Dual Random Lattice Modeling of Backward Erosion Piping

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Abstract

This manuscript provides a novel random lattice model to simulate the progressive degradation of soil embankments induced by the backward erosion piping (BEP) phenomenon. The progressive evolution of the piping process is described by expressing apparent diffusivity of the soil as a function of the local hydraulic gradient. In order to accurately compute local field gradients, a dual lattice approach that evaluates the response on both Delaunay and Voronoi nodes is formulated. The response computed at the Delaunay nodes is then employed to augment field gradient computation on the Voronoi grid. The proposed dual lattice model is numerically verified and validated by comparing numerical results with BEP experiments available in the literature.

Keywords: dual random lattice model, lattice modeling, backward erosion piping, internal erosion

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1 1. Introduction

Backward erosion piping (BEP) is an internal erosion mechanism that occurs when soil is gradually eroded from the foundation of a structure, creating a pipe that connects the upstream side with an exit condition generated by localized damage on the downstream side of the structure. The process is driven by seepage forces that progressively erode cohesionless material from the foundation of the structure. This phenomenon can have catastrophic consequences on dams and levees [7]. The topic is of interest, since internal erosion mechanisms are considered responsible for nearly half of all embankment dam failures and accidents [17]. BEP alone has been identified as the cause of nearly one third of all piping failures occurred in the last century [38].

In view of its criticality on safety of earthen embankments, the BEP phe-nomenon has been the focus of investigations for over a hundred years. The classical studies of BEP failure led to a number of design approaches, including the Bligh's method [4, 29], Terzaghi's method [49] and the blanket theory [52], among others. Sellmeijer [44] presented an analytical model which, based on the-oretical considerations, distinguishes between critical and non-critical conditions in the presence of sand boils. This model was the first to take into account theo-retical considerations of the BEP phenomenon, since the underground water flow equations were evaluated for all possible pipe lengths, furnishing design rules that were subsequently tuned by means of a multivariate analysis of a large number of experiments. Schmertmann [42] presented an empirical approach to assess the safety factor against piping mechanisms, incorporating several theoretically evaluated correction factors that take into account the specific geometrical and hydraulic features of the structure (e.g., grain size, inclination, layer thickness,

total pipe length). In this method, safety against piping phenomena is assessed by means of the point (local) gradients, highlighting the role of the hydraulic gradient intensity as the driving force for backward erosion of embankments. Foster et al. [16] developed an empirical procedure based on a historical database of 1,462 embankment dams. This procedure is based on the characteristics of the structure (i.e. the presence of filters, type of foundation, soil geology, core soil type and compaction, seepage observations and previous monitoring and surveillance) and is used to assess the likelihood of erosion.

BEP is a complex phenomenon, influenced by sediment transport, underground water flow, hydrodynamics and soil mechanics. Several authors performed a range of experiments to understand the driving mechanisms of BEP. Bendahmane et al. [3] proposed the use of an experimental device composed of three modified triaxial cells that are coupled to two air-water cells, capable of applying different levels of confinement to a soil sample subjected to a standard erosion test, allowing them to assess the influence of the percentage of clay in the soil, confining pressure and global hydraulic gradient. Van Beek et al. [54] performed small-, medium- and full-scale experiments on levee systems in order to validate Sellmeijer's models. The authors observed four phases: main seepage, backward erosion, widening of the pipe and ultimate levee failure, highlighting how failures take place in a short period of time, leading small sand boils to complete levee failure. Sellmeijer et al. [43] performed small-scale tests and analyzed the results by means of a multivariate regression in order to identify the influence of each variable present in a semi-theoretical model. Richards and Reddy [37] employed a triaxial piping test apparatus in order to study the initiation of piping in soils subjected to various levels of confining stress and seepage conditions. Fleshman and Rice [15] measured

the hydraulic conditions in soil samples during the development of piping under vertical flow conditions. Ke and Takahashi [26] experimentally investigated the behavior of non-cohesive soils during the onset and progression of internal erosion, while keeping track of the hydraulic characteristics of the soil as well as the change in mechanical behavior. Sharif et al. [45] employed laboratory flume experiments on mixtures of sand, silt and clay with different compaction rates and evaluated the effects of the erosion process visually by means of an image processing technique. A number of numerical approaches has also been proposed to study BEP. These approaches are classified in three categories [57]: 1) porous media flow models in which erosion is idealized by controlling the permeability of the elements in the

discretization of the domain (e.g. [39, 56]); 2) discrete element method (DEM) to describe soil deformation coupled with continuum description of the water flow (e.g. [11, 48, 58, 63]); and 3) multiphase flow modeling in which the fluidized particle density is evaluated by means of appropriate constitutive relationships and coupled to the flow description by kinematic constraints (e.g. [51, 18, 32, 62, 57, 30]). In the first approach, the permeability within the eroded zone is increased to simulate the progressive erosion process. The permeability is a function of a threshold value of a control variable (e.g. flow velocity, shear stress, hydraulic gradient), and is increased by a prescribed factor when the threshold value is reached at a material point within the domain. The second approach represents the behavior of the system by coupling the response of the soil phase (typically obtained from a DEM simulation) and the flow phase (typically obtained using finite element or finite volume models). This approach is computationally expensive, and often requires coarse graining of the soil phase to render the DEM approach feasible. In the third approach the potential for soil erosion is expressed

⁷⁶ by means of constitutive relations that allow for the evaluation of the fluidized
⁷⁷ particles and it is coupled to the hydraulic flow by imposing that the velocity of
⁷⁸ the particles and the fluid is equal at every point in the domain.

