R}_{11} D_{111}^{(1)}$ and $D_{1112}^{(2)} = \bar{R}_{11} \bar{R}_{11} \bar{R}_{12} D_{1112}^{(1)}$, indicating $D_{111}^{(1)} = 0$ and $D_{1112}^{(2)} = 0$, respectively.

Considering that the macroscopic behavior of the composite is orthotropic and using Remark 2, the macroscale momentum balance equations are written in indicial notation as:

$$O(1): \quad \rho_0 s^2 U^{(0)}(\mathbf{x}, s) - D^{(0)}_{kl}(s) U^{(0)}_{,kl} = 0 \tag{31a}$$

$$O(\zeta^{1}): \quad \rho_{0}s^{2}U^{(1)}(\mathbf{x},s) - D^{(0)}_{kl}(s)U^{(1)}_{,kl} = 0$$
(31b)

$$O(\zeta^2): \quad \rho_0 s^2 U^{(2)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(2)} = D_{klmn}^{(2)}(s) U_{,klmn}^{(0)}$$
(31c)

$$O(\zeta^3): \quad \rho_0 s^2 U^{(3)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(3)} = D_{klmn}^{(2)}(s) U_{,klmn}^{(1)}$$
(31d)

$$O(\zeta^4): \quad \rho_0 s^2 U^{(4)}(\mathbf{x}, s) - D^{(0)}_{kl}(s) U^{(4)}_{,kl} = D^{(2)}_{klmn}(s) U^{(2)}_{,klmn} + D^{(4)}_{klmnpq}(s) U^{(0)}_{,klmnpq}$$
(31e)

$$O(\zeta^{5}): \quad \rho_{0}s^{2}U^{(5)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(5)} = D_{klmn}^{(2)}(s)U_{,klmn}^{(3)} + D_{klmnpq}^{(4)}(s)U_{,klmnpq}^{(1)}$$
(31f)

$$O(\zeta^{6}): \quad \rho_{0}s^{2}U^{(6)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(6)} = D_{klmn}^{(2)}(s)U_{,klmn}^{(4)} + D_{klmnpq}^{(4)}(s)U_{,klmnpq}^{(2)} + D_{klmnpars}^{(6)}(s)U_{,klmnpars}^{(0)}$$
(31g)

$$O(\zeta^{7}): \quad \rho_{0}s^{2}U^{(7)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(7)} = D_{klmn}^{(2)}(s)U_{,klmn}^{(5)} + D_{klmnpq}^{(4)}(s)U_{,klmnpq}^{(3)} + D_{klmnpqrs}^{(6)}(s)U_{,klmnpqrs}^{(1)}$$
(31h)

where, $D^{(0)}$, $D^{(2)}$, $D^{(4)}$, $D^{(6)}$ have 2, 8, 32, 128 non-zero components, respectively. Moreover, only the symmetric parts of these tensors affect the solution, as a consequence of Remark 1. The

number of non-zero independent components in the symmetric part, $\operatorname{Sym}(\boldsymbol{D}^{(0)})$, $\operatorname{Sym}(\boldsymbol{D}^{(2)})$, $\operatorname{Sym}(\boldsymbol{D}^{(4)})$, $\operatorname{Sym}(\boldsymbol{D}^{(6)})$ are, 2, 3, 4, 5, respectively. $\operatorname{Sym}(\boldsymbol{D}^{(0)}) = \boldsymbol{D}^{(0)}$.

Equations 31a-h are combined into a single homogenized momentum balance equation by averaging the displacement field (Eq. 14) at each scale:

$$U(\mathbf{x}, s) = \sum_{i=0}^{7} \zeta^{i} U^{(i)}(\mathbf{x}, s) + O(\zeta^{8})$$
(32)

where, $U(\mathbf{x}, s)$ denotes the homogenized displacement field of $O(\zeta^8)$ accuracy. Combining the two lowest order balance equations (Eq. 31a and Eq. 31b) leads to the classical local homogenization model which is valid in the absence of wave dispersion (i.e., when the wavelength is much larger than the microstructure). The spatial nonlocal homogenization model previously studied in Refs. [20, 3, 28] incorporates additional two equations (Eq. 31c and Eq. 31d). It predicts wave dispersion in the long-wavelength regime in the first pass band. The prediction error increases as the wavelength becomes shorter, therefore, the initiation of the stop band and beyond is not well captured. Following the line of increasing asymptotic accuracy by incorporating higher order momentum balance equations, Eqs. 31a-h are combined to construct the proposed spatial-temporal nonlocal homogenization model. The direct weighted summation of Eqs. 31a-h results in a spatial nonlocal homogenization model with higher order spatial gradient terms $(D^{(4)}(.\nabla_x)^6 U$ and $D^{(6)}(.\nabla_x)^8 U)$ in addition to the fourth order spatial gradient term $(D^{(2)}(.\nabla_x)^4 U)$. While it is interesting to probe the role of $D^{(4)}(.\nabla_x)^6 U$ and $D^{(6)}(\nabla_x)^8 U$ terms in capturing wave dispersion, we limit our scope to the fourth order gradient nonlocal equation based on the observation that it captures wave dispersion and attenuation beyond the long-wavelength regime [23]. In what follows, we transform the higher order spatial gradient terms in Eqs. 31a-h to fourth order gradient terms (i.e., spatial nonlocal, temporal nonlocal, and mixed spatial-temporal nonlocal). It is observed that Eqs. 31a, c, e, g and Eqs. 31b, d, f, h, have identical equation form and coefficients. Therefore, all derivations on the former equations directly applies to the latter ones. Only the derivations on the even orders (i.e., Eqs. 31a, c, e, g) are presented in the remainder of this section.

Considering Remark 1 and without loss of generality, Eq. 31g is expressed as:

$$\rho_0 s^2 U^{(6)}(\mathbf{x}, s) - D^{(0)}_{kl}(s) U^{(6)}_{,kl} = D^{(2)}_{(klmn)}(s) U^{(4)}_{,klmn} + D^{(4)}_{(klmnpq)}(s) U^{(2)}_{,klmnpq} + \nu(s) D^{(6)}_{(klmnpqrs)}(s) U^{(0)}_{,klmnpqrs} + (1 - \nu(s)) D^{(6)}_{(klmnpqrs)}(s) U^{(0)}_{,klmnpqrs}$$
(33)

where, $\nu(s)$ is a scalar parameter. Sym $(\mathbf{D}^{(6)})$ in the third term on the right side of Eq. 33 is approximated as:

$$D_{(klmnpqrs)}^{(6)}(s) \approx A^{(1)}(s) \text{Sym} \left(D_{kl}^{(0)}(s) D_{(mnpqrs)}^{(4)}(s) \right)$$
(34)

where, $A^{(1)}(s)$ is a complex valued scalar that minimizes the discrepancy between the non-zero independent components of $\operatorname{Sym}(\mathbf{D}^{(6)})$ and $\operatorname{Sym}(\mathbf{D}^{(0)} \otimes \operatorname{Sym}(\mathbf{D}^{(4)}))$ in the Euclidean norm, and it is computed by the Moore-Penrose pseudo-inverse. Substituting Eq. 34 and Eq. 31e into the third term on the right side of Eq. 33 and considering Remark 1:

$$\nu(s)D_{(klmnpqrs)}^{(6)}(s)U_{,klmnpqrs}^{(0)} \approx \nu(s)A^{(1)}(s)\operatorname{Sym}\left(D_{kl}^{(0)}(s)D_{(mnpqrs)}^{(4)}(s)\right)U_{,klmnpqrs}^{(0)} = \\ \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{(mnpqrs)}^{(4)}(s)U_{,klmnpqrs}^{(0)} = \\ \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)\left[\rho_0s^2U^{(4)}(\mathbf{x},s) - D_{mn}^{(0)}(s)U_{,mn}^{(4)} - D_{(mnpq)}^{(2)}(s)U_{,mnpq}^{(2)}\right]_{,kl}$$
(35)

Substituting Eq. 35 for the the third term on the right side of Eq. 33:

$$\rho_{0}s^{2}U^{(6)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(6)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn}^{(4)} + \nu(s)A^{(1)}(s)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl}^{(4)} + \left(D_{(klmnpq)}^{(4)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{(mnpq)}^{(2)}(s)\right)U_{,klmnpq}^{(2)} + (36)\left(1 - \nu(s)\right)D_{(klmnpqrs)}^{(6)}(s)U_{,klmnpqrs}^{(0)}$$

Similar to Eq. 34, the non-zero independent components of $\text{Sym}(D^{(4)})$ is approximated as:

$$D_{(klmnpq)}^{(4)}(s) \approx A^{(2)}(s) \text{Sym}\left(D_{(klmn)}^{(2)}(s) D_{pq}^{(0)}(s)\right)$$
(37)

where $A^{(2)}(s)$ is computed by the Moore-Penrose pseudo-inverse. Substituting Eq. 37 into Eq. 36, considering Remark 1 and substituting Eq. 31c into the resulting equation for $D_{pq}^{(0)}(s)U_{,pq}^{(2)}$, we obtain:

$$\rho_{0}s^{2}U^{(6)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(6)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn}^{(4)} + \nu(s)A^{(1)}(s)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl}^{(4)} + \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)D_{(klmn)}^{(2)}\rho_{0}s^{2}U_{,klmn}^{(2)} + \left(1 - \nu(s)\right)D_{(klmnpqrs)}^{(6)}(s) - \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)D_{(klmn)}^{(2)}D_{(pqrs)}^{(2)}\right]U_{,klmnpqrs}^{(0)}$$
(38)

By substituting Eq. 31e into Eq. 38 and considering Remark 1 and Eq. 31a, we arrive at the macroscale momentum balance equation at $O(\zeta^6)$:

$$\rho_{0}s^{2}U^{(6)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(6)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn}^{(4)} + \left(2\nu(s)A^{(1)}(s) - A^{(2)}(s)\right)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl}^{(4)} + \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)\rho_{0}^{2}s^{4}U^{(4)} + E_{(klmnpqrs)}(\nu(s),s)U_{,klmnpqrs}^{(0)}$$
(39)

where the last term is denoted as the error term and E is expressed as:

$$E_{(klmnpqrs)}(\nu(s),s) = \left[(1-\nu(s))D_{(klmnpqrs)}^{(6)}(s) - \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right) \operatorname{Sym}\left(D_{(klmn)}^{(2)}(s)D_{(pqrs)}^{(2)}(s) + D_{(klmnpq)}^{(4)}(s)D_{rs}^{(0)}(s)\right) \right]$$
(40)

