

Multiscale Nonlocal Effective Medium Model for In-plane Elastic Wave Dispersion and Attenuation in Periodic Composites

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Abstract

This manuscript proposes a multiscale nonlocal homogenization and a nonlocal effective medium model for in-plane wave propagation in periodic composites accounting for dispersion and attenuation due to Bragg scattering. The nonlocal effective medium model is developed based on the spatial-temporal nonlocal homogenization model that is formulated to capture dispersion within the first Brillouin zone with particularly high accuracy along high symmetry directions. The homogenization model is derived by employing high order asymptotic expansions, extending the applicability of asymptotic homogenization to short wavelength regime, to capture wave dispersion and attenuation. The effective medium model is in the form of a second order PDE with a nonlocal effective moduli tensor that contains the nonlocal features of the homogenization model. The proposed models are derived and numerically verified for in-plane elastic wave propagation in two-dimensional periodic composites. It is shown that the dispersion curves of the spatial-temporal nonlocal homogenization model capture the acoustic branch, the first stop band and the optical branch of longitudinal and shear wave modes. The nonlocal effective medium model predicts transient elastic wave propagation in periodic composites and captures wave dispersion and attenuation within the limits of separation of scales.

Keywords: Nonlocal continuum; Wave dispersion; Transient wave propagation; High order homogenization; Phononic crystals

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1 Introduction

Periodic composites with tailored microstructures and material properties such as phononic crystals [55] and acoustic metamaterials [41] exhibit extraordinary capability in controlling acoustic and elastic waves by manipulating band gaps that forbid waves to propagate within targeted frequency ranges. The exotic properties of these composites [38, 72, 46] originate from the transient behaviors that occur across various scales. Modeling of these phenomena using direct numerical simulations by resolving all relevant scales is therefore computationally prohibitive for structural design and analysis. This motivated the development of multiscale methods in the past decade towards more efficient modeling of complex dynamic behaviors. Elastic waves in two-dimensional domain are classified as anti-plane shear and in-plane waves, where the wave fields are correspondingly scalar- and vector-valued. Compared to the anti-plane shear wave, the development of multiscale models for in-plane wave dispersion is more challenging due to the presence of coupled longitudinal and shear modes. This manuscript proposes a class of multiscale, asymptotic homogenization-based models for the in-plane elastic wave propagation in periodic composites.

The band gap phenomenon, due to either Bragg scattering or local resonance, is non-local in nature. It occurs when strong interactions are present between the wave and the microstructure. In order to capture wave dispersion in periodic composites, various nonlocal homogenization approaches have been proposed. Elastodynamic homogenization models based on Willis' theory [69, 68, 45, 51, 63, 53, 49, 42] construct a coupled nonlocal constitutive relation that links the stress to velocity, and momentum to strain, and have been shown to capture wave dispersion relations up to the first few optical branches. More recently, generalized homogenization models [50, 62] are being developed along the lines of Willis' theory by enriching the macroscale displacement with additional generalized degrees of freedom of Bloch modes. The resulting system of macroscale governing equations captures the dispersion curves of a much wider frequency range. The asymptotic expansion based homogenization approaches [14, 52, 4, 60, 7] have also demonstrated their capability in predicting band gaps at high frequencies. The homogenization models mentioned above focus on characterizing the dispersion relations of the periodic composites, and transient simulations using these models are typically not performed.

Computational homogenization [47, 44, 19] is a well established technique in modeling complex material behaviors across multiple scales. The scales (typically confined to two but generalizable to multiple scales) are posed as coupled and nested boundary value problems, which are related through the Hill-Mandel energy consistency condition. The constitutive behavior at an arbitrary macroscopic material point is obtained through numerical evaluation of the corresponding microscale boundary value problem over a unit cell or a representative

volume. In recent years, this approach has been further developed by Pham et al. [56], Liu and Reina [39], Roca et al. [58], where inertia effects are incorporated to capture transient dynamics of periodic composites. Due to the need to evaluate nested equation systems, the computational cost grows dramatically as a function of the size of the macroscopic structure. Sridhar et al. [61] proposed a model order reduction technique to evaluate the microscale initial-boundary value problems and improve numerical efficiency. The capability of computational homogenization models in capturing wave dispersion has been demonstrated for composites in which the band gaps are caused by local resonance, but not due to Bragg scattering.

An alternative approach to model the transient dynamics of periodic composites is by employing gradient elasticity models [18, 43, 6, 57, 10, 16]. The key idea in these models is to construct a gradient-type nonlocal governing equation for the structure such that the effects due to microstructural heterogeneities are incorporated without numerically evaluating the microscale problem. A challenge in this approach is the identification and quantification of the length-scale parameters associated with the nonlocal terms. Although several procedures have been proposed to characterize the length-scale parameters such as direct numerical simulations [25] and experimental measurements [17], they are typically limited to the low frequency regime or one-dimensional wave propagation problems. To the best of the authors' knowledge, appropriate procedures for the determination of length-scale parameters in the context of vector-field wave (e.g., in-plane elastic wave) dispersion accounting for the formation of the stop bands in a multi-dimensional domain are not yet available. In addition, numerical evaluation of nonlocal governing equations, whether derived using gradient elasticity or other homogenization approaches, requires high order initial and boundary conditions. How to define the high order conditions is usually not trivial [5].

Asymptotic homogenization models that result in gradient-type nonlocal form for the macroscale governing equations have been formulated by various researchers [11, 21, 3, 8, 67, 66]. A key benefit in this strategy is that, the length-scale parameters are directly identified in the homogenization process. Hui and Oskay [33, 31, 32] studied dispersion and attenuation of transient waves in periodic elastic and viscoelastic composites. This study demonstrated high accuracy in capturing the transient wave dispersion in the low-frequency acoustic regime with limited ability to predict the onset of the stop band. An important step towards predicting transient wave phenomena within the stop band and beyond is to incorporate temporal nonlocal terms into the governing equations as demonstrated in Refs. [26, 27]. The resulting spatial-temporal nonlocal homogenization model captures the acoustic branch, the first stop band and the first optical branch of the dispersion relation, as well as the transient wave propagation. However, the model proposed in Refs. [26, 27] is limited to scalar waves (e.g., anti-plane shear wave), where only a single wave mode exists and the polarization vector holds constant with or without wave dispersion. In case of in-plane waves, coupled longitudinal and

93 shear modes are present and polarization vectors vary as a function of wave dispersion. The
94 capability of asymptotic homogenization models in predicting vector-field wave dispersion has
95 so far been limited to capturing the behavior within the acoustic dispersive regime.

96 This manuscript proposes a spatial-temporal nonlocal homogenization and a nonlocal effective
97 medium model for transient in-plane wave propagation in periodic composites. The nonlo-
98 cal effective medium model is developed based on the spatial-temporal nonlocal homogenization
99 model that is formulated to accurately capture dispersion along high symmetry directions of
100 the first Brillouin zone. The two essential ingredients in developing the spatial-temporal non-
101 local homogenization model are: (1) asymptotic expansions with high order corrections; (2)
102 construction of the gradient-type spatial-temporal nonlocal governing equation. In the con-
103 text of statics, the role of high order corrections in asymptotic expansion has been investigated
104 by Gambin and Kröner [24] and Ameen et al. [1]. For wave propagation problems, the high
105 order expansions allow the asymptotic homogenization approach to be applied in the short
106 wavelength regime, where the first stop band and the first optical branch occur. The spatial-
107 temporal nonlocal governing equation is constructed directly from the momentum balance
108 equations of successive asymptotic orders, where the accuracy is controlled by the asymptotic
109 residual term. Through minimizing the asymptotic residual, the optimal set of model param-
110 eters are determined. The nonlocal homogenization model is then employed to formulate an
111 effective medium model that retains the nonlocal features in the form of a nonlocal effective
112 moduli tensor. It is second order in space, therefore does not require high order boundary
113 conditions for transient simulations. The dispersion curves for layered and matrix-inclusion
114 microstructures are numerically verified. Transient simulations of in-plane wave propagation
115 in an elastic waveguide are performed and compared with the direct numerical simulations.
116 We show that the proposed model captures wave dispersion of longitudinal and shear wave
117 modes up to the first optical branch. It is the first time, to the best of authors' knowledge,
118 the gradient-type nonlocal homogenization model captures vector-field elastic wave dispersion
119 beyond the acoustic regime and is applied for transient simulations of wave propagation.