This study adopts the first approach of simulating the backward erosion process using a new dual random lattice modeling approach. In recent years, random lattice models have emerged as an attractive alternative to continuum approaches to modeling various types of civil engineering problems [2, 5, 12, 14, 21, 23, 28, 33, 41, 13], including transport problems [1, 19, 20, 34, 40]. The basic idea of random lattice modeling dates back to the pioneering work of Hrennikoff [24]. In this approach, the response of a 3-dimensional continuous medium is evaluated on a discrete network (or lattice) of uniaxial elements, whose connectivity and inertial properties are retrieved from a Delaunay/Voronoi tessellation of the domain. This main advantages of this approach are: 1) the solution of the equations governing the problem is evaluated on simple 1-dimensional lattice elements; 2) the discrete nature of the model is inherently capable of representing localized phenomena such as localized damage or erosion; and 3) the random spatial arrangement of the lattice nodes introduces mesh independence of the simulated response [6]. The use of more traditional continuum approaches such as finite elements for erosion problems is challenging because of the difficulty in describing localized piping phenomena with a preferred direction and results in the need to force the pipe path in the horizontal plan (e.g. [39]) or mesh-induced biasing on the erosion paths. We demonstrate that the proposed approach is advantageous in alleviating the above-mentioned challenges.

⁹⁹ This manuscript presents a novel dual random lattice modeling approach for ¹⁰⁰ the simulation of the degradation processes that occur in soil systems due to

internal erosion. A key novelty and main contribution of this work is that the diffusion equation governing the underground water flow is independently solved on two sets of lattices. The lattices are constructed by exploiting the duality between the Delaunay and the Voronoi tessellations of a given domain [35]. To the best of the authors' knowledge, the present manuscript is the first attempt at using a dual lattice approach in the solution of the transport problem. An important capability of the proposed approach is the accurate calculation of the hydraulic gradient to enforce the stability criterion. Based on the values of the hydraulic head scalar field defined on both lattice systems, a scheme for the gradient calculation is obtained by exploiting the geometrical features of the Dual Random Lattice Model. The proposed approach is numerically verified by comparing the results of a mass transport problem with the closed form solution. The gradient calculation procedure is then tested against different polynomial functions to assess its accuracy. The procedure for the simulation of BEP in soil systems is compared with the experimental results reported in [10] and [39].

2. Governing Equations

The erosion process in the embankment is idealized using a nonlinear diffusion equation [22, 39, 56]. Consider the domain of a saturated earthen embankment denoted as Ω :

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nabla \cdot (D(h(\mathbf{x},t))\nabla h(\mathbf{x},t)) \qquad \mathbf{x} \in \mathbf{\Omega}, t \in (0,T)$$
(1)

where *h* represents the hydraulic head field and *T* the total time. The domain is subject to the following boundary conditions:

$$h = h_B(t) \qquad \text{on } \Gamma_b \subset \partial \Omega$$

$$\mathbf{q} \equiv -D \frac{\partial h}{\partial \mathbf{n}} = \mathbf{q}_B \qquad \text{on } \Gamma_q \subset \partial \Omega$$
(2)

with $\Gamma_b \cap \Gamma_q = \emptyset$. **q** is the outward flux orthogonal to the domain boundary of normal **n**, **q**_B the prescribed boundary flux and h_B is the time-dependent prescribed hydraulic head at the boundary. The initial conditions at t = 0 are:

$$h(\mathbf{x}, 0) = h_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega$$
$$k(\mathbf{x}, 0) = \begin{cases} k_0 & \mathbf{x} \in \Omega \setminus \Omega_{ini} \\ m_p k_0 & \mathbf{x} \in \Omega_{ini} \end{cases}$$
(3)

where $h_0(\mathbf{x})$ is a known function, Ω_{ini} is the portion of the domain where BEP initiation is imposed, k_0 is the initial value of the soil permeability and m_p is the amplification factor for the conductivity within the eroded zone.

The source of nonlinearity in the governing equation is that the soil permeability is a function of the hydraulic head h. According to [39], the hydraulic gradient ican be used to characterize the available flow energy. The following constitutive behavior is used in this study:

$$D(h(\mathbf{x},t)) = \frac{k(h(\mathbf{x},t))\rho g}{\mu S_s}$$
(4)

132 with:

$$k(h(\mathbf{x},t)) = k_0 \qquad \text{if } |\mathbf{i}| < i_{crit}$$

$$k(h(\mathbf{x},t)) = m_p k_0 \qquad \text{if } |\mathbf{i}| \ge i_{crit}$$
(5)

where **i** is the hydraulic gradient, i_{crit} its critical value at the onset of erosion, μ the water viscosity, ρ its volume mass, g is gravity and S_s the specific storage. Equation 5 represents the relationship between permeability and hydraulic gradient using a Heaviside function. Other, nonlinear forms of this constitutive relationship have also been proposed based on experimental investigations [8, 9].

The value of i_{crit} can either be evaluated by means of theoretical considerations [27, 25], or by experiments [53]. In this study, a numerical procedure for the evaluation of the critical gradient, proposed by [39], is used and described in the following sections, together with the definition of the other two parameters required to define the initial conditions in the domain: the spatial arrangement of the initiation zone, and the value of the conductivity amplification factor m_p .

3. Dual Random Lattice Model

The proposed approach exploits the geometrical features of the Delaunay and Voronoi tessellations of a 3-dimensional domain. In three dimensions, given a pointset (a set of points that lie within the domain or on the domain boundaries), the Delaunay tessellation is a triangulation such that none of the points lies inside the circumsphere of any tetrahedron in the triangulation. The Voronoi diagram of a given pointset is the geometrical tessellation that associates to each node a polyhedral cell composed of all the points that are closer to that point than any other in the set [35].