The momentum balance equations at $O(\zeta^4)$ and $O(\zeta^2)$ are obtained similarly and detailed derivations are provided in Appendix A. The momentum balance equations at $O(\zeta^7)$, $O(\zeta^5)$, $O(\zeta^3)$ are derived by the same procedure as $O(\zeta^6)$, $O(\zeta^4)$, $O(\zeta^2)$, respectively. The resulting macroscale momentum balance equation at order $O(\zeta^k)$ are summarized as follows:

$$\begin{aligned} \rho_{0}s^{2}U^{(k)}(\mathbf{x},s) &- D_{kl}^{(0)}(s)U_{,kl}^{(k)} = 0; \quad k = 0,1 \end{aligned} \tag{41a} \\ \rho_{0}s^{2}U^{(k)}(\mathbf{x},s) &- D_{kl}^{(0)}(s)U_{,kl}^{(k)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn}^{(k-2)} \\ &+ \left(2\nu(s)A^{(1)}(s) - A^{(2)}(s)\right)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl}^{(k-2)} + \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)\rho_{0}^{2}s^{4}U^{(k-2)}; \\ k = 2, 3, 4, 5 \end{aligned} \tag{41b} \\ \rho_{0}s^{2}U^{(k)}(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl}^{(k)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn}^{(k-2)} \\ &+ \left(2\nu(s)A^{(1)}(s) - A^{(2)}(s)\right)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl}^{(k-2)} + \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)\rho_{0}^{2}s^{4}U^{(k-2)} \\ &+ E_{(klmnpqrs)}(\nu(s),s)U_{,klmnpqrs}^{(k-6)}; \\ k = 6, 7 \end{aligned} \tag{41c}$$

Considering the definition for the homogenized displacement field (Eq. 32) and the summation of momentum balance equation at order $O(\zeta^k)$ multiplied by ζ^k , result in the homogenized momentum balance equation:

$$\rho_{0}s^{2}U(\mathbf{x},s) - D_{kl}^{(0)}(s)U_{,kl} = \left(D_{(klmn)}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)U_{,klmn} + \left(2\nu(s)A^{(1)}(s) - A^{(2)}(s)\right)\rho_{0}s^{2}D_{kl}^{(0)}(s)U_{,kl} + \left(A^{(2)}(s) - \nu(s)A^{(1)}(s)\right)\rho_{0}^{2}s^{4}U + \left(A^{2}C_{(klmnpqrs)}^{(0)}(\nu(s),s)U_{,klmnpqrs}^{(0)} + \zeta^{7}E_{(klmnpqrs)}(\nu(s),s)U_{,klmnpqrs}^{(1)} + O(\zeta^{8})\right)$$

$$(42)$$

Equation 42 represents a one-parameter family of nonlocal homogenization models as a function of $\nu(s)$. For arbitrarily chosen $\nu(s)$, these models achieve $O(\zeta^6)$ asymptotic accuracy. The detailed procedure for choosing $\nu(s)$ and obtaining a unique nonlocal model is provided in the next section.

We note that Eq. 42 is formally similar to the gradient elasticity models [44, 5, 55, 16] and the homogenization model [62], where spatial nonlocal, temporal nonlocal and mixed spatial-temporal nonlocal terms are present, and the length-scale parameters associated with these terms are calibrated. The proposed model is unique in that all the model parameters are

consistently derived from the homogenization process and it applies to viscoelastic composites.

4.2 Identification of $\nu(s)$

It is observed in Eq. 42 that the asymptotic error term depends on $\nu(s)$. We therefore seek to set $\nu(s)$ such that the error term is minimized. This requires knowledge of $U_{,klmnpqrs}^{(0)}$ which is not available a-priori. As an alternative, we pursue $\nu(s)$ such that all the independent components of the coefficient tensor of the error term (i.e., \mathbf{E}) are minimized. To be explicit, they are written in vector form: $\mathbf{e} = [E_{(1\times 8)} \ E_{(1\times 6^2\times 2)} \ E_{(1\times 4^2\times 4)} \ E_{(1\times 2^2\times 6)} \ E_{(2\times 8)}]^T$, where the subscript $(m_{\times n})$ denotes index m repeated n times. In addition, the dynamic stability of the general fourth order governing equation imposes constraints to the model parameters associated with the nonlocal terms. This poses a constrained minimization problem for the identification of $\nu(s)$.

Neglecting the $O(\zeta^6)$ and higher order error terms, Eq. 42 is rewritten as:

$$\rho_0 s^2 U(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl} = \alpha_{(klmn)}^{(1)}(s) U_{,klmn} + \rho_0 s^2 \alpha_{kl}^{(2)}(s) U_{,kl} + \rho_0^2 s^4 \alpha^{(3)}(s) U \tag{43}$$

where,

$$\alpha_{(klmn)}^{(1)}(s) = \operatorname{Sym}\left(D_{klmn}^{(2)}(s) - \nu(s)A^{(1)}(s)D_{kl}^{(0)}(s)D_{mn}^{(0)}(s)\right)$$
(44a)

$$\alpha_{kl}^{(2)}(s) = \left(2\nu(s)A^{(1)}(s) - A^{(2)}(s)\right)D_{kl}^{(0)}(s)$$
(44b)

$$\alpha^{(3)}(s) = A^{(2)}(s) - \nu(s)A^{(1)}(s) \tag{44c}$$

For elastic composites, $\mathbf{D}^{(0)}$, $\mathbf{\alpha}^{(1)}$, $\mathbf{\alpha}^{(2)}$ and $\mathbf{\alpha}^{(3)}$ are real-valued constant parameters. For viscoelastic composites, they are complex-valued and functions of the Laplace variable. Substituting a harmonic wave solution, $U = U_0(s)e^{i(k_1x_1+k_2x_2)}$, into Eq. 43, the resulting equation is expressed in polar coordinate system for the wave vector, $k_1 = k(s, \theta) \cos \theta$ and $k_2 = k(s, \theta) \sin \theta$. In this study, the imaginary and real parts of the wave vector are assumed to be co-linear, thus θ is taken as real-valued. The co-linearity assumption has been previously employed in the dispersion analysis of viscoelastic materials [10, 2]. The dispersion relation in terms of the Laplace variable is obtained as:

$$A(s,\theta)k^4 + B(s,\theta)k^2 + C(s,\theta) = 0$$
(45)

where,

$$A(s,\theta) = \alpha_{(1111)}^{(1)}(s)\cos^4\theta + \alpha_{(2222)}^{(1)}(s)\sin^4\theta + 6\alpha_{(1122)}^{(1)}(s)\cos^2\theta\sin^2\theta$$
(46a)

$$B(s,\theta) = -\left(\alpha_{11}^{(2)}(s)\rho_0 s^2 + D_{11}^{(0)}(s)\right)\cos^2\theta - \left(\alpha_{22}^{(2)}(s)\rho_0 s^2 + D_{22}^{(0)}(s)\right)\sin^2\theta \tag{46b}$$

$$C(s,\theta) = \alpha^{(3)}(s)\rho_0^2 s^4 - \rho_0 s^2 \tag{46c}$$

The solution of Eq. 45, $k(s, \theta)$, relates the wavenumber, the Laplace variable and the direction of wave vector as:

$$\left(k^{\{1,2\}}(s,\theta)\right)^2 = \frac{-B(s,\theta) - \sqrt{B^2(s,\theta) - 4A(s,\theta)C(s,\theta)}}{2A(s,\theta)}$$
(47a)

$$\left(k^{\{3,4\}}(s,\theta)\right)^2 = \frac{-B(s,\theta) + \sqrt{B^2(s,\theta) - 4A(s,\theta)C(s,\theta)}}{2A(s,\theta)}$$
(47b)

For elastic composites, the dynamic stability of Eq. 43 in terms of the constant model parameters is analyzed by the limit analysis in the frequency domain [44, 55]. Replacing s in Eq. 45 with $i\omega$, the dispersion relation in terms of frequency is obtained. Consider the limiting scenarios: 1) quasi-static (i.e., $\omega \to 0$); and 2) infinitely long wave (i.e., $k \to 0$), which respectively leads to the solutions:

$$k^{\{1,2\}} = 0, \quad k^{\{3,4\}} = \pm \sqrt{\frac{\left(D_{11}^{(0)}\cos^2\theta + D_{22}^{(0)}\sin^2\theta\right)}{A(\theta)}}$$
(48a)

$$\omega^{\{1,2\}} = 0, \quad \omega^{\{3,4\}} = \pm \sqrt{-\frac{1}{\alpha^{(3)}\rho_0}}$$
(48b)

In view of the harmonic wave solution in the frequency domain, $U = U_0 e^{i(k_1 x_1 + k_2 x_2 - \omega t)}$, the stability of wave propagation in the quasi-static limit requires that k is real; and infinitely long wave limit requires that ω is real. These are achieved by the following constraints:

$$A(\theta) > 0, \quad \alpha^{(3)} < 0 \tag{49}$$

In order to generalize these ideas to viscoelastic composites, the following constraints on the model parameters are proposed:

$$\operatorname{Re}\left[A(s,\theta)\right] > 0\tag{50a}$$

$$\operatorname{Re}\left[\alpha^{(3)}(s)\right] < 0, \quad \operatorname{Im}\left[\alpha^{(3)}(s)\right] = 0 \tag{50b}$$

In the quasi-static limit, viscoelastic phases within the composites behave as elastic ones, the parameters are independent of the Laplace variable s. Equation 50a therefore recovers the constraint for elastic composites. In the infinitely long wave limit, Eq. 50b is derived by analyzing the response of governing equation (Eq. 43) under an impulse load and examining the location of poles of the transfer function, as a common approach to analyze the system stability in the Laplace domain [21]. The detailed derivation is provided in Appendix B.

Employing the stability constraints introduced above and using the Euclidean norm of **e** as the objective function, the identification problem for $\nu(s)$ is stated in Box 2.

Given: Homogenized material properties, $\mathbf{D}^{(0)}(s)$, $\mathbf{D}^{(2)}(s)$, ρ_0 , and the approximation parameters $A^{(1)}(s)$ and $A^{(2)}(s)$.

- *Find*: $\nu(s)$, such that:
- Minimize: $\|\mathbf{e}(\nu, s)\|$
- Subject to constraints:

 $\operatorname{Re}\left[A(s,\theta)\right] > 0; \quad \operatorname{Re}\left[\alpha^{(3)}(s)\right] < 0, \quad \operatorname{Im}\left[\alpha^{(3)}(s)\right] = 0, \quad \theta \in [0,\pi/2]$

Box 2: Summary of the constrained minimization problem for the identification of $\nu(s)$.