120 The remainder of this manuscript is structured as follows: Section 2 provides an overview of
121 the multiscale analysis based on high order asymptotic expansions, which results in macroscale
122 balance equations of successive orders and solutions of homogenized moduli tensors. Section
123 3 constructs a spatial-temporal nonlocal homogenization model using the macroscale balance
124 equations, and formulates a multiscale nonlocal effective medium model for transient simu-
125 lations. Section 4 provides the numerical implementation procedure of the multiscale model.
126 Section 5 verifies the dispersion characteristics of the nonlocal homogenization model and the
127 nonlocal effective medium model. The verification of the proposed model in capturing tran-
128 sient response is also included in Section 5. The conclusions and future research directions are
129 presented in Section 6.

The following notation is used throughout this manuscript. Scalars are denoted by italic Roman or Greek characters; vectors by boldface Roman characters; second or higher order tensors by boldface italic Roman and Greek characters. Indicial notation is used when necessary and Einstein summation convention applies to repeated indices. Denoting vectors, second order tensors, n^{th} order tensors respectively as $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, the tensor operations are defined as follows. Dyadic product: $\mathbf{a} \otimes \mathbf{b} = a_i b_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, where $\hat{\mathbf{e}}_i$ is the Cartesian basis vector. Dot product: $\mathbf{A} \cdot \mathbf{b} = A_{ij} b_j \hat{\mathbf{e}}_i$, double contraction: $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ji}$ and n^{th} contraction: $\mathbf{C}(\cdot)^n \mathbf{D} = C_{ij\dots uv} D_{vu\dots ji}$.

2 Multiscale problem setting and asymptotic analysis

Consider the domain of a heterogeneous body in the Cartesian coordinate system, $\Omega \in \mathbb{R}^2$, constructed by periodic unit cells composed of two or more constituents, as illustrated in Fig. 1. Wave propagation within Ω is governed by the momentum balance equation:

$$\nabla_x \cdot \boldsymbol{\sigma}^\zeta(\mathbf{x}, t) = \rho^\zeta(\mathbf{x}) \ddot{\mathbf{u}}^\zeta(\mathbf{x}, t) \quad (1)$$

where, $\boldsymbol{\sigma}^\zeta$ denotes the Cauchy stress tensor; ρ^ζ the density; and $\mathbf{u}^\zeta$ the displacement vector. $\nabla_x \cdot$ is the divergence operator and superimposed dot denotes derivative with respect to time. $\mathbf{x}$ is the position vector of material points. The superscript, ζ , indicates that the response fields oscillate spatially due to the microstructural heterogeneity. Body forces are ignored in the present study.

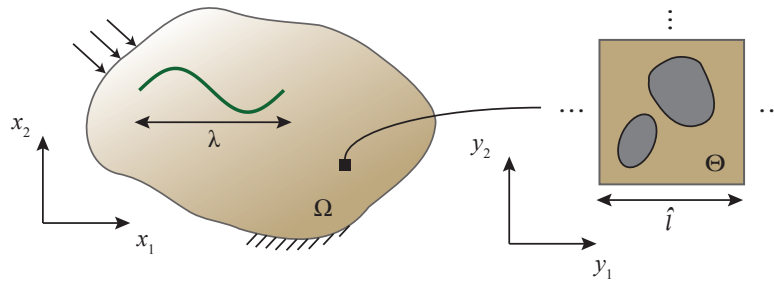


Figure 1: Schematic representation of the multiscale problem setting.

The constitutive response of the heterogeneous body is described by the generalized Hooke's law:

$$\boldsymbol{\sigma}^\zeta(\mathbf{x}, t) = \mathbf{C}^\zeta(\mathbf{x}) \cdot \boldsymbol{\epsilon}^\zeta(\mathbf{x}, t) \quad (2)$$

$\mathbf{C}^\zeta(\mathbf{x})$ is the elastic moduli tensor for the constituents and is taken to be strongly elliptic

with major and minor symmetries. $\epsilon^\zeta(\mathbf{x}, t)$ is the strain tensor under the assumption of small deformation:

$$\epsilon^\zeta(\mathbf{x}, t) = \nabla_x^s \mathbf{u}^\zeta(\mathbf{x}, t) = \frac{1}{2} \left[\nabla_x \mathbf{u}^\zeta(\mathbf{x}, t) + \left(\nabla_x \mathbf{u}^\zeta(\mathbf{x}, t) \right)^T \right] \quad (3)$$

where ∇_x and ∇_x^s are the gradient and the symmetric gradient operators, respectively.

The canonical unit cell domain, $\Theta \in \mathbb{R}^2$, is parameterized using the microscale coordinate, $\mathbf{y}$, which is related to the macroscale coordinate by $\mathbf{y} = \mathbf{x}/\zeta$, where $0 < \zeta < 1$ is the small scaling parameter. The smallness of the scaling parameter sets the premise for the homogenization approach. Asymptotic homogenization is not suitable when $\zeta \geq 1$. In the context of wave propagation, the scaling parameter is defined as the ratio between the size of microstructure, l , and the characteristic length of the deformation wave (i.e., $\zeta = l/\lambda$, where λ is the characteristic deformation wavelength). With the macro- and microscale coordinates, any response field, $f^\zeta(\mathbf{x}, t)$, is assumed to allow a two-scale description: $f^\zeta(\mathbf{x}, t) = f(\mathbf{x}, \mathbf{y}(\mathbf{x}), t)$. The material properties, i.e., elastic moduli tensor and density, are taken to depend on the microscale coordinate only, i.e., $\mathbf{C}^\zeta(\mathbf{x}) = \mathbf{C}(\mathbf{y})$ and $\rho^\zeta(\mathbf{x}) = \rho(\mathbf{y})$. Local periodicity is assumed for all response fields.

We consider periodic composites with low material property contrast between different phases. The mechanism of band gap formation of these composites is mainly due to destructive interaction of incident and scattered waves, i.e., Bragg scattering. The first band gap typically occurs when the macroscopic wavelength is of the same order as the size of the unit cell. Band gaps due to local resonance can occur with wavelength much larger than the size of the unit cell, which requires high contrast in constituent material properties. This type of composites is not considered in the manuscript.

2.1 High order two-scale asymptotic analysis

The formulation of the proposed model is based on mathematical homogenization with multiple spatial scales. The procedure for asymptotic analysis is rather standard and available in the classical texts [9, 59]. In this section, we provide a brief overview of the homogenization process that results in the set of equations from which the proposed model is derived. More details in the rigorous derivation in the general context is available in the above-mentioned references. Dynamic problems with large wavelengths can be accurately modeled based on two order asymptotic expansions [9, 59], which results in a local macroscopic balance equation. Models based on up to the fourth order expansions employ nonlocal macroscopic balance equations and have been shown to capture dispersive behavior [3, 21, 32]. The present approach considers asymptotic expansions of up to the eighth order. The details of the formulation in the context of transient dynamics for scalar-field wave is provided in Ref. [27]. The overview in this subsection is a straightforward extension to transient analysis of vector-field problems.