The proposed approach operates on the weak form of the initial boundary value problem. Employing the classical assumptions of smoothness and continuity of the hydraulic head field, the weak form of the problem is expressed as:



Figure 1: Graphical interpretation of the dual random lattice model approach

$$-\int_{\Omega} \left(D(\mathbf{x}, t) \nabla N_i \cdot \nabla N_j h \right) d\Omega + \int_{\Omega} N_i N_j \frac{\partial h}{\partial t} d\Omega - \int_{\Gamma_q} N_i q d\Omega_q = 0 \quad (6)$$

where N_i and N_j are the shape functions. Let $\mathbf{\Phi} = \{\phi_i\}_{i=1}^{n_D}$ denote a set of points (i.e., the Delaunay point cloud composed of n_D nodes) randomly distributed within the closure of the domain Ω (i.e., $\mathbf{x}(\phi_i) \in \overline{\Omega}$; $1 \leq i \leq n_D$). Ω^h denotes the approximation of the domain Ω generated by the Delaunay triangulation of Ω using the point cloud Φ . We further consider the Voronoi tessellation of the domain that is dual to the Delaunay triangulation, resulting in the Voronoi point cloud $\tilde{\Phi} = {\{\tilde{\phi}_i\}}_{i=1}^{n_V}$ (with n_V being the number of Voronoi points), as illustrated in Fig. 1. Employing the Delaunay triangulation, $\mathbf{\Omega}^h$ is decomposed into a set of effective volumes (areas in 2-D) $\{\Delta_e\}_{e=1}^{n_{el}^D}$, where each effective volume is associated with an element within the Delaunay lattice, h_e . A similar decomposition of the

domain is performed using the Voronoi tessellation $\{\tilde{\Delta}_e\}_{e=1}^{n_{el}^V}$, where an arbitrary $\tilde{\Delta}_e$ is associated with one Voronoi lattice element \tilde{h}_e .

We proceed with two separate discretizations of the hydraulic head field based on the two lattices:

$$h(\mathbf{x},t) = \sum_{a=1}^{n_D} N_a(\mathbf{x}) h_a(t); \qquad \tilde{h}(\mathbf{x},t) = \sum_{a=1}^{n_V} \tilde{N}_a(\mathbf{x}) \tilde{h}_a(t)$$
(7)

¹⁷⁰ in which $N_a(\mathbf{x})$ and $\tilde{N}_a(\mathbf{x})$ are piecewise linear, C^{-1} shape functions. The shape ¹⁷¹ functions have compact support over the Delaunay (or the Voronoi) elements with ¹⁷² direct connectivity with node *i*. We emphasize that a Delaunay (or a Voronoi) ¹⁷³ "element" and "effective volume" are different and the difference is illustrated in ¹⁷⁴ Fig. 1.

Employing Bubnov-Galerkin approach and substituting the hydraulic head field discretization into the weak form (Eq. 6):

$$-\sum_{b=1}^{n_D} \int_{\Omega^h} D(\mathbf{x}, t) \nabla N_a \cdot \nabla N_b d\Omega h_b(t) + \sum_{b=1}^{n_D} \int_{\Omega^h} N_a(\mathbf{x}) N_b(\mathbf{x}) d\Omega \frac{dh_b}{dt} + \int_{\Gamma_q^h} N_a(\mathbf{x}) \hat{\mathbf{q}} \cdot \mathbf{n} d\Gamma = 0 \qquad \forall b = 1, 2, ..., n_D$$
(8)

where Γ_q^h denotes the Delaunay approximation of the domain boundary Γ_q . A similar discretization expression is obtained for the Voronoi lattice, which is skipped for brevity. We proceed with the discretization of the domain integrals over the effective volumes of the Delaunay and Voronoi lattice. Within each effective volume the two shape functions are expressed as:



Figure 2: Typical arrangement of a Delaunay (left) and a Voronoi (right) edge, their dual resisting areas and effective volume

$$N_A^e = (\mathbf{x}_B - \mathbf{x}) \cdot \mathbf{s}; \qquad N_B^e = (\mathbf{x} - \mathbf{x}_A) \cdot \mathbf{s}; \qquad \mathbf{x} \in \Delta_e$$
(9)

¹⁸² in which, **s** denotes the unit vector along the lattice element ($\mathbf{s} = (\mathbf{x}_B - \mathbf{x}_A)/l_e$); ¹⁸³ l_e is the length of the lattice element; \mathbf{x}_A and \mathbf{x}_B are the coordinates of the two ¹⁸⁴ endpoints *A* and *B*. The shape functions are linear within the effective volume and ¹⁸⁵ admit the domain integrals defined above. Furthermore, the volume integrals can ¹⁸⁶ be expressed as line integrals over the lattice element; e.g.:

$$\int_{\Delta_e} D(\mathbf{x}, t) \nabla N_a \cdot \nabla N_b d\mathbf{\Omega} = A_e \int_{l_e} D(\mathbf{x}, t) \frac{d\hat{N}_a}{dx} \frac{d\hat{N}_b}{dx} dx$$
(10)

where, A_e is the cross-sectional area pertaining to the lattice element, \hat{N}_a denotes the shape functions defined over the lattice elements and x the local position along the lattice element:

$$\hat{N}_{A}^{e} = 1 - x/l_{e}; \qquad \hat{N}_{B}^{e} = x/l_{e}$$
 (11)

Remark 1: A number of differences exist regarding the topology of the Delau-nay and Voronoi lattices for 2-D and 3-D implementation of the proposed approach. In 2-D, the duality between the two tessellations is such that there exists a Voronoi lattice element orthogonal to each lattice element in the Delaunay grid and vice-versa (as shown in Fig. 1). The effective areas (the two triangles shown) of the two elements are one and the same. In 3-D, the effective volume of a Delaunay lattice element consists of a tetrahedron, while that of the Voronoi element has polygonal base as illustrated in Fig. 2. In a 2-D analysis, the number of lattice elements in the dual lattices are equal, whereas this is not the case in 3-D.