4.3 Reduced order nonlocal model

Straightforward implementation of the proposed fourth order PDE to simulate transient wave propagation is problematic since the numerical solution contains both the physical and non-physical wavenumber solutions. Furthermore, the evaluation of the fourth order governing equation requires setting high order boundary conditions, which is usually not trivial to obtain. In this section, we propose a reduced order (i.e., second order) spatial-temporal nonlocal homogenization model to simulate the transient behavior of the composite. This is achieved by identifying and retaining the physical wavenumber solutions of the original fourth order PDE, while eliminating the non-physical branches.

The identification and selection of the physical wavenumber is critical in obtaining stable and physically meaningful dynamics [55, 23]. Considering the harmonic solution form, $U = U_0(s)e^{i(k_1x_1+k_2x_2)}$, negative imaginary part of the wavenumber results in exponentially amplifying harmonic wave. The physically meaningful wavenumber is therefore identified as the one that has non-negative imaginary part. Examining Eq. 47 indicates that the physically meaningful wavenumber is:

$$\bar{k}^2(s,\theta) = \begin{cases} \left(k^{\{1,2\}}\right)^2, & \omega \le \omega_c \\ \left(k^{\{3,4\}}\right)^2, & \omega > \omega_c \end{cases}$$
(51)

where the switch, ω_c , between the two branches of the solutions is the imaginary part of the critical Laplace variable, $s_c = \sigma + i\omega_c$. In order to ensure that the continuity of the physical wavenumber at the switch, $\left(k^{\{1,2\}}(s,\theta)\right)^2 = \left(k^{\{3,4\}}(s,\theta)\right)^2$ is satisfied:

$$-\sqrt{B^2(s,\theta) - 4A(s,\theta)C(s,\theta)} = \sqrt{B^2(s,\theta) - 4A(s,\theta)C(s,\theta)}$$
(52)

The complex-valued equation above is equivalent to the constrained system of equations:

$$\operatorname{Im}\left[B^{2}(s,\theta) - 4A(s,\theta)C(s,\theta)\right] = 0$$
(53a)

$$\operatorname{Re}\left[B^{2}(s,\theta) - 4A(s,\theta)C(s,\theta)\right] \leq 0$$
(53b)

Substituting Eq. 46 and s_c into Eq. 53a results in an equation from which ω_c is obtained. ω_c is evaluated analytically for elastic composites. For viscoelastic composites, ω_c is solved numerically, since the model parameters are dependent on the Laplace variable, resulting in a nonlinear equation. The inequality condition in Eq. 53b provides the criterion to choose the appropriate ω_c among the multiple solutions of Eq. 53a.

Selecting only the physical wavenumber for the solution of Eq. 45 is captured by the following dispersion equation:

$$k^2 - \bar{k}^2(\theta, s) = 0 \tag{54}$$

multiplying Eq. 54 with $-\rho_0 s^2/\bar{k}^2(\theta, s)$, the dispersion relation becomes:

$$\phi(s,\theta) \left(D_{11}^{(0)}(s) \cos^2 \theta + D_{22}^{(0)}(s) \sin^2 \theta \right) k^2 + \rho_0 s^2 = 0$$
(55)

where,

$$\phi(s,\theta) = \frac{-\rho_0 s^2}{\bar{k}^2(\theta,s) \left(D_{11}^{(0)}(s) \cos^2 \theta + D_{22}^{(0)}(s) \sin^2 \theta \right)}$$
(56)

Equation 55 resembles the dispersion relation of the classical homogenization model (Eq. 31a) except for the factor $\phi(s, \theta)$, which results from the nonlocal terms in the fourth order governing equation. Compared to the dispersion relation of the original fourth order equation (Eq. 45), Eq. 55 is regularized in that the dispersion relation contains the physical branches of the solution only and the non-physical branches are suppressed. The regularized nonlocal momentum balance equation is then expressed as:

$$\rho_0 s^2 U(\mathbf{x}, s) - D^e_{kl}(s, \theta) U_{,kl} = 0$$
(57)

in which, $D_{kl}^e(s,\theta) = \phi(s,\theta)D_{kl}^{(0)}(s)$. The nonlocal factor, $\phi(s,\theta)$, depends on the Laplace variable and the direction of wave propagation. The direction-dependence reveals the anisotropic

nature of wave dispersion, which is also observed by Phani et al. [54] and Wautier and Guzina [62].

Equation 57 is a second order PDE that shares the same structure as the classical homogenization model but expressed in terms of nonlocal effective stiffness that is frequencyand direction-dependent. In a transient dynamic analysis, computing the direction-dependent nonlocal factor requires knowledge of the direction of local wave vector at each material point which evolves as a function of time. Since the wave vector is normal to the wave front, $\phi(s, \theta)$ can be obtained by solving for the evolution of wave front for any given s. A comprehensive study of techniques to track wave front can be found in [17]. In this manuscript, we use a simplified approach that prescribes all the material points to have the same property that captures wave propagation of Eq. 57 exactly in $\theta = 0$ and $\theta = \pi/2$ directions only. Substituting test harmonic wave $U = U_0(s)e^{i(k_1x_1)}$ and $U = U_0(s)e^{i(k_2x_2)}$ into Eq. 57, the effective stiffness are obtained: $D_{11}^e(s) = -\rho_0 s^2/\bar{k}^2(s, 0), D_{22}^e(s) = -\rho_0 s^2/\bar{k}^2(s, \pi/2)$. The resulting momentum balance equation is:

$$\rho_0 s^2 U(\mathbf{x}, s) - D_{kl}^e(s) U_{,kl} = 0 \tag{58}$$

This operation in fact reduces the direction-dependent nonlocal effective stiffness to a directionindependent one. As a result, the reduced dispersion relation matches that of Eq. 55 exactly in $\theta = 0$ and $\theta = \pi/2$ directions, whereas the solutions for arbitrary wave direction θ are approximated. The effect of this approximation is discussed in the Section 6.1. The resulting boundary value problem to evaluate the macroscale displacement is summarized in Box 3.

- Given: Effective nonlocal stiffness, $D_{kl}^{e}(s)$, and homogenized density, ρ_{0} .
- *Find*: Macroscale displacement $U(\mathbf{x}, s) : \Omega \times \mathbb{C} \to \mathbb{C}$, such that:
- Momentum balance:

$$p_0 s^2 U(\mathbf{x}, s) - D_{kl}^e(s) U_{kl} = 0$$

- Boundary conditions:
- $$\begin{split} U(\mathbf{x},s) &= \tilde{u}(\mathbf{x},s), \quad \mathbf{x} \in \Gamma^u \\ D^e_{kl} U_l n_k &= \tilde{t}(\mathbf{x},s), \quad \mathbf{x} \in \Gamma^t \end{split}$$

Box 3: Boundary value problem for macroscale displacement $U(\mathbf{x}, s)$.

5 Numerical implementation

In this section, the numerical evaluation of the microscale influence functions, computation of the homogenized shear moduli and model parameters, and the solution to the macroscale momentum balance equation are presented. The analyses are performed in the Laplace domain using complex algebra. For a fixed but arbitrary Laplace variable, the microscale influence functions are evaluated using the standard finite element method (FEM) with C^0 -continuous shape functions. Isogeometric analysis (IGA) with C^1 -continuous NURBS (Non-Uniform Rational B-Splines) basis functions is employed to evaluate the macroscale momentum balance equation. The time domain response is then obtained by numerical inverse Laplace transform.

5.1 Microscale problem

The numerical implementation of the first two influence functions, $H^{(1)}$ and $H^{(2)}$, have been previously provided in Refs. [29, 30], and skipped herein for brevity. In what follows, we provide a general procedure to compute the influence functions at higher orders. Considering Eq. 24 for $G^{(n)}(\mathbf{y}, s)$, the weak form of equilibrium equation for the $(n+1)^{th}$ influence function (Box 1, $n \geq 1$) is written as:

$$\int_{\Theta} G(\mathbf{y}, s) \nabla_{y} w(\mathbf{y}) \cdot \nabla_{y} H^{(n+1)}(\mathbf{y}, s) d\mathbf{y} = -\int_{\Theta} G(\mathbf{y}, s) \nabla_{y} w(\mathbf{y}) \cdot \left(H^{(n)}(\mathbf{y}, s) \otimes I \right) d\mathbf{y} + \int_{\Theta} w(\mathbf{y}) G^{(n-1)}(\mathbf{y}, s) d\mathbf{y} - \int_{\Theta} w(\mathbf{y}) \left[\theta(\mathbf{y}) \sum_{j=0}^{n-1} H^{(j)}(\mathbf{y}, s) \otimes D^{(n-1-j)}(s) \right] d\mathbf{y}$$
(59)

where, $w \in \mathcal{W}_{\text{per}} \subset H^1(\Theta, \mathbb{R})$ is the weighting function; \mathcal{W}_{per} is space of sufficiently smooth functions that is periodic along the microstructure boundary, Γ_{Θ} ; and $H^1(\Theta, \mathbb{R})$ is the Sobolev space of scalar-valued functions with square integrable first derivative over the domain of microstructure, Θ . Rewriting $\mathbf{H}^{(n+1)}(\mathbf{y},s) = \mathbf{H}_{\text{re}}^{(n+1)} + i\mathbf{H}_{\text{im}}^{(n+1)}$, where $\mathbf{H}_{\text{re}}^{(n+1)}$ and $\mathbf{H}_{\text{im}}^{(n+1)}$ are real. The solutions are sought in the function space, $\mathcal{H}_{\text{per}\{\text{re,im}\}} \subset H^1(\Theta, \mathbb{R}^{n+2})$:

$$\mathcal{H}_{\text{per}\{\text{re,im}\}} := \left\{ \boldsymbol{H}_{\{\text{re,im}\}}^{(n+1)}(\mathbf{y},s) \mid \boldsymbol{H}_{\{\text{re,im}\}}^{(n+1)}(\mathbf{y},s) \text{ is periodic } \forall \mathbf{y} \in \Gamma_{\Theta}; \ \langle \boldsymbol{H}_{\{\text{re,im}\}}^{(n+1)}(\mathbf{y},s) \rangle = 0 \right\}$$
(60)

In the discrete approximation of $\boldsymbol{H}_{\{\text{re,im}\}}^{(n+1)}$, periodicity is imposed by coupling the nodes at the opposing edges as the master and slave nodes. The degrees of freedom associated with the slave nodes are eliminated through static condensation. To implement this coupling, the microstructure is discretized such that the nodal positions at the opposing boundaries match exactly. In addition, the value of the influence function is set to zero at corner nodes to eliminate rigid body modes. The normalization condition is imposed by subtracting the microstructural average of the influence functions from each nodal value, which is done as a post-processing step. The finite dimensional subspace for the trial solutions, $\mathcal{H}_{\text{per}}^h \subset \mathcal{H}_{\text{per}}$, is constructed as:

$$\mathcal{H}_{\text{per}\{\text{re,im}\}}^{h} \coloneqq \left\{ \boldsymbol{H}_{\{\text{re,im}\}}^{(n+1),h}(\mathbf{y},s) \mid \boldsymbol{H}_{\{\text{re,im}\}}^{(n+1),h}(\mathbf{y},s) = \sum_{A=1}^{M} N^{[A]}(\mathbf{y}) \mathbf{H}_{\{\text{re,im}\}}^{(n+1)[A]}(s); \mathbf{H}_{\{\text{re,im}\}}^{(n+1)[c]}(s) = 0 \right\}$$
(61)

where, $N^{[A]}(\mathbf{y})$ is the shape function, M the number of nodes, superscript h indicates discretization, and $\mathbf{H}_{\{\text{re,im}\}}^{(n+1)[A]}(s)$ denotes the influence function matrix at node A. $\mathbf{H}_{\{\text{re,im}\}}^{(n+1)[c]}(s)$ is the influence function at the corner nodes.