The displacement field is approximated using the following decomposition:

$$\mathbf{u}^\zeta(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, \mathbf{y}, t) = \mathbf{u}^{(0)}(\mathbf{x}, t) + \sum_{i=1}^8 \zeta^i \mathbf{u}^{(i)}(\mathbf{x}, \mathbf{y}, t) + O(\zeta^9) \quad (4)$$

where, $\mathbf{u}^{(0)}$ denotes the macroscopic displacement field and is dependent on the macroscale coordinate only [23] and $\mathbf{u}^{(i)}$, $i = 1, \dots, 8$, are the displacement fields of high orders which depend on both macroscale and microscale coordinates. We note that for composites that have constituents with high property contrast (e.g., when the tensors of elastic moduli of the constituents exhibit double porosity-type scaling [60, 7]), $\mathbf{u}^{(0)}$ may depend on both micro- and macroscale coordinates due to local resonance within the microstructures.

The displacement field at each order is decomposed into a macroscopically constant field and summation of a series of locally varying fields with zero mean over the unit cell [11]:

$$\mathbf{u}^{(i)}(\mathbf{x}, \mathbf{y}, t) = \mathbf{U}^{(i)}(\mathbf{x}, t) + \sum_{k=0}^{i-1} \tilde{\mathbf{U}}^{(i,k)}(\mathbf{x}, \mathbf{y}, t) \quad (5)$$

where, $\tilde{\mathbf{U}}^{(i,k)}(\mathbf{x}, \mathbf{y}, t)$ is the k^{th} locally varying field of $\mathbf{u}^{(i)}(\mathbf{x}, \mathbf{y}, t)$ and it is assumed to be related to the successive gradients of macroscopic strain of an inferior order by a time-invariant, locally periodic influence function defined over the unit cell:

$$\tilde{\mathbf{U}}^{(i,k)}(\mathbf{x}, \mathbf{y}, t) = \mathbf{H}^{(k+1)}(\mathbf{y}) \cdot (\cdot \nabla_{\mathbf{x}})^k \nabla_{\mathbf{x}}^s \mathbf{U}^{(i-k-1)}(\mathbf{x}, t) \quad (6)$$

where, $\mathbf{H}^{(k+1)}$ is the periodic microstructural influence function at order $(k+1)$ and it is symmetric in the last two indices that are contracted with $\nabla_{\mathbf{x}}^s \mathbf{U}^{(i-k-1)}$. $(\cdot \nabla_{\mathbf{x}})^k$ is the $(k)^{th}$ gradient with respect to the macroscale coordinate, $\mathbf{x}$, with k contractions to the microstructural influence function. This construction allows separate evaluation of the microscale influence functions and macroscale momentum balance equations. Employing Eqs. 1-6, the equilibrium equations for microscale influence functions and macroscale momentum balance equations at each order are successively derived (see Appendix A). The microscale equilibrium equations are:

$$O(\zeta^{-1}) : \quad \nabla_{\mathbf{y}} \cdot \mathbf{C}^{(0)}(\mathbf{y}) = 0 \quad (7a)$$

$$O(\zeta^\alpha) : \quad \nabla_{\mathbf{y}} \cdot \mathbf{C}^{(\alpha+1)}(\mathbf{y}) = \theta(\mathbf{y}) \sum_{j=0}^{\alpha} \mathbf{H}^{(j)}(\mathbf{y}) \cdot \mathbf{D}^{(\alpha-j)} - \mathbf{C}^{(\alpha)}(\mathbf{y}) \quad (7b)$$

where $\theta(\mathbf{y}) = \rho(\mathbf{y})/\rho_0$, and $\rho_0 = \overline{\rho(\mathbf{y})}$ is the homogenized density. Overbar indicates averaging

operator over the unit cell domain. $\mathbf{C}^{(\alpha)}(\mathbf{y})$ ($\alpha = 0, 1, \dots, 7$) is expressed as:

$$\mathbf{C}^{(\alpha)}(\mathbf{y}) = \mathbf{C}(\mathbf{y}).. \left[\mathbf{H}^{(\alpha)}(\mathbf{y}) \otimes \mathbf{I} + \nabla_{\mathbf{y}} \mathbf{H}^{(\alpha+1)}(\mathbf{y}) \right] \quad (8)$$

where $\mathbf{I}$ is the second order identity tensor. The homogenized moduli at each order $\mathbf{D}^{(\alpha)}$ is written as:

$$\mathbf{D}^{(0)} = \overline{\mathbf{C}^{(0)}(\mathbf{y})} \quad (9a)$$

$$\mathbf{D}^{(\alpha)} = \overline{\mathbf{C}^{(\alpha)}(\mathbf{y})} - \sum_{j=1}^{\alpha} \overline{\theta(\mathbf{y}) \mathbf{H}^{(j)}(\mathbf{y})} . \mathbf{D}^{(\alpha-j)} \quad (9b)$$

$\mathbf{D}^{(0)}$ is a 4th order tensor that defines the effective moduli in the quasi-static limit and is a function of microstructural geometry and elastic properties only. It possesses major and minor symmetries. The high order moduli, $\mathbf{D}^{(\alpha)}$, are $(\alpha + 4)^{th}$ order tensors. These moduli are not only functions of microstructural geometry and elastic properties, but also the densities of the constituents of the microstructure and they contain length-scale of $O(\hat{l}^\alpha)$. For simplicity of formulation below, we consider microstructures that possess symmetry aligned with the Cartesian coordinate planes. For these microstructures, orthotropic macroscopic properties are observed. As a consequence, $\mathbf{D}^{(\alpha)} = \mathbf{0}$ for odd α and the components with odd number of repeated indices are zero for even α [27]. Therefore, the balance equations of odd orders have identical form and parameters as the even order ones of an order inferior.

The macroscale balance equation at $O(\zeta^\alpha)$ is written as:

$$\nabla_x . \sum_{j=0}^{\alpha} \mathbf{D}^{(j)} .. (\nabla_x)^j \nabla_x^s \mathbf{U}^{(\alpha-j)} = \rho_0 \ddot{\mathbf{U}}^{(\alpha)} \quad (10)$$

At the leading order, $\alpha = 0$, the macroscale momentum balance equation is local and describes non-dispersive wave propagation in the long wavelength limit. The balance equations at higher asymptotic orders successively include nonlocal stresses that are of the second order form, $\mathbf{D}^{(j)} .. (\nabla_x)^j \nabla_x^s \mathbf{U}^{(\alpha-j)}$, which are functions of strain gradients of lower asymptotic orders. The contribution of the nonlocal stresses is the volume source disturbance resulting from the lower order displacements. It balances with the stress and momentum of the current order.

3 Multiscale models

In this section, we first propose a fourth-order spatial-temporal nonlocal homogenization model. The structure of the resulting governing equation is similar to those proposed in Refs. [26, 27] for scalar-field wave, and include spatial nonlocal, temporal nonlocal and mixed

spatial-temporal nonlocal terms. Yet the proposed model differs from the existing approaches in the way the model parameters are consistently derived, and is applicable to problems in which the wave field is vector-valued. The model parameters are determined using response in high symmetry directions of the first Brillouin zone. Next, we formulate a nonlocal effective medium model that retains the nonlocal characteristics of the corresponding nonlocal homogenization model in high symmetry directions but in the form of a second-order PDE with a frequency-dependent moduli tensor. The nonlocal effective medium model is used for simulation of transient wave propagation.

The proposed formulation consists of 5 steps: (1) restatement of the governing equations at various asymptotic orders using a series of weighting tensors; (2) application of projection operations to the high order homogenized moduli to achieve a consistent model of the spatial-temporal nonlocal form; (3) identification of the projection tensors based on particular plane wave solutions; (4) identification of the weighting tensors; and (5) model order reduction to devise the effective medium model for transient dynamics simulations. The key ideas, assumptions and concepts in the proposed five-step formulation are discussed below. For clarity of the presentation, we skip the algebraic details. A detailed derivation of the formulation is provided in Appendix B.