Remark 2: Previous investigations focused on characterization of transport behavior using either the Delaunay [40] or the Voronoi lattice [19], but not both. In fact, the response approximations predicted by the two lattices are complementary. The local response gradients predicted by the lattice shape functions vanish along the direction normal to the lattice element. The dual lattice approximates the gradi-ent field at the orthogonal direction, augmenting the local gradient approximation. The proposed approach leverages this observation for accurate computation of the hydraulic gradients.



$$\Psi \equiv \mathbf{M} \frac{d\mathbf{h}}{dt} + \mathbf{K}(h)\mathbf{h} - \mathbf{f} = 0$$
(12)

where **M** is the global mass matrix, **h** the hydraulic head vector, $\mathbf{K}(h)$ the global diffusion matrix, **f** the force vector.

The expressions for the relevant element matrices are:

$$\mathbf{K}_{e} = \int_{\Omega} \mathbf{B}^{T} D \mathbf{B} d\Omega = \int_{l_{e}} DA_{e}(\mathbf{x}) \mathbf{B}^{T} \mathbf{B} dx = \frac{DA_{e}^{*}}{l_{e}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\mathbf{M}_{e} = \int_{\Omega} \mathbf{N}^{T} \mathbf{N} d\Omega = \int_{l_{e}} A_{e}(\mathbf{x}) \mathbf{N}^{T} \mathbf{N} dx = \frac{V_{e}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
(13)
$$\mathbf{f}_{e} = -\int_{\Omega_{q}} \mathbf{q} \mathbf{N}^{T} d\Omega_{q} = \begin{bmatrix} -q_{I} A_{e} \\ -q_{J} A_{e} \end{bmatrix}$$

where, V_e is the value of the effective volume of the current element, **N** is the vector containing the element shape functions and **B** the vector of their derivatives.

It is important to note that the forms of the element matrices in Eq. 13 are identical to those reported in [19, 20, 40]. In these works, a correction parameter was used in the calculation of the matrix M to ensure mass conservation. In the present formulation, by exploiting the concept of dual effective areas and volume, the exact value of V_e is calculated by connecting the points defining the resisting area with the two ends of the lattice elements (see Fig. 2). The expression for M is therefore consistently derived from the weak form without the need of a correction parameter, still satisfying conservation of mass.

²²¹ Discretization in time is performed by means of the Crank-Nicolson method [31]:

$$\mathbf{M}\frac{\mathbf{h}^{n+1} - \mathbf{h}^n}{\Delta t} + \frac{1}{2}\left(\mathbf{K}^{n+1}\mathbf{h}^{n+1} + \mathbf{K}^n\mathbf{h}^n - \mathbf{f}^{n+1} - \mathbf{f}^n\right) = 0$$
(14)

where Δt is the time step size and superscripts indicate the time step count (1 $\leq n \leq n_s t$, with $n_s t$ the total number of steps). Eq. 14 yields:

$$\left(\mathbf{M} + \frac{1}{2}\mathbf{K}^{n+1}\Delta t\right)\mathbf{h}^{n+1} - \left(\mathbf{M} - \frac{1}{2}\mathbf{K}^n\Delta t\right)\mathbf{h}^n - \frac{1}{2}\left(\mathbf{f}^{n+1} + \mathbf{f}^n\right) = 0$$
(15)

The Crank-Nicolson method is a semi-implicit scheme and unconditionally stable [50]. Due to the presence of decaying spurious oscillation in the solution of the transient problem, the maximum allowable time step size is set to:

$$\Delta t = \frac{l_{min}^2}{2D} \tag{16}$$

where l_{min} is the minimum value of the lattice element length in the mesh.

In this study, the dual random lattice model is applied to the simulation of BEP. The erosion process typically occurs in cohesionless materials which are overlain by relatively impervious core of the embankment. The modeling approach stated above is general and applicable to arbitrary materials that undergo transport processes.

233 3.1. Computational Implementation

The proposed dual random lattice model has been implemented to simulate the progressive degradation of saturated soil media. In what follows, the implementation steps for the construction of the dual lattice model, imposition of the boundary conditions, computation of the field gradient, and the implementation algorithm are provided.

The generation of a random lattice model requires the following steps (see Fig. 3):

1. Construction of the 3-D model of the domain;

242 2. Insertion of randomly arranged nodal sites in the domain;

²⁴³ 3. Delaunay tetrahedralization of the previously defined pointset;

4. Voronoi tessellation of the same set of points.



Figure 3: Graphical interpretation of the steps required for the construction of a dual random lattice model: a) 3-dimensional domain, b) random points generation, c) Delaunay tetrahedralization, d) Voronoi diagram

The three-dimensional model of the domain is first constructed by means of a CAD software. The domain is then saturated with randomly generated points while enforcing a minimum distance criterion [59]. The Delaunay tessellation is constructed to retrieve the connectivity between the previously defined pointset. The tetrahedralization of non-convex domains is performed using the TetGen li-brary [47]. Finally, the Voronoi diagram is constructed by connecting the centers of the circumspheres for every tetrahedron obtained from the Delaunay tetrahedral-ization. Both the Delaunay and Voronoi edges are used to create two independent lattice systems. The duality that exists between the two tessellations allows for the definition of the resisting areas and effective volume of all the elements in the domain, as well as granting the interesting characteristic that every edge has an orthogonal facet in the dual diagram (see Fig. 2).