Substituting the discretization of the influence function and the weighting function into the weak form and expressing the terms in matrix-vector form using the Voigt notation yields the following discrete system:

$$\mathbf{Kd}^{(n+1)} = \mathbf{F}^{(n+1)} \tag{62}$$

which is formed by assembling the element matrices:

$$\mathbf{K} = \mathop{\mathcal{A}}_{e=1}^{N_e} \mathbf{K}^e; \quad \mathbf{d}^{(n+1)} = \mathop{\mathcal{A}}_{e=1}^{N_e} \mathbf{d}^{(n+1)e}; \quad \mathbf{F}^{(n+1)} = \mathop{\mathcal{A}}_{e=1}^{N_e} \mathbf{F}^{(n+1)e}$$
(63)

 \mathcal{A} is the assembly operator, superscript *e* denotes the element index and N_e the number of elements. The element stiffness matrix is expressed as:

$$\mathbf{K}^{e} = G(\mathbf{y}, s) \int_{\Theta^{e}} \mathbf{B}^{e} \left(\mathbf{y}\right)^{T} \mathbf{B}^{e} \left(\mathbf{y}\right) d\mathbf{y}$$
(64)

where Θ^e denotes the domain of element e, T the matrix transpose, and

$$\mathbf{B}^{e} = \begin{bmatrix} \mathbf{B}^{e[1]} & \mathbf{B}^{e[2]} & \dots & \mathbf{B}^{e[M_{e}]} \end{bmatrix}$$
(65a)

$$\mathbf{B}^{e[A]} = \begin{bmatrix} N_{,y_1}^{e[A]}(\mathbf{y}) & N_{,y_2}^{e[A]}(\mathbf{y}) \end{bmatrix}^T$$
(65b)

For the convenience and efficiency of numerical implementation, the same discretization of microstructure and shape functions are used for all influence functions, therefore, the stiffness matrix is only assembled and factorized (e.g., LU factorization) once. The right-hand-side force matrices are assembled for each influence function. The components of the influence functions are listed as follows:

$$\mathbf{H}^{(1),h}(\mathbf{y},s) = \begin{bmatrix} H_1^{(1),h}(\mathbf{y},s) & H_2^{(1),h}(\mathbf{y},s) \end{bmatrix}^T$$
(66a)

$$\mathbf{H}^{(2),h}(\mathbf{y},s) = \begin{bmatrix} H^{(2),h}_{(11)}(\mathbf{y},s) & H^{(2),h}_{(12)}(\mathbf{y},s) & H^{(2),h}_{(22)}(\mathbf{y},s) \end{bmatrix}^T$$
(66b)

$$\mathbf{H}^{(7),h}(\mathbf{y},s) = \begin{bmatrix} H^{(7),h}_{(1\times7)}(\mathbf{y},s) & H^{(7),h}_{(1\times62)}(\mathbf{y},s) & H^{(7),h}_{(1\times52\times2)}(\mathbf{y},s) & H^{(7),h}_{(1\times42\times3)}(\mathbf{y},s) \\ H^{(7),h}_{(1\times32\times4)}(\mathbf{y},s) & H^{(7),h}_{(1\times22\times5)}(\mathbf{y},s) & H^{(7),h}_{(1\times12\times6)}(\mathbf{y},s) & H^{(7),h}_{(2\times7)}(\mathbf{y},s) \end{bmatrix}^{T}$$
(66c)

The element matrix of influence function $\mathbf{H}^{(n+1),h}(\mathbf{y},s)$ is written as:

$$\mathbf{d}_{H}^{(n+1)e} = \begin{bmatrix} \mathbf{H}^{(n+1)e[1]} & \mathbf{H}^{(n+1)e[2]} & \dots & \mathbf{H}^{(n+1)e[M_{e}]} \end{bmatrix}^{T}$$
(67)

where, M_e is the number of nodes in the element. The force matrix for element e is written as a summation of three components:

$$\mathbf{F}_{H}^{(n+1)e} = \mathbf{F}_{H,1}^{(n+1)e} + \mathbf{F}_{H,2}^{(n+1)e} + \mathbf{F}_{H,3}^{(n+1)e}$$
(68)

where,

$$\mathbf{F}_{H,1}^{(n+1)e} = -G(\mathbf{y}, s) \int_{\Theta^e} \mathbf{B}^e \left(\mathbf{y}\right)^T \operatorname{Sym}\left(\mathbf{P}^{(n)e}\left(\mathbf{y}, s\right)\right) d\mathbf{y}$$
(69a)

$$\mathbf{F}_{H,2}^{(n+1)e} = \int_{\Theta^e} \mathbf{N}^e \left(\mathbf{y} \right)^T \operatorname{Sym} \left(\mathbf{G}^{(n-1)e} \left(\mathbf{y}, s \right) \right) d\mathbf{y}$$
(69b)

$$\mathbf{F}_{H,3}^{(n+1)e} = -\int_{\Theta^e} \mathbf{N}^e \left(\mathbf{y}\right)^T \operatorname{Sym}\left(\mathbf{Q}^{(n-1)e}\left(\mathbf{y},s\right)\right) d\mathbf{y}$$
(69c)

Sym $(\mathbf{P}^{(n)e}(\mathbf{y},s))$, Sym $(\mathbf{G}^{(n-1)e}(\mathbf{y},s))$ and Sym $(\mathbf{Q}^{(n-1)e}(\mathbf{y},s))$ are defined in Eq. 75. Assembling the force matrix, imposing periodic boundary condition and setting the degree of freedom at corner nodes to be 0, the boundary value problem is solved with standard finite element method. Upon normalization, the components of influence function $\mathbf{H}^{(n+1),h}(\mathbf{y},s)$ are obtained:

$$H_{(\cdot)}^{(n+1),h}(\mathbf{y},s) = \sum_{A=1}^{M} N^{[A]}(\mathbf{y}) H_{(\cdot)}^{(n+1)[A]}(s)$$
(70)

where, the normalization is computed as:

$$H_{(\cdot)}^{(n+1)[A]}(s) = H_{(\cdot)0}^{(n+1)[A]}(s) - \frac{1}{|\Theta|} \int_{\Theta} \sum_{A=1}^{M} N^{[A]}(\mathbf{y}) H_{(\cdot)0}^{(n+1)[A]}(s) d\mathbf{y}$$
(71)

in which, $H_{(\cdot)0}^{(n+1)[A]}(s)$ are the nodal values obtained by solving Eq. 62 before normalization.

In view of Eq. 59, computation of element force matrices requires information from the inferior orders, which results in a successive solution of the influence functions. The detailed evaluation procedure is provided in what follows.

5.1.1 Evaluation of $H^{(1)}(\mathbf{y},s)$ and $D^{(0)}(s)$

At n = 0, the expression for $\mathbf{P}^{(0)e}(\mathbf{y}, s)$ is:

$$\mathbf{P}^{(0)e}\left(\mathbf{y},s\right) = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$$
(72)

whereas $\mathbf{G}^{(-1)e} = \mathbf{0}$ and $\mathbf{Q}^{(-1)e} = \mathbf{0}$. Substituting these expressions into Eq. 69, $H_k^{(1),h}(\mathbf{y},s)$ is obtained from the procedure described above. $G_{kl}^{(0)}(\mathbf{y},s)$ is obtained by substituting Eq. 70 for $H_k^{(1)}(\mathbf{y},s)$ into Eq. 24:

$$G_{kl}^{(0)}(\mathbf{y},s) = G(\mathbf{y},s) \left[\delta_{kl} + \sum_{A=1}^{M} B_l^{[A]}(\mathbf{y}) H_k^{(1)[A]}(s) \right]$$
(73)

where, $B_l^{[A]}(\mathbf{y}) = N_{,l}^{[A]}(\mathbf{y})$. According to Eq. 21, the symmetric part is computed as:

$$G_{(kl)}^{(0)}(\mathbf{y},s) = \frac{1}{2} \left(G_{kl}^{(0)}(\mathbf{y},s) + G_{lk}^{(0)}(\mathbf{y},s) \right)$$
(74)

and $D_{(kl)}^{(0)}(s)$ is computed by Eq. 26 accordingly.