3.1 Spatial-temporal nonlocal homogenization model

In what follows, we employ the momentum balance equations, Eq. 10 with $\alpha = 0, 2, 4, 6$, to derive the fourth-order spatial-temporal nonlocal model. The balance equations at odd orders do not contribute to the resulting model under the symmetry conditions discussed above [27]. The homogenized displacement field is expressed as the additive sum of contributions from all even orders:

$$\mathbf{U}(\mathbf{x}, t) = \sum_{i=0}^3 \zeta^{2i} \mathbf{U}^{(2i)}(\mathbf{x}, t) + O(\zeta^8) \quad (11)$$

We start by restating the macroscale balance equations, Eq. 10 with $\alpha = 2, 4, 6$, in alternative forms. Let $\boldsymbol{\nu}^{(j)}$, $j = 1, 2, 3$, denote three weighting tensors that are applied to the nonlocal stress terms in Eq. 10. Without loss of generality, the macroscale balance equations become:

$$\begin{aligned} \rho_0 \ddot{\mathbf{U}}^{(\alpha)} - \nabla_{\mathbf{x}} \cdot \left(\mathbf{D}^{(0)} \cdot \nabla_{\mathbf{x}}^s \mathbf{U}^{(\alpha)} \right) &= \nabla_{\mathbf{x}} \cdot \sum_{j=1}^{\alpha/2} \boldsymbol{\nu}^{(j)} \cdot \mathbf{D}^{(2j)} \cdot (\nabla_{\mathbf{x}})^{2j} \nabla_{\mathbf{x}}^s \mathbf{U}^{(\alpha-2j)} \\ &+ \nabla_{\mathbf{x}} \cdot \sum_{j=1}^{\alpha/2} \left(\boldsymbol{\delta} - \boldsymbol{\nu}^{(j)} \right) \cdot \mathbf{D}^{(2j)} \cdot (\nabla_{\mathbf{x}})^{2j} \nabla_{\mathbf{x}}^s \mathbf{U}^{(\alpha-2j)} \end{aligned} \quad (12)$$

The additive decomposition in Eq. 12 does not introduce any approximation and is valid for

any form of the weighting tensors. This weighted form of the balance equations subsequently allows transformation of weighted spatial nonlocal terms to a temporal nonlocal term and a mixed spatial-temporal nonlocal term as described below. The temporal and mixed spatial-temporal nonlocal terms are derived from the first term on the right hand side of Eq. 12, while the second term is referred as the asymptotic residual term. The fractions of spatial nonlocal terms that contribute to the formulation are controlled by $\boldsymbol{\nu}^{(j)}$, where the spatial nonlocal terms are fully incorporated when $\boldsymbol{\nu}^{(j)} = \boldsymbol{\delta}$ and not incorporated when $\boldsymbol{\nu}^{(j)} = \mathbf{0}$. In this work, the weighting tensors are taken to be second-order and diagonal. The choice for the form of the weighting tensors is made to ensure that we introduce as small number of independent parameters as possible, while capturing wave dispersion with reasonable accuracy. The choice of diagonal weighting tensor implies that the contribution of the relevant high order asymptotic term to momentum balance in each direction is weighted using a separate weighting parameter. By this approach, two independent parameters need to be identified for each weighting tensor. These diagonal weighting tensors are uniquely determined by examining the characteristics of the nonlocal governing equation in each coordinate direction as discussed in Section 3.3.

The next step in the formulation is the projection of high order moduli as follows:

$$\nabla_x \cdot \left[\boldsymbol{D}^{(6)} \cdot (\cdot \nabla_x)^6 \nabla_x^s \mathbf{U}^{(0)} \right] \approx \nabla_x \cdot \left[\left(\boldsymbol{A}^{(1)} \cdot \boldsymbol{D}^{(0)} \cdot \boldsymbol{D}^{(4)} \right) \cdot (\cdot \nabla_x)^6 \nabla_x^s \mathbf{U}^{(0)} \right] \quad (13a)$$

$$\nabla_x \cdot \left[\left(\boldsymbol{D}^{(0)} \cdot \boldsymbol{D}^{(2)} \right) \cdot (\cdot \nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)} \right] \approx \nabla_x \cdot \left[\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(2)} \cdot \boldsymbol{D}^{(0)} \right) \cdot (\cdot \nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)} \right] \quad (13b)$$

$$\nabla_x \cdot \left[\boldsymbol{D}^{(4)} \cdot (\cdot \nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)} \right] \approx \nabla_x \cdot \left[\left(\boldsymbol{A}^{(3)} \cdot \boldsymbol{D}^{(2)} \cdot \boldsymbol{D}^{(0)} \right) \cdot (\cdot \nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)} \right] \quad (13c)$$

where $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$ and $\boldsymbol{A}^{(3)}$ are the projection tensors. The projection tensors are second-order and diagonal. A straightforward rearrangement of Eq. 13 shows that the moduli tensors $\boldsymbol{D}^{(6)}$, $\boldsymbol{D}^{(0)} \cdot \boldsymbol{D}^{(2)}$ and $\boldsymbol{D}^{(4)}$ on the left hand side are projected onto the corresponding tensors $\boldsymbol{D}^{(0)} \cdot \boldsymbol{D}^{(4)}$, $\boldsymbol{D}^{(2)} \cdot \boldsymbol{D}^{(0)}$ and $\boldsymbol{D}^{(2)} \cdot \boldsymbol{D}^{(0)}$, respectively, for arbitrary high order strain gradient fields (e.g., $(\nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)}$ for Eq. 13c). In Ref. [27], the projection tensors were identified by Moore-Penrose pseudo inversion, which provides the closest point projection of the projected tensors onto the corresponding tensors. In the current manuscript, we employ an alternative strategy for the identification of the projection tensors. The projection equations are expressed in the form shown in Eq. 13 and the tensors are identified based on strain fields associated with certain plane wave solutions as described in Section 3.2.

Employing the high order moduli projection, Eq 13, the first spatial nonlocal terms on the right hand side of the restated macroscale balance equations, Eq. 12, are transformed into the fourth-order spatial-temporal nonlocal form by replacing spatial nonlocal terms with the terms in balance with them in the macroscale balance equations, Eq. 10 with $\alpha = 0, 2, 4$. Employing

the definition for the homogenized displacement field (Eq. 11), performing the summation of the spatial-temporal nonlocal macroscale balance equations at $O(\zeta^\alpha)$ with $\alpha = 0, 2, 4, 6$, the homogenized momentum balance equation is obtained as:

$$\begin{aligned} \rho_0 \ddot{\mathbf{U}} - \nabla_x \cdot \left(\mathbf{D}^{(0)} \cdot \nabla_x^s \mathbf{U} \right) = & \nabla_x \cdot \left(\boldsymbol{\alpha} \cdot (\nabla_x)^2 \nabla_x^s \mathbf{U} \right) + \nabla_x \cdot \left(\boldsymbol{\beta} \cdot \nabla_x^s \ddot{\mathbf{U}} \right) + \gamma \cdot \ddot{\ddot{\mathbf{U}}} \\ & + \sum_{i=1}^3 \zeta^{2i} \mathbf{R}^{(2i)} \end{aligned} \quad (14)$$

where, the coefficients of the PDE are:

$$\boldsymbol{\alpha} = \boldsymbol{\nu}^{(1)} \cdot \mathbf{D}^{(2)} - \boldsymbol{\nu}^{(3)} \cdot \mathbf{A}^{(1)} \cdot \mathbf{D}^{(0)} \cdot \mathbf{D}^{(0)} \quad (15a)$$

$$\boldsymbol{\beta} = \rho_0 \left[\boldsymbol{\nu}^{(3)} \cdot \mathbf{A}^{(1)} \cdot \left(\mathbf{I} + \mathbf{A}^{(2)} \right) - \boldsymbol{\nu}^{(2)} \cdot \mathbf{A}^{(3)} \right] \cdot \mathbf{D}^{(0)} \quad (15b)$$