257 3.2. Boundary Meshing

Special attention must be devoted to the discretization of the domain boundaries. Since the boundary conditions are independently applied on the two lattice systems, the goal is to obtain external surface discretizations on both the Delaunay and Voronoi tessellation, while retaining the duality of the two diagrams. This is



Figure 4: Procedure for the meshing of external surfaces: retained and discarded parts of a Voronoi facet dual to a Delaunay element

achieved in the following way: 1) External surfaces are first saturated with points while enforcing the minimum allowed distance; 2) internal points are added in the domain while enforcing the same minimum distance criterion; 3) the Voronoi diagram for the set of points is constructed; 4) all the tetrahedra that lie on an exter-nal face of the domain are flagged as 'external'; 5) the circumcenter pertaining to external tetrahedra are mirrored with respect to the external surface they represent; 6) the part of the resulting Voronoi diagram that lies outside of the domain is discarded while the internal one is retained.

The discretization of the domain boundary is performed as illustrated in Fig. 4, which shows a Delaunay lattice pertaining to an external surface of the boundary to be meshed. While the proposed procedure is derived from [19] and [59], it does not require the mirroring of the entire pointset in order to construct the Voronoi tessellation.

The random lattice modeling approach does not have any difficulty in representing Dirichlet boundary conditions. Some difficulties exist in the representation of Neumann boundary conditions. In the problem considered in this manuscript,



Figure 5: A Voronoi lattice element and its dual triangular facet

the component of the prescribed flux normal to the boundary contributes to transport. The Voronoi lattice elements at the boundary are by construction orthogonal
to the surface, and therefore inherently capable of handling Neumann boundary conditions. The Delaunay lattice requires special treatment to accommodate
Neumann-type boundary conditions as described in [19].

283 3.3. Gradient Calculation

The evaluation of the transport problem using the Voronoi or the Delaunay lattice provides the values of the hydraulic head field values at the lattice nodes. This information by itself is not sufficient to accurately compute the gradient of the function. Fig. 5 illustrates the Voronoi lattice element $A_V B_V$ and its dual triangular facet $A_D B_D C_D$, obtained from the Delaunay triangulation. By solving the diffusion problem on the Voronoi assembly, it is only possible to obtain information on the variation of the h field in the **n** direction. To be able to evaluate the gradient of the hydraulic head, information on the spatial variation of the h field in the two directions **l** and **m** is needed. To do so, the component of the gradient in the **n** direction is calculated on the Voronoi element (hence, i_V), while information on

the plane orthogonal to the element (i.e. the plane of the resisting area) is obtained

²⁹⁵ from the values the function assumes at the Delaunay nodes (hence, \mathbf{i}_D).

The component \mathbf{i}_V pertaining the **n** direction is calculated by dividing the jump in hydraulic head by the length of the element:

$$\mathbf{i}_V = \mathbf{n} \frac{h_{B_V} - h_{A_V}}{l_e} \tag{17}$$

The component \mathbf{i}_D is evaluated on the triangular resisting area of the conductivity element, and calculated as:

$$\mathbf{i}_D = \frac{1}{2A_e} \mathbf{n} \times (h_{A_D} \mathbf{e}_{BC} + h_{B_D} \mathbf{e}_{CA} + h_{C_D} \mathbf{e}_{AB})$$
(18)

where \mathbf{e}_{AB} , \mathbf{e}_{BC} , \mathbf{e}_{CA} are the unit vectors defined on the facet.

³⁰¹ By summing the two contributions, the value of the gradient of the hydraulic ³⁰² head field acting in the element is obtained as:

$$\mathbf{i} = \mathbf{i}_V + \mathbf{i}_D = \mathbf{n} \frac{h_{B_V} - h_{A_V}}{l_e} + \frac{1}{2A_e} \mathbf{n} \times (h_{A_D} \mathbf{e}_{BC} + h_{B_D} \mathbf{e}_{CA} + h_{C_D} \mathbf{e}_{AB})$$
(19)

303 3.4. Numerical Procedure for the Simulation of Backward Erosion Piping

In this study, the analysis starts and predicates upon the onset of the backward erosion piping process. The onset of the piping process is introduced through the initial conditions (Eq. 3). We proceed with a three-step numerical procedure to evaluate the BEP process as described below:

 a steady-state mass transport analysis is performed to obtain hydraulic equilibrium conditions in the specimen for the given values of the imposed heads.
 No degradation zone is assumed as initial condition (i.e., only Eq. 3a is employed as initial condition). In this analysis, the permeability of the soil is assumed to be constant and does not degrade with hydraulic gradient; 2. Under the steady-state hydraulic head field conditions, the conductivity of all the lattice elements within the pipe initiation zone is amplified (i.e., Eq. 3b). The increased permeability channels more flow towards the initiation zone, increasing the hydraulic gradient in the nearby lattice elements;

3. the piping simulation is then performed by evaluating the nonlinear transport problem considering step (2) as the initial condition. At every time incre-ment, the hydraulic gradient is recalculated to compute the permeability at each lattice element. Every element undergoing a gradient higher than the critical value is flagged as piped and its conductivity is increased according to Eq. 5. In case the initial state provided in Step (2) does not introduce a local gradient that exceeds the critical value, the solution is trivial and piping does not progress through the domain.