5.1.2 Evaluation of higher order influence functions and homogenized shear moduli

The procedure to evaluate higher order influence functions, $\boldsymbol{H}^{(n+1)}(\mathbf{y},s)$ and homogenized shear moduli, $\boldsymbol{D}^{(n)}(s)$, $n = \{1, 2, ..., 6\}$, is provided in a general form. Sym $\left(\mathbf{P}^{(n)e}(\mathbf{y},s)\right)$,

 $\operatorname{Sym}\left(\mathbf{G}^{(n-1)e}\left(\mathbf{y},s\right)\right)$ and $\operatorname{Sym}\left(\mathbf{Q}^{(n-1)e}\left(\mathbf{y},s\right)\right)$ are written as:

$$Sym\left(\mathbf{P}^{(n)e}\left(\mathbf{y},s\right)\right) = \frac{1}{(n+1)!H_{(1\times n)}^{(n)e} \quad n(n!)H_{(1\times (n-1)^2)}^{(n)e} \quad \cdots \quad n!H_{(2\times n)}^{(n)e} \quad 0}{0 \qquad n!H_{(1\times n)}^{(n)e} \quad \cdots \quad n(n!)H_{(12\times (n-1))}^{(n)e} \quad (n+1)!H_{(2\times n)}^{(n)e}}\right]$$
(75a)
$$Sym\left(\mathbf{G}^{(n-1)e}\left(\mathbf{y},s\right)\right) =$$

$$\begin{bmatrix} G_{(1\times n)}^{(n-1)e}(\mathbf{y},s) & G_{(1\times (n-1)^2)}^{(n-1)e}(\mathbf{y},s) & \cdots & G_{(12\times (n-1))}^{(n-1)e}(\mathbf{y},s) & G_{(2\times n)}^{(n-1)e}(\mathbf{y},s) \end{bmatrix}$$
(75b)
Sym $\left(\mathbf{Q}^{(n-1)e}(\mathbf{y},s) \right) =$

$$\theta(\mathbf{y}) \begin{bmatrix} Q_{(1\times n)}^{(n-1)e}(\mathbf{y},s) & Q_{(1\times (n-1)^2)}^{(n-1)e}(\mathbf{y},s) & \cdots & Q_{(12\times (n-1))}^{(n-1)e}(\mathbf{y},s) & Q_{(2\times n)}^{(n-1)e}(\mathbf{y},s) \end{bmatrix}$$
(75c)

where, the expressions for $G_{(\cdot)}^{(n-1)e}(\mathbf{y},s)$ and $Q_{(\cdot)}^{(n-1)e}(\mathbf{y},s)$ are provided in Appendix C. Substituting Eq. 75 into Eq. 69, $H_{(\cdot)}^{(n+1),h}(\mathbf{y},s)$ is obtained using the same procedure as for $H_k^{(1),h}(\mathbf{y},s)$. The homogenized moduli $D_{(\cdot)}^{(n)}(s)$ is obtained from Eq. 29. The expressions for $D_{(klmn)}^{(2)}(s)$, $D_{(klmnpq)}^{(4)}(s)$ and $D_{(klmnpqrs)}^{(6)}(s)$ are obtained as follows:

$$D_{(klmn)}^{(2)}(s) = \bar{G}_{(klmn)}^{(2)}(\mathbf{y}, s) - \text{Sym}\left(\overline{\rho(\mathbf{y})}H_{(kl)}^{(2)}(\mathbf{y}, s)D_{(mn)}^{(0)}(s)\right)$$
(76a)
$$D_{(klmnpq)}^{(4)}(s) = \bar{G}_{(klmnpq)}^{(4)}(\mathbf{y}, s) - \text{Sym}\left(\overline{\rho(\mathbf{y})}H_{(kl)}^{(2)}(\mathbf{y}, s)D_{(mnpq)}^{(2)}(s)\right)$$
(76a)

$$-\operatorname{Sym}\left(\overline{\rho(\mathbf{y})}H^{(4)}_{(klmn)}(\mathbf{y},s)D^{(0)}_{(pq)}(s)\right)$$
(76b)

$$D_{(klmnpqrs)}^{(6)}(s) = \bar{G}_{(klmnpqrs)}^{(6)}(\mathbf{y}, s) - \operatorname{Sym}\left(\overline{\rho(\mathbf{y})}H_{(kl)}^{(2)}(\mathbf{y}, s)}D_{(mnpqrs)}^{(4)}(s)\right) - \operatorname{Sym}\left(\overline{\rho(\mathbf{y})}H_{(klmn)}^{(4)}(\mathbf{y}, s)}D_{(pqrs)}^{(4)}(s)\right) - \operatorname{Sym}\left(\overline{\rho(\mathbf{y})}H_{(klmnpq)}^{(6)}(\mathbf{y}, s)}D_{(rs)}^{(0)}(s)\right)$$
(76c)

5.2 Macroscale problem

The weak form of the boundary value problem defined in Box 3 is written as:

$$\int_{\Omega} w(\mathbf{x})_{,x_k} D^e_{kl}(s) U_{,x_l}(\mathbf{x},s) d\mathbf{x} + \int_{\Omega} w(\mathbf{x}) \rho_0 s^2 U(\mathbf{x},s) d\mathbf{x} = \int_{\Gamma^t} w(\mathbf{x}) \tilde{t}(\mathbf{x},s) d\mathbf{x}$$
(77)

The macroscale displacement is rewritten as: $U(\mathbf{x}, s) = U_{re} + iU_{im}$, where U_{re} and U_{im} are respectively the real and imaginary parts. Using Galerkin's approximation, the finite dimensional solution space for the macroscale displacement is written as:

$$\mathcal{U}^{h}_{\{\text{re,im}\}} \coloneqq \left\{ U^{h}_{\{\text{re,im}\}}(\mathbf{x}, s) \in H^{1}(\Theta, \mathbb{R}) \mid U^{h}_{\{\text{re,im}\}}(\mathbf{x}, s) = \tilde{u}_{\{\text{re,im}\}}(s) \quad \forall \mathbf{x} \in \Gamma^{u} \right\}$$
(78)

Equation 77 is evaluated by Isogeometric Analysis (IGA) with NURBS basis functions. Among the various desirable characteristics, e.g., exact geometric representation [25], its application in wave propagation problems [27, 26, 63, 15] has demonstrated high rate of convergence due to the so-called k-refinement, by which C^{p-1} -continuity (*p* is the polynomial order of the basis functions) could be achieved. It is noted that standard FEM with C^0 -continuous shape functions can also be employed to numerically evaluate Eq. 77, provided that a sufficiently fine mesh is used to avoid numerical dispersion.

In the context of IGA, the basis functions that approximate the solution field are ones representing the geometry of the physical domain. In two-dimensions, the physical domain is represented by the NURBS basis functions through the mapping:

$$\mathbf{x}(\boldsymbol{\xi}) = \sum_{A=1}^{M_{bf}} N^{[A],p}(\boldsymbol{\xi}) \mathbf{P}^{[A]}$$
(79)

where, $\{\mathbf{P}^{[A]}\}_{A=1}^{M_{bf}} \in \mathbb{R}^2$ is a set of control points in the physical domain, $N^{[A],p}$ is the A^{th} NURBS basis function that is constructed from B-spline basis functions of polynomial order p. C^1 -continuous basis functions are used for all the numerical examples in Section 6, therefore p = 2. The basis function is parameterized by knot vectors that are defined in the parametric domain $\hat{\Omega} \in \mathbb{R}^2$. The parametric domain is partitioned into elements by knots, which forms the elements in physical domain through mapping Eq. 79. A detailed discussion of the NURBS basis functions and IGA are provided in Ref. [12]. By virtue of the isogeometric concept, the macroscale displacement in the parametric domain is approximated by:

$$\hat{U}^{h}(\boldsymbol{\xi},s) = \sum_{A=1}^{M_{bf}} N^{[A],p}(\boldsymbol{\xi}) U^{[A]}(s)$$
(80)

The macroscale displacement in the physical domain is obtained by considering the inverse of the geometric mapping (Eq. 79):

$$U^{h}(\boldsymbol{\xi}, s) = \hat{U}^{h}(\boldsymbol{\xi}, s) \circ \mathbf{x}^{-1}$$
(81)

5.3 Uncoupled multiscale solution strategy

In this section, we present the solution strategy to the multiscale system. The microand macroscale problems are uncoupled because the microscale boundary value problems are independent of the macroscale solution. The microscale boundary value problems are successively evaluated in the Laplace domain to compute the homogenized shear moduli, after which the nonlocal effective stiffness is computed. The macroscale boundary value problem





Figure 1: Computational flowchart.

is then solved to obtain the macroscale displacement in the Laplace domain. Numerical inverse Laplace transform is applied to obtain the solution in time domain. The computational flowchart is shown in Fig. 1.

A numerical inverse Laplace transform algorithm [13, 8] based on the Fast Fourier Transform is used to transform the solution from the Laplace domain to the time domain. In this algorithm, the accuracy of the numerical inverse transformation is controlled by an algorithm parameter. It is set to provide sufficient numerical accuracy in all simulations. Fast convergence is achieved by employing the ϵ -algorithm [40]. A comprehensive study of this algorithm compared to others can be found in Ref. [14]. With this transformation, the key steps towards obtaining the macroscale displacement, $U(\mathbf{x}, t)$ are described as follows:

1. Given an observation time period, discrete time steps (i.e., t_{β} where $\beta = 1, 2, \dots, N_t$ and N_t is the number of time steps) are generated. The Laplace variable, s, is sampled by s_{α} where $\alpha = 1, 2, \dots, N_s$ and N_s is the number of samples, within the convergence region

of the inverse Laplace transform integration, along a vertical line (i.e., of the same real part) in the complex plane.

- 2. For each sample, s_{α} , influence functions, $\boldsymbol{H}^{(n+1)}(\mathbf{y}, s_{\alpha})$, and homogenized shear moduli, $\boldsymbol{D}^{(n)}(s_{\alpha}), n = 1, 2, ..., 6$, are successively evaluated through the procedure in Section 5.1.
- 3. Provided with $\mathbf{D}^{(0)}(s_{\alpha})$, $\mathbf{D}^{(2)}(s_{\alpha})$, $\mathbf{D}^{(4)}(s_{\alpha})$ and $\mathbf{D}^{(6)}(s_{\alpha})$, the approximation parameters are computed by the Moore-Penrose pseudo-inverse: $A^{(1)}(s_{\alpha}) = \mathbf{d}^{(6)}(s_{\alpha}) \left(\mathbf{d}^{(0,4)}(s_{\alpha})\right)^{(-m)}$ and $A^{(2)}(s_{\alpha}) = \mathbf{d}^{(4)}(s_{\alpha}) \left(\mathbf{d}^{(2,0)}(s_{\alpha})\right)^{(-m)}$, where $\mathbf{d}^{(6)}(s_{\alpha})$, $\mathbf{d}^{(0,4)}(s_{\alpha})$, $\mathbf{d}^{(4)}(s_{\alpha})$ and $\mathbf{d}^{(2,0)}(s_{\alpha})$ are vectors of the non-zero independent components of $\operatorname{Sym}(\mathbf{D}^{(6)})$, $\operatorname{Sym}\left(\mathbf{D}^{(0)} \otimes \operatorname{Sym}(\mathbf{D}^{(4)})\right)$, $\operatorname{Sym}(\mathbf{D}^{(4)})$ and $\operatorname{Sym}\left(\operatorname{Sym}(\mathbf{D}^{(2)}) \otimes \mathbf{D}^{(0)}\right)$. The parameter, $\nu(s_{\alpha})$, is obtained by the constrained minimization problem defined in Box 2.
- 4. With all the homogenized moduli and model parameters computed in the higher order gradient nonlocal formulation, the nonlocal effective stiffness, $D^e(s_{\alpha})$, for the reduced order model is obtained as detailed in Section 4.3.
- 5. The macroscale displacement, $U(\mathbf{x}, s_{\alpha})$, is obtained by evaluating the macroscale boundary value problem in Section 5.2.
- 6. Steps 2-5 are repeated N_s times to compute the macroscale displacements for all sampled Laplace variables. The complex valued macroscale displacements, $U(\mathbf{x}, s_{\alpha})$ are provided to the numerical inverse Laplace transform algorithm and the macroscale displacements for all the time steps in the time domain are obtained.