$$\gamma = \rho_0^2 \left(\boldsymbol{\nu}^{(2)} \cdot \mathbf{A}^{(3)} - \boldsymbol{\nu}^{(3)} \cdot \mathbf{A}^{(1)} \cdot \mathbf{A}^{(2)} \right) \quad (15c)$$

and the residual vector $\mathbf{R}^{(\alpha)}$ at each order are:

$$\mathbf{R}^{(2)} = \nabla_x \cdot \left(\mathbf{E}^{(2)} \cdot (\nabla_x)^2 \nabla_x^s \mathbf{U}^{(0)} \right) \quad (16a)$$

$$\mathbf{R}^{(4)} = \nabla_x \cdot \left(\mathbf{E}^{(2)} \cdot (\nabla_x)^2 \nabla_x^s \mathbf{U}^{(2)} + \mathbf{E}^{(4)} \cdot (\nabla_x)^4 \nabla_x^s \mathbf{U}^{(0)} \right) \quad (16b)$$

$$\mathbf{R}^{(6)} = \nabla_x \cdot \left(\mathbf{E}^{(2)} \cdot (\nabla_x)^2 \nabla_x^s \mathbf{U}^{(4)} + \mathbf{E}^{(4)} \cdot (\nabla_x)^4 \nabla_x^s \mathbf{U}^{(2)} + \mathbf{E}^{(6)} \cdot (\nabla_x)^6 \nabla_x^s \mathbf{U}^{(0)} \right) \quad (16c)$$

in which, the residual coefficient tensors are:

$$\mathbf{E}^{(2)} = (\boldsymbol{\delta} - \boldsymbol{\nu}^{(1)}) \cdot \mathbf{D}^{(2)} \quad (17a)$$

$$\mathbf{E}^{(4)} = (\boldsymbol{\delta} - \boldsymbol{\nu}^{(2)}) \cdot \mathbf{D}^{(4)} \quad (17b)$$

$$\mathbf{E}^{(6)} = (\boldsymbol{\delta} - \boldsymbol{\nu}^{(3)}) \cdot \mathbf{D}^{(6)} - \left(\boldsymbol{\nu}^{(2)} \cdot \mathbf{A}^{(3)} - \boldsymbol{\nu}^{(3)} \cdot \mathbf{A}^{(1)} \cdot \mathbf{A}^{(2)} \right) \cdot \left(\mathbf{D}^{(2)} \cdot \mathbf{D}^{(2)} + \mathbf{D}^{(4)} \cdot \mathbf{D}^{(0)} \right) \quad (17c)$$

Equation 14 represents a family of nonlocal homogenization models with the coefficient tensors $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$, $\boldsymbol{\nu}^{(1)}$, $\boldsymbol{\nu}^{(2)}$ and $\boldsymbol{\nu}^{(3)}$ to be determined. In addition to the terms (left hand side of the equation) pertain to the classical local homogenization model that characterize non-dispersive wave propagation, three nonlocal terms, i.e., spatial nonlocal, temporal nonlocal and mixed spatial-temporal nonlocal terms, are present to capture wave dispersion and attenuation. The asymptotic accuracy of Eq. 14 is controlled by the residual term. When wavelength is much larger than the size of microstructures, wave propagation is non-dispersive, the contribution of the nonlocal terms compared to local terms is negligible. Compared to the existing gradient-type nonlocal homogenization models [21, 8, 32] for in-plane elastic wave, Eq. 14 incorporates the temporal nonlocal term by employing the balance equations of higher

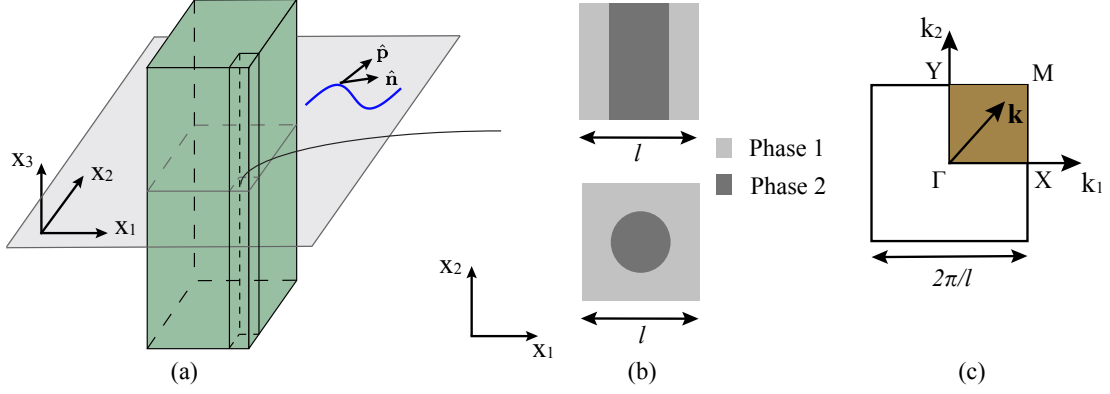


Figure 2: (a) Schematic representation of in-plane wave propagation in a macroscopic structure. (b) Layered and matrix-inclusion unit cells as examples for microstructures that possess symmetry with respect to the Cartesian planes. (c) Sketch of the first Brillouin zone for considered microstructures.

orders. The asymptotic residual term is a function of the weighting tensors and homogenized moduli, which is essential in enforcing the uniqueness of the spatial-temporal nonlocal homogenization model. Through minimizing this term, the optimal set of weighting tensors can be obtained that achieves the highest asymptotic accuracy.

3.2 Projection tensors

The procedure to identify the projection tensors, $\mathbf{A}^{(j)}$, is based on examining the propagation of characteristic plane waves in the homogenized medium described by the nonlocal model (Eq. 14). Consider three plane waves of the same mode propagating along the same direction as characteristic waves:

$$\hat{\mathbf{U}}^{(i)}(\mathbf{x}, t) = \hat{U}^{(i)} \hat{\mathbf{p}} e^{i(k^{(i)} \hat{\mathbf{n}} \cdot \mathbf{x} - \omega t)}, \quad i = 0, 2 \quad (18a)$$

$$\hat{\mathbf{U}}(\mathbf{x}, t) = \hat{U} \hat{\mathbf{p}} e^{i(k \hat{\mathbf{n}} \cdot \mathbf{x} - \omega t)} \quad (18b)$$

where $\hat{\mathbf{n}}$ and $\hat{\mathbf{p}}$ are unit vectors in fixed directions of wave propagation and polarization, respectively. Without loss of generality, the plane waves are taken to reside in the $[x_1, x_2]$ plane, as shown in Fig. 2(a). ω is a positive real-valued scalar denoting the frequency of the plane waves. $\hat{U}$ and $\hat{U}^{(i)}$ are the amplitudes of the homogenized displacement and the two low order components, respectively. k and $k^{(i)}$ denote the wavenumbers. The wavenumbers are complex-valued when the wave frequency is within the stop band, where the positive imaginary part results in the exponential decay of the displacement amplitude. Substituting Eq. 18a into

Eq. 13 (considering equality), the components of the approximation tensors are computed as:

$$A_{[ii]}^{(1)} = d_{[i]}^{(6)} / d_{[i]}^{(04)} \quad (19a)$$

$$A_{[ii]}^{(2)} = d_{[i]}^{(02)} / d_{[i]}^{(20)} \quad (19b)$$

$$A_{[ii]}^{(3)} = d_{[i]}^{(4)} / d_{[i]}^{(20)} \quad (19c)$$

in which,

$$\mathbf{d}^{(6)} = \mathbf{D}^{(6)}(.)^9 [\hat{\mathbf{p}}(\otimes \hat{\mathbf{n}})^8] \quad (20a)$$

$$\mathbf{d}^{(04)} = (\mathbf{D}^{(0)}.\mathbf{D}^{(4)}) (.)^9 [\hat{\mathbf{p}}(\otimes \hat{\mathbf{n}})^8] \quad (20b)$$

$$\mathbf{d}^{(4)} = \mathbf{D}^{(4)}(.)^7 [\hat{\mathbf{p}}(\otimes \hat{\mathbf{n}})^6] \quad (20c)$$

$$\mathbf{d}^{(02)} = (\mathbf{D}^{(0)}.\mathbf{D}^{(2)}) (.)^7 [\hat{\mathbf{p}}(\otimes \hat{\mathbf{n}})^6] \quad (20d)$$

$$\mathbf{d}^{(20)} = (\mathbf{D}^{(2)}.\mathbf{D}^{(0)}) (.)^7 [\hat{\mathbf{p}}(\otimes \hat{\mathbf{n}})^6] \quad (20e)$$