4. Numerical Verification

This section presents the assessment of performance of the proposed model. The presented simulations are divided as follows: 1) the verification of the dual transport model is presented by solving a linear diffusion 1-dimensional problem is presented and compared with the exact solution, to show accuracy of the dual random lattice transport model; 2) accuracy of the proposed gradient calculation approach is demonstrated by constructing different dual lattice assemblies with increasing mesh density and assigning polynomial function at each node, compar-ing the numerical results for the gradient field with the exact solution; 3) results of the nonlinear transport problem defined in the previous sections, and core of the present work, are reported for the simulation of the experimental tests by Robbins [39] and De Wit et al. [10].



Figure 6: Numerical test setup for the verification of the transport model: a) schematic representation, b) dual random lattice assemblies employed

337 4.1. Transport Model Verification

The value of the local gradient is calculated using Eq. 19 by taking into account the contributions of the Delaunay and Voronoi nodes. To this end, it is important that the results obtained from the two systems are consistent with theoretical results as well as with each other. Accuracy of the calculation on the two different lattice systems is demonstrated on a simple one-dimensional filtration case (see Fig. 6) solved by means of a transient linear flow analysis, by comparison with the exact solution. The two ends of a bar are subjected to constant values of the hydraulic head, creating a seepage flow in the direction parallel to the bar axis. The values of the model parameters used in the example are reported in Table 1, together with the number of nodes and lattice elements in both the Voronoi and Delaunay lattice assemblies. The numerical solution obtained on the two lattice assemblies is compared with the theoretical solution (Fig. 7) and the L-2 norm of error is reported in Table 2. As expected, errors are lower in the Voronoi pointset, since the tessellation produces denser lattice systems (see the Voronoi/Delaunay points ratios reported in Table 4). Figure 7 illustrates that both lattices very accurately captures the flow behavior within the transient and steady-state regions of the solution. The simulated results approximate the exact solution better getting

Parameters used in the transport model verification

L	h(x=0)	h(x=1000)	Diffusivity	Delaunay Nodes	Voronoi Nodes
mm	mm	mm	mm^2/s	-	-
1000	0	100	6824	2151	11227

Table 2

Error calculation for the dual random lattice model at different time steps

Time	Delaunay Nodes	Voronoi Nodes
S	-	-
1	1.66%	1.22%
10	0.92%	0.76%
20	0.67%	0.53%
50	0.48%	0.34%

closer to the steady state, since the hydraulic head distribution tends to be linear.

356 4.2. Gradient Calculation Verification

Next, we investigate the accuracy in computing the field gradients using the proposed approach, by evaluating gradients of known fields over the random lattices. The gradient accuracy assessment is performed over the domain described in Fig. 6. We consider that the field is defined by three different polynomial functions (i.e., linear, 3rd order and 5th order). The exact form of the functions are stated in Fig. 8. The values of the functions at the corresponding positions



Figure 7: Comparison between theoretical and numerical results at different time steps: a) 1s, b) 10s, c) 20s, d) 50s



Figure 8: L-2 norm of the error on the evaluated gradient versus number of Delaunay nodes for different test functions

are assigned to the lattice point sets. The gradients of the function were then evaluated and checked against the exact solutions. Convergence as a function of mesh refinement is demonstrated in Fig. 8 by comparing the results obtained from different meshes with increased point density. The calculated values agree with theory to machine precision for the case of linear function. In the case of non-linear functions, refinement leads to an increase of the accuracy.

5. Backward Erosion Piping Simulations

The previously described dual random lattice model approach and the gradient calculation procedure were then used to simulate the progression of BEP. Results are provided following the steps reported in Section 3.4: having validated the proposed dual transport model (see previous Section), a sensitivity analysis study is performed to evaluate the influence of the model parameters on the piping initiation procedure. Piping simulations are then performed by using the so evaluated



Figure 9: Flume tests experimental setup by Robbins [39]. All dimensions in millimeters

Table 3

Soil characteristics in experiments by Robbins [39]

Туре	d_{50}	Uniformity Coefficient	Porosity	Conductivity
-	mm	-	-	mm/s
Sand	2.50	1.31	0.42	40

	1
	2
	3
	4
	5
	6
	7
	/
	8
	9
1	0
1	1
1	2
1	3
1	4
1	5
1	6
1	7
1	0
1	0
1	9
2	0
2	1
2	2
2	3
2	4
2	5
2	6
2	7
2	/
2	8
2	9
3	0
3	1
3	2
3	3
3	4
3	5
2 2	6
с С	7
с С	/
-	Ø
5	~
3	9
3 4	9 0
3 4 4	9 0 1
3 4 4 4	9 0 1 2
3 4 4 4 4	9 0 1 2 3
3 4 4 4 4 4 4	9 0 1 2 3 4
3 4 4 4 4 4 4 4 4	9 0 1 2 3 4 5
3 4 4 4 4 4 4 4 4 4 4	9 0 1 2 3 4 5 6
3 4 4 4 4 4 4 4 4 4 4 4 4 4	901234567
3 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4	901234567°
3 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4	90123456780
3 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4	90123456789
3 4 4 4 4 4 4 4 4 4 4 4 5	901234567890
3 4 4 4 4 4 4 4 4 5 5	9012345678901
3 4 4 4 4 4 4 4 4 5 5 5	90123456789012
3 4 4 4 4 4 4 4 4 4 5 5 5 5	901234567890123
3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5	9012345678901234
3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5	90123456789012345
3 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5	901234567890123456
) 3 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5	9012345678901234567
3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5	90123456789012345678
3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5	901234567890123456780
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3 3 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5	9012345678901234567890
) 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5	90123456789012345678901
) 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5	901234567890123456789012
) 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5	9012345678901234567890123
3 3 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5	90123456789012345678901234
3 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5	901234567890123456789012345
3 3 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5	901234567890123456789012345