The uncoupled solution strategy allows an off-line computation of the required nonlocal effective stiffness for the solution of macroscale displacement for a given microstructure. In particular, microscale problems are evaluated once in the case of composites with elastic constituents, since they are independent of the Laplace variable. For viscoelastic composites, the repeated evaluation of Steps 2-5 are implemented in a parallel environment since all the Laplace variable samples are independent of each other. They are evaluated once for a given period of simulation time.

6 Model verification

In this section, we assess the capability of the proposed spatial-temporal nonlocal homogenization model (STNHM) in capturing wave dispersion and attenuation in both elastic and viscoelastic composites. Two numerical examples are presented to evaluate the proposed model for elastic layered composites and viscoelastic composites with circular inclusions. The dispersion relation and transient wave propagation in elastic layered composite are investigated in



Figure 2: (a) Aluminum-steel layered microstructure. (b) Sketch of irreducible Brillouin zone (shaded square) and the probed domain of wave vector (square OACD). (c) Macrostructure and boundary conditions.

the first example. The second example focuses on transient wave propagation in the viscoelastic composite. Time domain response and wave transmission characteristics are examined and compared with the direct numerical simulations. In all numerical examples, the number of Laplace variable samples are chosen large enough such that further increase of the resolution does not significantly alter the simulation results.

6.1 Elastic layered composite

6.1.1 Low material property contrast

We consider a two-dimensional bi-material layered microstructure composed of aluminum and steel, as shown in Fig. 2(a). The size of the microstructure is 0.02 m \times 0.02 m and the volume fraction of steel is 0.5. The shear modulus and density are 26.2 GPa and 2700 kg/m³ for aluminum, and 80.8 GPa and 7900 kg/m³ for steel, respectively.

The steady-state wave propagation characteristics of the proposed model is evaluated by comparing the dispersion curves with those obtained from the Bloch wave expansion [32], where an eigenvalue problem of frequency ω is formulated by discretizing the unit cell with finite element method and sampling the wave vector **k** within the irreducible Brillouin zone. As shown in Fig. 2(b), this approach samples the wave vector within the domain $[0, \pi/l] \times [0, \pi/l]$. The dispersion relation for wave vectors outside of the irreducible Brillouin zone but within $[0, 2\pi/l] \times [0, 2\pi/l]$ is obtained by shifting the corresponding branch obtained in the irreducible Brillouin zone by π/l . The dispersion curves of the proposed model (STNHM) is computed



Figure 3: Dispersion curves of aluminum-steel layered composite.

by sampling the wave vector in $[0, 2\pi/l] \times [0, 2\pi/l]$ and solving for the frequency ω by Eq. 54 (s is replaced with $i\omega$).

Figure 3 shows the dispersion curves of wave vectors sampled in four directions, i.e., OA, OB, OC, OD. The dispersion relations of the spatial nonlocal homogenization model (SNHM) proposed in Refs. [29, 19], STNHM with $\nu = 0$ and STNHM, which respectively correspond to the nonlocal homogenization models employing asymptotic expansions of up to the 4^{th} , 6^{th} and 8^{th} order, are plotted for comparison. STNHM accurately captures the dispersion in the first pass band, and the initiation and size of the first stop band in all directions. In Fig. 3(a) and (b), SNHM behaves as a low-pass filter which prohibits any wave of frequency higher than the cut-off frequency from propagating. In addition, it does not predict the onset of the stop band. By setting $\nu = 0$, STNHM recovers the homogenization model that is derived based on Eqs. 31a-f. This model predicts the optical branch due to the presence of temporal nonlocal term, but it is less accurate at the end of the first stop band compared to STNHM.

Figure 4 shows the effect of reducing the direction-dependent nonlocal effective stiffness to a direction-independent one as employed in the reduced order approximation. The wavenumber as a function of direction computed by the regularized STNHM (Eq. 57), k_{reg} , and reduced order STNHM (Eq. 58), k_{red} , are compared with that from Bloch wave expansion in Fig. 4(a).



Figure 4: (a) Polar plots of the wavenumber $kl/(2\pi)$ as a function of direction θ . Dotted line, '. ', Bloch wave expansion; Solid line, '--', regularized STNHM; Dashed line, '--', reduced order STNHM. (b) Polar plots of the error caused by model reduction. From the center outward, the diagrams are computed for $\omega = 2\pi\{10, 20, 30, 40, 50\}$ kHz.

Results obtained at frequencies, $\omega = 2\pi\{10, 20, 30, 40, 50\}$ kHz, are plotted. At low frequencies, both STNHM models predict the wavenumber in all directions accurately. As frequency increases, where significant wave dispersion occurs, the reduced order model captures the dispersion in 0° and 90°, and introduces a slight discrepancy in other directions. The error induced by approximating the direction-dependent nonlocal effective stiffness with one obtained from $\theta = 0$ and $\theta = \pi/2$ is shown in Fig. 4(b). The error, $k_{err} = |k_{reg} - k_{red}|/k_{reg}$, measures the normalized absolute difference between wavenumber computed from the regularized STNHM and reduced order STNHM, as a function of frequency and wave propagation direction. The maximum error is about 0.05 around 30°, 150°, 210°, 330° and when the frequency is near the onset of the stop band. The regularized STNHM predicts the anisotropic wave dispersion characteristics because of the direction-dependent nonlocal effective stiffness. As a result of the model reduction, this feature is captured but with a slightly increased error.

Transient wave propagation is investigated by considering a macrostructure that is composed of 20 microstructures as shown in Fig. 2(a). The macrostructure and boundary conditions are shown in Fig. 2(c). Sinusoidal out-of-plane displacement load with amplitude M, $\tilde{u}_3(t) = M \sin(2\pi f t)$, is applied at the left boundary of the structure and the right boundary is fixed. Periodic displacement boundary condition is applied to the top and bottom edges. In Fig. 5, the normalized displacement (U/M) along the bottom edge of the structure computed from STNHM is examined against the reference solution at two time instances, i.e., t=0.5T



Figure 5: Displacement snapshots along the measurement line.

and t=T. The reference solution is obtained from direct numerical simulations with all the heterogeneities fully resolved by IGA and Newmark-beta method for time integration. It is noted that pointwise comparison of displacement between the homogenization model and direct numerical simulations is only one-to-one at the corner points of the microstructure, where the microscale influence functions are zero (see Section 5.1). Because the heterogeneous displacement is equal to the homogenized displacement at these points according to Eqs. 14, 19, 20. Therefore, only the displacements at corner points of each microstructure from the direct numerical simulations are plotted as the reference solution. As is shown in Fig. 5, STNHM predicts the non-dispersive wave propagation (Fig. 5(a)) and wave dispersion (Fig. 5(b)) in the first pass band. Further increasing the loading frequency, wave is significantly attenuated in the first stop band (Fig. 5(c)). The proposed model captures the attenuated wave reasonably well. Wave propagation in the second pass band is shown in Fig. 5(d). At f = 120 kHz, a phase shift between STNHM and direct numerical simulations is observed. The cause of this error is linked to the fundamental assumption of separation of scales. Although higher order terms in the asymptotic expansion contributes to a more accurate approximation of the heterogeneous displacement field and allows STNHM to predict wave dispersion in the first pass band and attenuation in the first stop band, the accuracy of the proposed model deteriorates



Figure 6: Dispersion curves of elastic layered composite with different material property contrasts. (a) Low stiffness contrast with $r_{\rho} = 3$, (b) High stiffness contrast with $r_{\rho} = 3$, (c) High density contrast with $r_G = 200$.

as the wavelength decreases. When the wavelength approaches the size of the microstructure, the microscale and macroscale responses are inseparable, and the homogenization model is no longer accurate.

6.1.2 The effect of material property contrast

A parametric study examining the accuracy of STNHM for different material property contrasts is performed for the layered microstructure. The volume fraction of each phase is set to be 0.5. The dispersion curves for wave propagation in the k_1 direction are shown in Fig. 6 for various stiffness contrasts, $r_G = G_{(1)}/G_{(2)}$, and density contrasts, $r_{\rho} = \rho_{(1)}/\rho_{(2)}$, where $G_{(1)}, G_{(2)}$ and $\rho_{(1)}, \rho_{(2)}$ are the shear moduli and densities of the two phases, respectively. $\omega l/(2\pi c_1)$ is the normalized frequency and $c_1 = \sqrt{D_{11}^{(0)}/\rho_0}$ is the homogenized wave velocity in the k_1 direction.

For low material property contrast (Fig. 6(a)), STNHM accurately captures the acoustic branch. The accuracy in prediction of the end of the stop band decreases as the stiffness contrast increases. Figure 6(b) shows the dispersion curves of composites with high stiffness contrast which is typical of the contrast between polymers and metals. In these cases, STNHM accurately predicts wave dispersion in the acoustic regime and the initiation of the stop band. The end of the stop band is over predicted and the optical branch is shifted to higher frequency compared to the Bloch wave solutions. The effect of density contrast is shown in Fig. 6(c). As the density contrast increases, the accuracy of STNHM in the short wavelength regime of the acoustic branch decreases. STNHM predicts the trend that the width of the stop band increases as the density contrast increases. However, similar to the behavior when high stiffness contrast is present, the predicted end of the stop band and the optical branch occur at higher frequency.