Square brackets around indices, $[ii]$, imply that Einstein summation convention is not applied to the subscript. The equality in Eq. 19 that fully defines the projection tensors are strictly valid for plane waves associated with a fixed pair $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$. Employing the projection tensors back to Eqs. 13 for a different and arbitrary pair of wave and polarization directions $(\mathbf{n}, \mathbf{p})$ would not satisfy the equality and constitute an approximation. In other words, Eqs. 13 are equalities only when $\mathbf{n} = \hat{\mathbf{n}}$ and $\mathbf{p} = \hat{\mathbf{p}}$.

In general, the directions of wave propagation and polarization are not necessarily related for an arbitrary plane wave, but the eigenvalues and eigenvectors of a plane wave are related to the direction of wave propagation and the material properties. For plane wave propagation in homogeneous anisotropic materials, the eigenvalues and eigenvectors are obtained by solving the Christoffel equation [48]. In order to characterize the projection tensors and weighting tensors, the characteristic polarization vector, $\hat{\mathbf{p}}$, is taken to be an eigenvector of wave propagation in a selected direction, $\hat{\mathbf{n}}$, thus $\hat{\mathbf{p}} = \hat{\mathbf{p}}(\hat{\mathbf{n}})$. The selection of the characteristic pair $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$ constitutes two steps, i.e., choose $\hat{\mathbf{n}}$ first and then compute $\hat{\mathbf{p}}$ along the selected direction.

Wave dispersion in periodic composites is typically characterized by the Bloch wave expansion approach. This approach evaluates an eigenvalue problem of the unit cell with periodic boundary conditions, either by solving for the frequency with the wavevector sampled within the first Brillouin zone (i.e., $\omega(\mathbf{k})$ method [34, 35]) or by solving for the wavenumber along a prescribed direction with frequency sampled (i.e., $\mathbf{k}(\omega)$ method [37, 2, 36]). The unit cell averaged Bloch mode shapes describe the effective polarization of the media when the frequency and wavenumber are within the homogenizable regime [49, 64], including the acoustic branch

and the first optical branch of the dispersion curves. Therefore, the averaged Bloch mode shapes are used as the characteristic polarization vector, $\hat{\mathbf{p}}$.

Although there are infinite number of Bloch modes within or on the edges of the first Brillouin zone, the Bloch modes at high symmetry points are typically selected as the projection basis for reduced representation of the dispersion characteristics of periodic composites. For example, Hussein [34] used high symmetry modes as basis for model order reduction of unit cell dispersion analysis. Sridhar et al. [62] employed high symmetry modes to formulate a generalized homogenization model. In order to capture the dispersion behavior at the high symmetry points, we select the directions of wave propagation from the center to the high symmetry points on the edges of the first Brillouin zone (i.e., high symmetry directions) as the characteristic wave propagation directions $\hat{\mathbf{n}}$. For instance, $\Gamma - X$, $\Gamma - M$ and $\Gamma - Y$ directions in Fig. 2(c) are chosen as $\hat{\mathbf{n}}$ for the microstructures shown in Fig. 2(b). Since the nonlocal terms in Eq. 14 do not contribute to non-dispersive wave propagation, i.e., lower acoustic branches, Bloch modes of high symmetry points on the edges of the first Brillouin zone are used. At these points, wave propagation is dispersive. However, only the modes of acoustic branches (constituent phases moving in-phase) and the first optical branches (constituent phases moving out-of-phase [13]) can be employed, because higher modes correspond to wave propagation of wavelengths shorter than the size of microstructures, violating the scale separation assumption. In addition, the prediction of the optical branch is more sensitive to the magnitude of the asymptotic residual (Eq. 14), since the residual grows as a function of the wavenumber. Therefore, the lowest optical Bloch mode is used to compute the characteristic polarization vectors, $\hat{\mathbf{p}}$.

Selecting $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$ in high symmetry directions results in spatial-temporal nonlocal homogenization models that are in the same form, with different model parameters. The projection tensors of each model are uniquely computed in a direction of high symmetry. As a result, the homogenization model most accurately characterizes wave dispersion in that direction. As the direction of wave propagation migrates away from the selected direction, the error of prediction increases.

3.3 Weighting tensors

In this section, we propose two approaches to identify the weighting tensors $\nu^{(j)}$: asymptotic residual minimization, and band gap size matching. The asymptotic residual minimization approach determines the weighting tensors through a constrained optimization problem that minimizes the asymptotic residual term in Eq. 14, where the constraint is imposed by considering the dispersion characteristics of the nonlocal governing equation. In the band gap size matching approach, the weighting tensors are identified by minimizing the discrepancy be-

tween the band gap size predicted by the nonlocal homogenization model and those computed based on the Bloch wave analysis in prescribed directions.

3.3.1 Asymptotic residual minimization

The first approach we propose in the identification of the weighting tensors, $\boldsymbol{\nu}^{(j)}$, is the idea of minimizing the asymptotic residual term in Eq. 14. The order of the residual is $O(\zeta^2)$ and is expressed as a function of all three weighting tensors. In view of Eqs. 16 and 17, the $O(\zeta^2)$ and $O(\zeta^4)$ terms in the expression of the residual is eliminated by simply setting $\boldsymbol{\nu}^{(1)} = \mathbf{I}$ and $\boldsymbol{\nu}^{(2)} = \mathbf{I}$, resulting in a residual of $O(\zeta^6)$. The weighting tensor $\boldsymbol{\nu}^{(3)}$ is selected such that the remaining residual is minimized, while considering constraints imposed by physical considerations of wave dispersion.

Considering plane wave propagation with the direction and polarization vectors set to $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$, substituting Eq. 18b into Eq. 14 and neglecting the residual term, the characteristic equations of the nonlocal homogenization model are obtained as:

$$A_i k^4 + B_i(\omega) k^2 + C_i(\omega) = 0 \quad (21a)$$

$$\tilde{A}_i \omega^4 + \tilde{B}_i(k) \omega^2 + \tilde{C}_i(k) = 0 \quad (21b)$$

where,

$$A_i = \alpha_{ijpqmn} \hat{p}_n \hat{n}_m \hat{n}_q \hat{n}_p \hat{n}_j, \quad \tilde{A}_i = \gamma_{in} \hat{p}_n \quad (22a)$$

$$B_i = (\omega^2 \beta_{ijmn} - D_{ijmn}^{(0)}) \hat{p}_n \hat{n}_m \hat{n}_j, \quad \tilde{B}_i = (k^2 \beta_{ijmn} \hat{n}_m \hat{n}_j + \rho_0 \delta_{in}) \hat{p}_n \quad (22b)$$

$$C_i = (\omega^4 \gamma_{in} + \rho_0 \omega^2 \delta_{in}) \hat{p}_n, \quad \tilde{C}_i = (k^4 \alpha_{ijpqmn} \hat{n}_m \hat{n}_q \hat{n}_p \hat{n}_j - k^2 D_{ijmn}^{(0)} \hat{n}_m \hat{n}_j) \hat{p}_n \quad (22c)$$

The dispersion relation between ω and k for the pair $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$ is obtained either by solving for k given ω (Eq. 21a), or by solving for ω given k (Eq. 21b). According to the plane wave solution form, Eq. 18b, the stop band occurs at frequencies that result in complex-valued wavenumbers when k is solved in terms of ω . ω is required to be real-valued when ω is solved in terms of k . The expressions:

$$\phi_i(\omega) = B_{[i]}^2(\omega) - 4A_{[i]}C_{[i]}(\omega) = a_i \omega^4 + b_i \omega^2 + c_i \quad (23a)$$

$$\tilde{\phi}_i(k) = \tilde{B}_{[i]}^2(k) - 4\tilde{A}_{[i]}\tilde{C}_{[i]}(k) = a_i k^4 + \tilde{b}_i k^2 + \tilde{c}_i \quad (23b)$$

determine the roots of Eqs. 21, k^2 and ω^2 , being real- or complex-valued. a_i , b_i , c_i , $\tilde{b}_i$ and $\tilde{c}_i$ are functions of $\boldsymbol{\nu}^{(3)}$ only and their expressions are provided in Appendix C. In order to constrain the solutions of k when k is solved in terms of ω , and ω when ω is solved in terms of k , constraints on $\boldsymbol{\nu}^{(3)}$ are imposed through considering the behaviors of $\phi_i(\omega)$ and $\tilde{\phi}_i(k)$.