Figure 10: Influence of the initiation zone volume on the maximum hydraulic gradient attained in the mesh

parameters and results are used to investigate the convergence properties of the approach. Accuracy of the method is demonstrated by comparing results of the BEP simulations with experimental evidence reported in [39], as well with the numerical results the author provides in his paper. The experimental campaign consisted of different specimens under saturated conditions, subjected to increasing differential heads until piping started at the downstream side. The test setup is schematically illustrated in Fig. 9 and the material parameters are reported in Table 3. Moreover, the classical tests by De Wit et al. [10] are simulated and results are compared, showing accuracy of the method.

385 5.1. Sensitivity Analysis

With respect to the setup reported in Fig. 9, different simulations have been performed in order to assess the influence of the model parameters on the evaluated

response. In particular, the sensitivity to the extension of the initiation domain Ω_{ini} and the value of the amplification factor m_p has been investigated. Fig. 10 shows the results of different simulations with varying initiation zone volume, exhibiting the expected trend of decreasing values of the maximum gradient with increasing initiation zone volume. This results are consistent with previous findings from other authors [39, 55]. The variability in the obtained maximum gradient decreases with increasing initiation zone volume. The value used in the simulations presented in the next sections is of 2×10^4 mm³. This value, selected by means of numerical considerations, is in agreement with experimental tests exhibiting pipe widths of 15 to 30 times the average grain size [55, 56].

The influence of the parameter m_p has been investigated on the same numerical setup. Different values of the parameter have been tested in order to assess the model sensitivity to the parameter. We show in Fig. 11 the variation of the maximum hydraulic gradient as a function of m_p , which exhibits a hyperbolic relationship.

403 5.2. BEP analysis and comparison to previous findings

This section presents the results of BEP simulations obtained on different sets of numerical models together with comparison with experimental findings and numerical results reported in [39]. In order to investigate the influence of the mesh density on the numerical results, 3 meshes have been constructed (see Table 4) and the obtained results compared.

As explained in detail in Section 3.4, the first phase of the procedure consists of a linear seepage analysis, in order to obtain the hydraulic head profiles in the specimen prior to the piping routine. The initiation of piping is then enforced by increasing the conductivity of the elements in a user-defined zone. The piping



Figure 11: Influence of the amplification factor m_p on the maximum hydraulic gradient attained in the mesh

Table 4

Different meshes used for the BEP simulations

Mesh	Delaunay Nodes	Voronoi Nodes	Voronoi/Delaunay Ratio	<i>i</i> _{crit}
1	1153	5684	4.93	0.6301
2	3766	19884	5.28	0.6248
3	5518	30634	5.55	0.6211



Figure 12: BEP initiation procedure: gradient contours for three meshes with increasing mesh density

simulation is run until equilibrium conditions are attained, and the maximum gradient value in the mesh is chosen as the critical gradient. The results of this step on the three different meshes are reported in Fig. 12, and indicate that the value of the so obtained critical gradient is not affected by the mesh and shows a small variation among different simulations (see Table 4).

Once the BEP process is started in the domain and the value of the critical gradient is evaluated, the piping simulation starts. The analysis is run until equi-librium conditions are attained, i.e. when no element undergoes piping for a given, user-defined number of time steps. In all the simulations reported, this number was set to 1000. Results of this phase for the 3 different meshes are reported in Fig. 13. The results from the three meshes indicate that even with a different level of detail, the numerically evaluated eroded paths are consistent and show similarities among the different mesh densities considered. The denser meshes exhibit higher tortuosity of the evaluated erosion path, due to the lattice assembly density.

In the experiments, the formation of a shallow pipe is observed, running from the downstream to the upstream side of the specimen. In [39], the constitutive law (i.e. the local gradient check) was restricted to the horizontal plane of the pipe



Figure 13: Eroded path at equilibrium in three specimens with different mesh densities

progression to avoid the erosion to grow in the direction of gravity. This step is not
needed in the current formulation, as the spatial gradient is used in the constitutive
law for the lattice elements. The discrete nature of the model is able to localize
the erosion path in the direction observed in the experiments.

Convergence of the normalized pipe progression time with mesh refinement is demonstrated by Fig. 14. As the number of nodes is increased, the evaluated response tends to a constant value of the total time BEP requires to fully develop (i.e. the time it takes for the front of the pipe to reach the upstream side).

The influence of the randomness of the computed pointset has been evaluated by generating four different models with same mesh densities, and comparing the obtained response. Fig 15 shows the results of the different simulations, which show that the overall response is consistent with eroded paths exhibiting the same global behavior, which is in agreement with the experiments. The differences in the evaluated damage patterns are due to the randomness of the lattice assembly, which guides the progression of the pipe locally when enforcing the gradient-dependent constitutive law.

The sensitivity of the BEP analysis to the value of the amplification factor m_p



Figure 14: Normalized time of BEP progression as a function of the number of Delaunay nodes

has also been investigated. Different simulations have been performed on Mesh #2 (Table 4) with different values of the parameter. Since the parameter m_p controls the contrast between the conductivity of the eroded elements and the undamaged ones, it has influence on the progression of BEP in the simulations. This influence is summarized in Fig. 16, which shows that lower values of m_p lead to evaluated pipe paths that do not extend all the way to the upstream side of the specimen.