6.2 Viscoelastic matrix-fiber composite

In this section, we investigate wave propagation in viscoelastic composite with the microstructure that has elastic circular inclusion embedded in viscoelastic matrix and size of 0.02 m $\times 0.02$ m. The elastic phase has volume fraction of 0.2. The constitutive relation of the viscoelastic phase is given by Eq. 4, where the shear modulus is represented with Prony series in time domain. After Laplace transformation, the modulus function is expressed as:

$$G_{(2)}(s) = G_0 s \left(1 - \sum_{i=1}^n p_i \left(1 - \frac{s}{s+1/q_i} \right) \right)$$
(82)

where, G_0 is the instantaneous shear modulus describing the elastic behavior, and p_i and q_i are parameters that define the Prony series expansion representing the effect of damping. G_0 and the density are taken as 1.48 GPa and 1142 kg/m³, according to the experimentally fabricated two-dimensional elastic phononic crystal [61]. Damping effect is introduced by considering the Prony series provided in Table 1 [1].

Table 1: Prony series of the viscoelastic phase.

p_1	p_2	p_3	p_4
0.074	0.147	0.313	0.379
$q_1 [\mathrm{ms}]$	$q_2 [\mathrm{ms}]$	$q_3 [\mathrm{ms}]$	$q_4 [\mathrm{ms}]$
463.4	0.06407	1.163×10^{-4}	7.321×10^{-7}

6.2.1 Dispersion analysis

In this section, we perform dispersion analysis of viscoelastic composites and examine the characteristics of STNHM in capturing wave dispersion and attenuation. The dispersion relation of the proposed model is compared to the Bloch expansion approach [2]. The dispersion curves of both STNHM and the reference are obtained in the frequency domain (replacing s with $i\omega$) by sampling the frequency and solving for the wavenumber. The analysis is performed for microstructures with different elastic phase properties, while the viscoelastic phase properties are fixed.

Figure 7 shows the dispersion curves (in the k_1 direction) of viscoelastic composites with various property contrasts. The shear modulus of the viscoelastic phase at the frequency, f = 15 kHz, is used to evaluate the stiffness contrast, r_G . Both real and imaginary parts of the wavenumber are shown, and they represent the spatial variation of wavefield and attenuation, respectively. Due to the presence of viscoelastic dissipation, wave attenuation increases monotonically as frequency increases in the pass bands. The stop band is featured by the elevated



Figure 7: Dispersion curves of viscoelastic matrix-fiber composite with different material property contrasts. (a) Low stiffness contrast with $r_{\rho} = 1.5$, (b) High contrast.



Figure 8: Dispersion curves of viscoelastic composite with aluminum inclusion.

attenuation within certain frequency ranges. In Fig. 7(a), STNHM predicts both real and imaginary parts of wavenumber in the acoustic branch. Similar to the elastic case (Fig. 6(a)) the error in the prediction of the end of the stop band increases as the stiffness contrast increases. Figure 7(b) shows the dispersion curves of the acoustic branch and a part of the stop band for viscoelastic composite with high material property contrast. STNHM predicts wave dispersion and attenuation in the acoustic regime and the initiation of the stop band for high stiffness contrast. The accuracy decreases as the density contrast increases. This observation is consistent with the example of elastic composite in Fig. 6(b)(c).

Figure 8 shows the dispersion curves for viscoelastic composite in two wave vector ($\mathbf{k} = k_1\hat{\mathbf{e}}_1 + k_2\hat{\mathbf{e}}_2$) directions, $k_2 = 0$ and $k_2 = 0.5k_1$. The elastic phase is taken as aluminum. SNHM, STNHM with $\nu = 0$ and STNHM are compared to demonstrate the effect of high order asymptotic expansions. While both SNHM and STNHM capture wave dispersion in the acoustic regime, SNHM does not capture the initiation of the stop band accurately. By incorporation of the temporal nonlocal term, STNHM with $\nu = 0$ improves the prediction of



Figure 9: Multiscale system of the viscoelastic composite.

the initiation of the stop band with respect to the frequency compared to SNHM, however, it has significant error in predicting wave dispersion within the first pass band. This manifests the importance of appropriate computation of the parameter ν proposed in Section 4.2.

6.2.2 Transient wave propagation in viscoelastic composite

A numerical example of transient wave propagation in a viscoelastic composite is provided in this section. A macroscopic structure composed of 20×20 microstructures is considered as shown in Fig. 9. The material properties are identical to those used in Fig. 8. Out-of-plane sinusoidal displacement load is applied at the center of left boundary within 10*l* range and the right boundary is fixed.

Figure 10 shows the normalized displacement along the measurement line at the end of the simulation, t=T, for four loading frequencies. The results of STNHM are compared to the classical homogenization model (CHM) and direct numerical simulation (reference). The classical homogenization model, which does not capture the dispersion caused by heterogeneities, is provided to distinguish the dispersion caused by material heterogeneity from material damping. At low frequencies, the wavelength is significantly larger than the microstructure and the dispersion caused by material heterogeneity is negligible. Both STNHM and CHM predict wave propagation with progressively reduced amplitude due to wave spreading and viscoelas-



Figure 10: Displacement snapshots along the measurement line at t=T.

tic dissipation in Fig. 10(a). Increasing the loading frequency leads to shorter deformation wavelength, and dispersion occurs due to interactions between macroscopic wave and the microstructure. As a consequence, the macroscopic wave speed and amplitude decrease. It is shown in Fig. 10(b) that STNHM predicts wave speed and the displacement along the measurement line. The first stop band occurs when the macroscopic wavelength is about twice of the size of microstructure. As shown in Fig. 10(c), the wave is significantly attenuated as a result of stop band formation combined with viscoelastic dissipation. When the frequency is further increased to 24 kHz (Fig. 10(d)), the result of direct simulation is within the second pass band. STNHM deviates from the reference result.

Figure 11 shows the contours of wave propagation in viscoelastic composite compared to the elastic counterpart at fixed loading frequency, f = 12 kHz, at two time instances, t=0.5T and t=T. The elastic composite is modeled by replacing the modulus function of the viscoelastic phase, $G_{(2)}(s)$, with the instantaneous shear modulus, G_0 . As a result, the matrix material is stiffer than the viscoelastic matrix. Wave propagates at lower speed in the viscoelastic composite compared to the elastic one, leading to reduced wavelength at given frequency, which results in stronger interaction with the microstructure thus more significant dispersion. At t=T, the shear wave propagating in the viscoelastic composite just reaches the



Figure 11: Displacement contours of STNHM compared to the reference at frequency f = 12 kHz. (a), (b), (c), (d) are for viscoelastic composite, and (e), (f), (g), (h) are for elastic composite. (a), (c), (e), (g) are at t=0.5T, and (b), (d), (f), (h) are at t=T.

right boundary of the domain, whereas in the elastic composite, the reflected and incoming wave superimpose, creating an intensified wave field near the boundary. Within the stop band, the traveling shear wave is subject to significant attenuation. Figure 12 shows the displacement contour of viscoelastic and elastic composites, at loading frequencies f = 18 kHz and f = 35 kHz, respectively, within the stop band. Compared to wave propagating in the elastic composite, more significant attenuation is observed in viscoelastic composite, featured by the almost complete attenuation after two wavelength distance from the left boundary. In Fig. 11 and 12, the overall wave pattern predicted by STNHM matches with the direct simulation for both viscoelastic and elastic composites.

The wave transmission spectrum provides a more intuitive way to identify and understand wave propagation characteristics in viscoelastic composites in terms of loading frequency. We consider wave propagation in a viscoelastic composite composed of a row of 20 microstructures, as sketched in Fig. 2(c). Sinusoidal load is applied to the left boundary with a range of frequencies, $f \in [1, 36]$ kHz, and the displacement is measured 5 unit cells away from the left boundary, i.e., $x_1 = 0.1$ m. Two viscoelastic microstructures are investigated, i.e., matrix-fiber and bi-material layered. The bi-material layered microstructure is shown in Fig. 2(a) with steel phase replaced by the viscoelastic phase for which, the material properties are provided earlier in this section. Figure 13 shows the normalized maximum transmitted wave amplitude, U_t/M , at the measurement point for the considered two microstructures. As CHM does not capture wave dispersion due to microstructures, it characterizes the viscoelastic dissipation, which



Figure 12: Displacement contours of viscoelastic and elastic composites at t=T for loading frequencies within the stop band. (a), (c) are the reference, and (b), (d) are results of STNHM.

monotonically increases as a function of loading frequency. At low frequency, only viscoelastic dissipation causes wave attenuation, therefore, STNHM and CHM predict identical wave transmission. As frequency increases, dispersion induced by material heterogeneity enhances wave attenuation. Formation of the stop band is featured by the dip around f = 13 kHz, contributing significant wave attenuation. STNHM predicts the transmitted wave amplitude up to f = 22 kHz and f = 31 kHz respectively for matrix-fiber and layered viscoelastic composites, where the wavelength is about 1.5 times of the size of microstructure. The second pass band starts to form at these frequencies featured by the bump that covers about 5 kHz frequency range. Frequencies higher than 36 kHz are not plotted, because the wavelength is smaller than the size of microstructure within the matrix-fiber composite and asymptotic homogenization is no longer applicable. Comparing Figs. 13(a) and (b), the stop band of viscoelastic composite with bi-material layered microstructure covers a much wider frequency range than the one with matrix-fiber microstructure. Moreover, more pronounced wave attenuation is observed within the stop band.

7 Conclusion

This manuscript presented a spatial-temporal nonlocal homogenization model for transient anti-plane shear wave propagation in viscoelastic composite. The proposed model is derived based on asymptotic expansions of up to the eighth order. The homogenized momentum balance equation has higher order gradient terms, i.e., fourth order spatial, fourth order temporal and mixed spatial-temporal derivatives. Transient shear wave propagation is evaluated by a reduced order homogenization model that shares the same equation structure as the classical homogenization model, while the nonlocal characteristics are retained through the nonlocal effective stiffness. The major conclusions are summarized as follows:



Figure 13: Maximum transmitted wave amplitude after 5 unit cells. (a) Matrix-fiber viscoelastic composite. (b) Bi-material layered viscoelastic composite.

(1) The spatial-temporal nonlocal homogenization model captures wave dispersion within the first pass band, the initiation of the first stop band and wave attenuation within it for both elastic and viscoelastic composites.

(2) A major contribution of the proposed model is that all model parameters are computed uniquely from the microstructural equilibrium and dependent on the material properties and microstructural geometry only. The computation of model parameters is performed as an off-line process uncoupled from the macroscale solution.

(3) Higher order asymptotic expansions and the derived temporal nonlocal term is critical in extending the applicability of the asymptotic homogenization to shorter wavelength regime, nevertheless, it cannot go beyond the limit of separation of scales.