In view of Eqs. 23, the stop band (k being complex) appears in the frequency range that corresponds to $\phi_i(\omega) < 0$ and the existence of real-valued solution of ω requires $\tilde{\phi}_i(k) > 0$. The bounds on the selection of $\boldsymbol{\nu}^{(3)}$ are found by constraining the characteristics of the solutions of Eqs. 21 as follows: (1) a stop band exists and it has finite size, (2) for any given k , there exists a real-valued solution of ω . Since c_i and $\tilde{c}_i$ are positive, in order to satisfy the two conditions, the following inequality constraints are imposed on the selection of $\boldsymbol{\nu}^{(3)}$:

$$a_i > 0 \quad (24a)$$

$$b_{[i]}^2 - 4a_{[i]}c_{[i]} \geq 0 \quad (24b)$$

$$\tilde{b}_{[i]}^2 - 4a_{[i]}\tilde{c}_{[i]} < 0 \quad (24c)$$

Equation 24a ensures that the size of the stop band is not infinite. We make a distinction between the existence of a stop band with zero width (Eq. 24b in equality form) and nonexistence of a stop band (i.e., $b_{[i]}^2 - 4a_{[i]}c_{[i]} < 0$). The conditions where "apparent" lack of stop band is therefore captured by Eq. 24b in its equality form. Compared to those imposed based on stability arguments discussed in Ref. [27] in the context of anti-plane shear wave, the proposed constraints are conceptually more restrictive. This is because the stability arguments proposed in Ref. [27] are applied at frequencies and wavenumbers associated with the quasistatic and infinitely long wave conditions only. The proposed constraints affect the behavior of dispersion for all frequencies and wavenumbers. Subject to the constraints in Eq. 24, the weighting tensor $\boldsymbol{\nu}^{(3)}$ is uniquely determined by minimizing Eq. 16c in the Euclidean norm under the condition of plane wave propagation.

3.3.2 Band gap size matching

Similar to above, the weighting tensors $\boldsymbol{\nu}^{(1)}$ and $\boldsymbol{\nu}^{(2)}$ are determined by eliminating the $O(\zeta^2)$ and $O(\zeta^4)$ residuals (i.e., $\boldsymbol{\nu}^{(1)} = \boldsymbol{\nu}^{(2)} = \mathbf{I}$). Considering the characteristic equation of the nonlocal governing equation for plane wave propagation in a prescribed direction, the wavenumbers and polarization vectors for quasi-longitudinal (P) and vertically polarized quasi-shear (SV) modes in $[x_1, x_2]$ plane in Fig. 2(a) satisfy the following relation:

$$B_{in}(\omega, k, \hat{\mathbf{n}}, \boldsymbol{\nu}^{(3)})p_n = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0 \quad (25)$$

where,

$$B_{in} = k^4 \alpha_{ijpqmn} \hat{n}_m \hat{n}_q \hat{n}_p \hat{n}_j + k^2 (\omega^2 \beta_{ijmn} - D_{ijmn}^{(0)}) \hat{n}_m \hat{n}_j + (\omega^4 \gamma_{in} + \rho_0 \omega^2 \delta_{in}) \quad (26)$$

For any given ω and $\hat{\mathbf{n}}$, the wavenumbers for P and SV modes are computed by equating the determinant of $\mathbf{B}$ in Eq. 25 to zero and the corresponding polarization vectors, $\mathbf{p}$, are obtained by substituting the wavenumbers back into Eq. 25. The P and SV modes are generally coupled. In the special case when the wave is propagating along the x_1 or x_2 planes (i.e., $\hat{\mathbf{n}} = [1, 0]^T$ or $\hat{\mathbf{n}} = [0, 1]^T$), the off-diagonal terms of $\mathbf{B}$ vanish and the resulting two polarization vectors are along and orthogonal to the wave vector, resulting in the uncoupling of the longitudinal and shear modes. Equating the determinant of $\mathbf{B}$ in Eq. 25 to zero results in the following expression:

$$B_{11}(\omega, k, \boldsymbol{\nu}^{(3)})B_{22}(\omega, k, \boldsymbol{\nu}^{(3)}) = 0 \quad (27)$$

The equality in Eq. 27 can be satisfied by either setting $B_{11} = 0$ or $B_{22} = 0$. These expressions correspond to the dispersion relations for the uncoupled longitudinal and shear wave modes, allowing us to determine the band structure of each mode given an arbitrary weighting tensor. The frequency range of the stop band for each mode is obtained in the analytical form:

$$\varphi_i = \frac{\sqrt{\bar{b}_{[i]}^2 - 4\bar{a}_{[i]}\bar{c}_{[i]}}}{\bar{a}_{[i]}} \quad (28)$$

where, $\bar{a}_{[i]}$, $\bar{b}_{[i]}$ and $\bar{c}_{[i]}$ are respectively computed using the expressions for a_i , b_i and c_i with $\hat{\mathbf{n}} = [1, 0]^T$ or $\hat{\mathbf{n}} = [0, 1]^T$. With the analytical expression of the width of the stop bands, $\boldsymbol{\nu}^{(3)}$ is obtained by solving the constrained minimization problem with the objective function: $|\varphi - \varphi^{(B)}|$. $\varphi^{(B)}$ is the frequency range of the first stop band of the P and SV modes calculated using the Bloch wave expansion approach. The constraints in Eq. 24 are imposed in this minimization problem. Wave propagation in more general situations (arbitrary $\hat{\mathbf{n}}$) couples the P and SV modes. Performing the calibration described above may be significantly more involved, since the analytical expression for the frequency range of the stop band for each mode may not be readily available.

3.4 Nonlocal effective medium model

Direct numerical implementation of the fourth-order initial-boundary value problem governed by Eq. 14 for transient wave propagation in a composite medium poses two challenges. First, high order boundary conditions need to be imposed due to the presence of the fourth order spatial term. The form of these high order boundary conditions has been subject to a number of investigations, but remain to be controversial [5]. Second, both physical and non-physical wave number solutions are present in Eq. 14. The non-physical wavenumbers may result in severe instability in the numerical solution [27]. In order to avoid these issues, we propose a second-order nonlocal effective medium model for transient wave propagation simula-

tions. The model is formulated in the Laplace domain and by employing the spatial-temporal nonlocal homogenization model with projection tensors and weighting tensors computed in different high symmetry directions. The nonlocal features of the homogenization model are retained by a nonlocal moduli tensor that is dependent on the Laplace variable (i.e., frequency).