454 5.3. Influence of the type of exit condition

In a set of physical experiments, De Wit et al. [10] investigated the influence of the type of exit conditions on the progression of BEP. The different experiments were run for plane, ditch, and hole exit conditions, while retaining all the other characteristics intact. The tests consisted of an increasing hydraulic head applied at the upstream side, until piping occurred. The findings of this study clearly showed



Figure 15: Evaluated erosion paths for four different dual random lattice models with same mesh density

Table 5

Soil characteristics in experiments by De Wit et al. [10]

Туре	d_{50}	Uniformity Coefficient	Porosity	Conductivity	Amplification Factor m_p
-	mm	-	-	mm/s	-
Sand	0.4	-	0.36	0.5	100



Figure 16: Normalized maximum pipe length as a function of the amplification factor m_p



Figure 17: Experimental setup for the experiments by De Wit et al. [10]: a) plane type,b) ditch type and c) hole type exit conditions

how an increase of the exit condition area reduces the likelihood of occurrence of
BEP and that the applied global gradient must be increased for erosion to progress.
Smaller, localized exit conditions such as holes, in fact, channel more flow towards
the head of the pipe and therefore create higher local gradients and greater potential
for erosion. Table 5 reports the mechanical parameters used in the study.

The aforementioned set of tests were simulated by means of the proposed dual random lattice modeling approach. The approach for the simulations was as follows: the model was calibrated to replicate results from the "plane" exit condition reported in [10]. An initiation zone of 8×10^3 mm³ was used, based on the grain size distribution curves reported in [10] and by assuming a pipe head width of 25 times d_{50} . The results of this first simulation, in terms of highest local hydraulic gradient in the mesh, were used in the subsequent simulations. By keeping all the mechanical parameters constant, the applied hydraulic head was increased until the local gradient matched the value obtained from the reference model. The results of this investigations in terms of global applied gradients (i.e. the difference between the upstream and downstream hydraulic head values divided by the seepage length), are reported in Fig. 18 and discussion is provided in the next section.

478 5.4. Discussion on the numerical results

The results presented in the previous section indicate that the proposed dual random lattice modeling approach is capable of representing BEP phenomena in saturated embankments. The discrete nature of the model is inherently capable of localizing the progression of erosion in the domain. The proposed model allows for a prediction of the piping process that is compatible with the one observed in experiments (a shallow pipe running in the direction of the applied head difference,



Figure 18: Experimental and numerical applied global gradients for different exit conditions (experiments De Wit et al. [10])

connecting the downstream and the upstream side of the specimen). The results indicate that the amplification factor m_p is an important parameter and governs the progression of the pipe. For m_p values lower than 100, the simulations exhibit a total pipe length less than the extent of the specimen. In view of these results, a value for m_p greater or equal to 100 appears to be appropriate in the present case. Previous experimental investigations have shown that the value of the amplification factor is related to soil parameters such as type and grain size distribution [46], as well as the stress state [8, 9]. The proposed degradation model will be further developed in the near future to account for the effects of soil type and the stress state on m_p .

⁴⁹⁵ Mesh refinement provides an increase in the detail of the evaluated response, ⁴⁹⁶ as the tortuosity of the computational lattice increases, leading to a higher number ⁴⁹⁷ of possible directions for the pipe progression and branching. All the simulations

performed, however, proved capable of capturing the essential characteristics of BEP, and they show the same value of the maximum hydraulic gradient obtained from the described initiation procedure.

The influence of the type of exit condition was also investigated and the results obtained show good agreement with experimental evidence. In particular, as the size of the exit conditions decreases, the results show an increases in the erosion potential, due to increased local hydraulic gradient. This finding is consistent with experimental findings by De Wit et al. [10].

6. Conclusions

A novel random lattice modeling approach has been presented. The solution of the mass transport problem is retrieved on the two lattice assemblies independently. The information obtained on the two systems is then combined in order to define a procedure to evaluate the gradient of a scalar field defined at the lattice nodes. Accuracy of the approach has been verified by comparing the obtained numerical results with the theoretical solution. The so calculated hydraulic gradient is, in fact, a suitable measure to quantify the energy flow that is available for erosion to progress. Results show that the random lattice approach is capable of simulating BEP processes and the independence of the response on the mesh was demonstrated by simulating experimental tests available in literature by means of different mesh densities. The results from these analyses are in good agreement with each other, indicating that the eroded path is independent on the mesh size (such finding is compatible with the substantial independence of the crack path in random lattice simulations observed by previous authors [6]).



The influence of the model parameters has been investigated through a sen-

sitivity analysis, and suggestions on the values to adopt are given. Efficiency of
the method is demonstrated by showing how the results for the hydraulic gradient
calculation yield values in good agreement with classical theories, while the model
is able to predict realistic eroded paths without the need to restrict the constitutive
law to the horizontal plane.

The presented model is suitable for the solution of nonlinear diffusion prob-lems where nonlinearity is governed by the hydraulic gradient field. The dual representation allows for an efficient and accurate evaluation of the gradient of a scalar field defined at the lattice nodes. Ongoing work also considers the full coupling of the transport model and the mechanical response, in order to simulate the change in strength of the material in the presence of internal degradation, in which mechanical parameters of the soil can be obtained from local identification techniques [36, 60, 61]. The coupling of the seepage problem with the mechanical response of the soil will allow for the quantification of loss in stability earthen embankments experience as a result of BEP phenomena.

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