For future work, the following challenges will be addressed. First, this framework will be extended to model in-plane wave propagation in viscoelastic composites, where the vector field displacement needs to be investigated, instead of the scalar field as in the current work. Second, as viscoelastic dissipation naturally introduces temperature rise when the material is subject to high frequency excitation, incorporation of heating effect in the viscoelastic constitutive model [31, 24] is important towards a better understanding of wave dispersion and attenuation in viscoelastic composites. Third, moderate material property contrast is assumed in the current work, the generalization of the proposed model to consider high property contrast would extend its capability to acoustic metamaterials, where local resonance plays a major role in wave dispersion and attenuation. Extending the proposed model beyond the limit of separation of scales is challenging if not impossible. In that regime, multiscale methods without assuming separation of scales, e.g., variational multiscale enrichment method [51, 67, 68], could be more appropriate for modeling the transient dynamics of phononic crystals and acoustic metamaterials.

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The derivation of macroscale momentum balance equations at $O(\zeta^4)$ and $O(\zeta^2)$ are provided in this appendix.

Equation 31e is rewritten as:

$$\rho_0 s^2 U^{(4)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(4)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) \right) U_{,klmn}^{(2)} + \nu(s) A^{(1)}(s) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) U_{,klmn}^{(2)} - \nu(s) A^{(1)}(s) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + D_{(klmnpq)}^{(4)}(s) U_{,klmnpq}^{(0)}$$
(A.1)

Multiplying Eq. 31c with $\rho_0 s^2$ and considering Eq. 31a:

$$\rho_0^2 s^4 U^{(2)}(\mathbf{x},s) - \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} = \rho_0 s^2 D_{(klmn)}^{(2)}(s) U_{,klmn}^{(0)} = D_{(klmn)}^{(2)}(s) D_{pq}^{(0)}(s) U_{,klmnpq}^{(0)}$$
(A.2)

we recall Eq. 37 and plug Eq. A.2 into Eq. A.1:

$$\rho_0 s^2 U^{(4)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(4)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) \right) U_{,klmn}^{(2)} + \nu(s) A^{(1)}(s) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + A^{(2)}(s) \left(\rho_0^2 s^4 U^{(2)} - \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} \right) - \nu(s) A^{(1)}(s) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) U_{,klmn}^{(2)}$$
(A.3)

Taking two spatial derivatives of Eq. 31c and substituting the resulting expression for the last two terms of Eq. A.3 and collecting terms:

$$\rho_0 s^2 U^{(4)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(4)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) \right) U_{,klmn}^{(2)} + \left(\nu(s) A^{(1)}(s) - A^{(2)}(s) \right) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + A^{(2)}(s) \rho_0^2 s^4 U^{(2)} - \nu(s) A^{(1)}(s) D_{kl}^{(0)} D_{mnpq}^{(2)}(s) U_{,klmnpq}^{(0)}$$
(A.4)

Substituting Eq. 31a into the last term of Eq. A.4 and plugging Eq. 31c into the resulting expression, we obtain the momentum balance equation at $O(\zeta^4)$:

$$\rho_0 s^2 U^{(4)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(4)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) \right) U_{,klmn}^{(2)} + \left(2\nu(s) A^{(1)}(s) - A^{(2)}(s) \right) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(2)} + \left(A^{(2)}(s) - \nu(s) A^{(1)}(s) \right) \rho_0^2 s^4 U^{(2)}$$
(A.5)

In view of Eq. 31a, the momentum balance equation at $O(\zeta^2)$ is obtained by rewriting Eq. 31c:

$$\rho_0 s^2 U^{(2)}(\mathbf{x}, s) - D_{kl}^{(0)}(s) U_{,kl}^{(2)} = \left(D_{(klmn)}^{(2)}(s) - \nu(s) A^{(1)}(s) D_{kl}^{(0)}(s) D_{mn}^{(0)}(s) \right) U_{,klmn}^{(0)} + \left(2\nu(s) A^{(1)}(s) - A^{(2)}(s) \right) \rho_0 s^2 D_{kl}^{(0)}(s) U_{,kl}^{(0)} + \left(A^{(2)}(s) - \nu(s) A^{(1)}(s) \right) \rho_0^2 s^4 U^{(0)}$$
(A.6)

Appendix B Laplace domain stability analysis for infinitely long wave

The stability of Eq. 43 is analyzed by moving the nonlocal terms to the left hand side and applying a spatial-harmonic-time-impulse load at the right hand side, which is expressed as $Fe^{i(k_1x_1+k_2x_2)}$ in the Laplace domain. Assuming the spatial response is harmonic, $U(s) = U_0(s)e^{i(k_1x_1+k_2x_2)}$, substituting the impulse load expression and spatial-harmonic displacement into Eq. 43 and using the polar representation of the wave vector, the transfer function is written as:

$$T(s) = \frac{U_0(s)}{F} = \frac{-1}{A(s,\theta)k^4 + B(s,\theta)k^2 + C(s,\theta)}$$
(B.1)

where, $A(s,\theta)$, $B(s,\theta)$, $C(s,\theta)$ are given in Eq. 46a-c. Taking the infinitely long wave limit, i.e., $k \to 0$, the poles of the rational transfer function are found by $C(s,\theta) = 0$:

$$s^{\{1,2\}} = 0, \quad s^{\{3,4\}} = \pm \sqrt{\frac{1}{\alpha^{(3)}(s)\rho_0}}$$
 (B.2)

In the complex plane, any pole of the transfer function that has a positive real part results in exponentially unstable response [21]. Therefore, in order to avoid exponential instability, $s^{\{3,4\}}$ must have real part to be zero. According to the rule of computing square root of complex number [48], Eq. 50b has to be satisfied.

 $\textbf{Appendix C} \quad \textbf{Expressions for } G^{(n-1)e}_{(\cdot)}(\mathbf{y},s) \textbf{ and } Q^{(n-1)e}_{(\cdot)}(\mathbf{y},s)$

The expression for $G_{klm}^{(1)e}(\mathbf{y},s)$ is:

$$G_{klm}^{(1)e}(\mathbf{y},s) = G(\mathbf{y},s) \left[\sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_k^{(1)[B]}(s) \delta_{lm} + \sum_{B=1}^{M_e} B_m^{[B]}(\mathbf{y}) H_{(kl)}^{(2)[B]}(s) \right]$$
(C.1)

According to Eq. 21, the symmetric part is computed as summation of components of all permutations divided by the total number of permutations:

$$G_{(klm)}^{(1)e}(\mathbf{y},s) = \frac{1}{6} \left(G_{klm}^{(1)e}(\mathbf{y},s) + G_{lkm}^{(1)e}(\mathbf{y},s) + G_{lmk}^{(1)e}(\mathbf{y},s) + G_{mlk}^{(1)e}(\mathbf{y},s) + G_{mlk}^{(1)e}(\mathbf{y},s) + G_{mlk}^{(1)e}(\mathbf{y},s) + G_{mkl}^{(1)e}(\mathbf{y},s) \right)$$
(C.2)

The expression for $Q_{klm}^{(1)e}(\mathbf{y},s)$ is:

$$Q_{klm}^{(1)e}(\mathbf{y},s) = \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_k^{(1)e[B]}(s) D_{(lm)}^{(0)}(s)$$
(C.3)

The symmetric part of $Q_{klm}^{(1)e}(\mathbf{y},s)$ is computed by replacing $G_{(\cdot)}^{(1)e}(\mathbf{y},s)$ with $Q_{(\cdot)}^{(1)e}(\mathbf{y},s)$ in Eq. C.2. We provide the expressions for $G_{(\cdot)}^{(n-1)e}(\mathbf{y},s)$ and $Q_{(\cdot)}^{(n-1)e}(\mathbf{y},s)$ in what follows, and the symmetric part is computed by Eq. 21.

$$G_{klmn}^{(2)e}(\mathbf{y},s) = G(\mathbf{y},s) \left[\sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(kl)}^{(2)[B]}(s) \delta_{mn} + \sum_{B=1}^{M_e} B_n^{[B]}(\mathbf{y}) H_{(klm)}^{(3)[B]}(s) \right]$$
(C.4)

$$Q_{klmn}^{(2)e}(\mathbf{y},s) = \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(kl)}^{(2)e[B]}(s) D_{(mn)}^{(0)}(s) + D_{(klmn)}^{(2)}(s)$$
(C.5)

$$G_{klmnp}^{(3)e}(\mathbf{y},s) = G(\mathbf{y},s) \left[\sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klm)}^{(3)[B]}(s) \delta_{np} + \sum_{B=1}^{M_e} B_p^{[B]}(\mathbf{y}) H_{(klmn)}^{(4)[B]}(s) \right]$$
(C.6)

$$Q_{klmnp}^{(3)e}(\mathbf{y},s) = \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klm)}^{(3)e[B]}(s) D_{(nq)}^{(0)}(s) + \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_k^{(1)e[B]}(s) D_{(lmnp)}^{(2)}(s)$$
(C.7)

$$G_{klmnpq}^{(4)e}(\mathbf{y},s) = G(\mathbf{y},s) \left[\sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klmn)}^{(4)[B]}(s) \delta_{pq} + \sum_{B=1}^{M_e} B_q^{[B]}(\mathbf{y}) H_{(klmnp)}^{(5)[B]}(s) \right]$$
(C.8)

$$Q_{klmnpq}^{(4)e}(\mathbf{y},s) = \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klmn)}^{(4)e[B]}(s) D_{(pq)}^{(0)}(s) + \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(kl)}^{(2)e[B]}(s) D_{(mnpq)}^{(2)}(s) + D_{(klmnpq)}^{(4)}(s) \quad (C.9)$$

$$G_{klmnpqr}^{(5)e}(\mathbf{y},s) = G(\mathbf{y},s) \left[\sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klmnp)}^{(5)[B]}(s) \delta_{qr} + \sum_{B=1}^{M_e} B_r^{[B]}(\mathbf{y}) H_{(klmnpq)}^{(6)[B]}(s) \right]$$
(C.10)

$$Q_{klmnpqr}^{(5)e}(\mathbf{y},s) = \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klmnp)}^{(5)e[B]}(s) D_{(qr)}^{(0)}(s) + \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_{(klm)}^{(3)e[B]}(s) D_{(npqr)}^{(2)}(s) + \sum_{B=1}^{M_e} N^{[B]}(\mathbf{y}) H_k^{(1)e[B]}(s) D_{(lmnpqr)}^{(4)}(s) \quad (C.11)$$

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