Applying the Laplace transformation to Eq. 14 and neglecting the residual term, the spatial-temporal nonlocal equation is expressed in the Laplace domain:

$$\rho_0 s^2 \mathbf{U} - \nabla_x \cdot \left(D^{(0)} \cdot \nabla_x^s \mathbf{U} \right) = \nabla_x \cdot \left(\alpha \cdot (\nabla_x)^2 \nabla_x^s \mathbf{U} \right) + s^2 \nabla_x \cdot (\beta \cdot \nabla_x^s \mathbf{U}) + s^4 \gamma \cdot \mathbf{U} \quad (29)$$

where $s = \sigma + i\omega$ is the complex-valued Laplace variable. We seek the nonlocal effective medium model in the form of a second-order PDE:

$$\rho_0 s^2 \mathbf{U} - \nabla_x \cdot \left(D^{(e)}(s) \cdot \nabla_x^s \mathbf{U} \right) = \mathbf{0} \quad (30)$$

where, $D_{ijmn}^{(e)}(s)$ is the nonlocal effective moduli tensor that has minor symmetry in the last two indices, $D_{ijmn}^{(e)} = D_{ijnm}^{(e)}$, due to the symmetry of local strain tensor. Considering macroscopic orthotropy, it has 6 non-zero independent components for 2D in-plane wave propagation problems. Inserting a plane wave, $\mathbf{U}(\mathbf{x}, s) = U(s) \mathbf{p} e^{i\mathbf{k} \cdot \mathbf{x}}$, into Eq. 29 and 30, the following relations are satisfied respectively for the homogenization model and the effective medium model:

$$\begin{aligned} \tilde{B}_{in}(s, k, \mathbf{n}) p_n = \\ \left[\alpha_{ijprmn} n_m n_r n_p n_j k^4 + (-s^2 \beta_{ijmn} - D_{ijmn}^{(0)}) n_m n_j k^2 + (s^4 \gamma_{in} - \rho_0 s^2 \delta_{in}) \right] p_n = 0 \end{aligned} \quad (31a)$$

$$\bar{B}_{in}(s, k, \mathbf{n}, D^{(e)}) p_n = \left[\rho_0 s^2 \delta_{in} + D_{ijmn}^{(e)} n_m n_j k^2 \right] p_n = 0 \quad (31b)$$

For any given s and $\mathbf{n}$, the wavenumbers and polarization vectors of the nonlocal homogenization model are computed by taking the determinant of $\tilde{\mathbf{B}}$ to be zero. Both physical and non-physical wavenumber solutions exist, and the non-physical wavenumbers are identified as those that have negative imaginary parts, which result in amplifying plane waves and instability in transient simulations. Only the physical wavenumbers that have non-negative imaginary parts are employed in the derivation of the nonlocal effective medium model. The nonlocal effective moduli tensor is determined such that the discrepancy of wavenumbers computed from the nonlocal effective medium model, Eq. 31b, and the physical wavenumbers of the nonlocal homogenization model, Eq. 31a, is minimized for any s along high symmetry directions, $\hat{\mathbf{n}}$. For each direction, the model parameters of the nonlocal homogenization model is computed as discussed in Section 3.2 and 3.3.

The components of the nonlocal effective moduli tensor are determined in two steps. First,

the wavenumbers of directions along the x_1 and x_2 planes computed by Eq. 31b are matched to those computed by Eq. 31a, since P and SV modes are uncoupled in these directions and the wavenumber of each mode of the nonlocal effective medium model is uniquely determined by one component of $D_{ijmn}^{(e)}(s)$ that $j = m$:

$$D_{1111}^{(e)} = -\rho_0 s^2 / \tilde{k}_{(11)}^2, \quad D_{2112}^{(e)} = -\rho_0 s^2 / \tilde{k}_{(12)}^2 \quad (32a)$$

$$D_{1221}^{(e)} = -\rho_0 s^2 / \tilde{k}_{(21)}^2, \quad D_{2222}^{(e)} = -\rho_0 s^2 / \tilde{k}_{(22)}^2 \quad (32b)$$

where $\tilde{k}_{(ij)}^2$ is the square of the physical wavenumber of j -direction-polarized wave propagating in the i -direction, computed from Eq. 31a. Equation 32 ensures that the wave dispersion predicted by the nonlocal homogenization model in these orthogonal high symmetry directions (e.g., $\Gamma - X$ and $\Gamma - Y$ directions in Fig. 2(c)) is captured by the nonlocal effective medium model. Second, in order to compute the components that $j \neq m$ (i.e., $D_{1122}^{(e)}$, $D_{2211}^{(e)}$), we incorporate the high symmetry direction that is not aligned in the coordinate planes (e.g., $\Gamma - M$ in Fig. 2(c)) and minimize the discrepancy between wavenumbers computed from Eq. 31b and Eq. 31a. Denote the wavenumbers and polarization vectors computed from equating the determinant of $\tilde{\mathbf{B}}$ to zero as: $\tilde{\mathbf{k}} = [\tilde{k}_{(P)}, \tilde{k}_{(SV)}]^T$ and $\tilde{\mathbf{P}} = [\tilde{\mathbf{p}}_{(P)}, \tilde{\mathbf{p}}_{(SV)}]$. Substituting $\tilde{\mathbf{p}}_{(\cdot)}$, $(\cdot)$ indicating P mode or SV mode, into Eq. 31b, the wavenumber of the nonlocal effective medium model, $\bar{k}_{(\cdot)}^2$, is respectively related to $D_{1122}^{(e)}$ and $D_{2211}^{(e)}$:

$$\bar{k}_{(\cdot)}^2 \left(D_{1122}^{(e)} \right) = \frac{-\rho_0 s^2 \tilde{p}_{(\cdot)1}}{\left(D_{1111}^{(e)} \hat{n}_1 \hat{n}_1 + D_{1221}^{(e)} \hat{n}_2 \hat{n}_2 \right) \tilde{p}_{(\cdot)1} + \left(D_{1122}^{(e)} \hat{n}_2 \hat{n}_1 + D_{1212}^{(e)} \hat{n}_1 \hat{n}_2 \right) \tilde{p}_{(\cdot)2}} \quad (33a)$$

$$\bar{k}_{(\cdot)}^2 \left(D_{2211}^{(e)} \right) = \frac{-\rho_0 s^2 \tilde{p}_{(\cdot)2}}{\left(D_{2112}^{(e)} \hat{n}_1 \hat{n}_1 + D_{2222}^{(e)} \hat{n}_2 \hat{n}_2 \right) \tilde{p}_{(\cdot)2} + \left(D_{2211}^{(e)} \hat{n}_1 \hat{n}_2 + D_{2121}^{(e)} \hat{n}_2 \hat{n}_1 \right) \tilde{p}_{(\cdot)1}} \quad (33b)$$

$D_{1122}^{(e)}$ and $D_{2211}^{(e)}$ are respectively solved by eliminating the total normalized error of wavenumbers in the two modes:

$$\Phi_1 \left(D_{1122}^{(e)} \right) = \left| \frac{\tilde{k}_{(P)}^2 - \bar{k}_{(P)}^2 \left(D_{1122}^{(e)} \right)}{\tilde{k}_{(P)}^2} \right| + \left| \frac{\tilde{k}_{(SV)}^2 - \bar{k}_{(SV)}^2 \left(D_{1122}^{(e)} \right)}{\tilde{k}_{(SV)}^2} \right| = 0 \quad (34a)$$

$$\Phi_2 \left(D_{2211}^{(e)} \right) = \left| \frac{\tilde{k}_{(P)}^2 - \bar{k}_{(P)}^2 \left(D_{2211}^{(e)} \right)}{\tilde{k}_{(P)}^2} \right| + \left| \frac{\tilde{k}_{(SV)}^2 - \bar{k}_{(SV)}^2 \left(D_{2211}^{(e)} \right)}{\tilde{k}_{(SV)}^2} \right| = 0 \quad (34b)$$

Compared to the local homogenization model which has static moduli tensor, the effective medium is nonlocal and its moduli tensor depends on the Laplace variable, which captures wave dispersion caused by the microstructures. Taking the real part of the Laplace variable

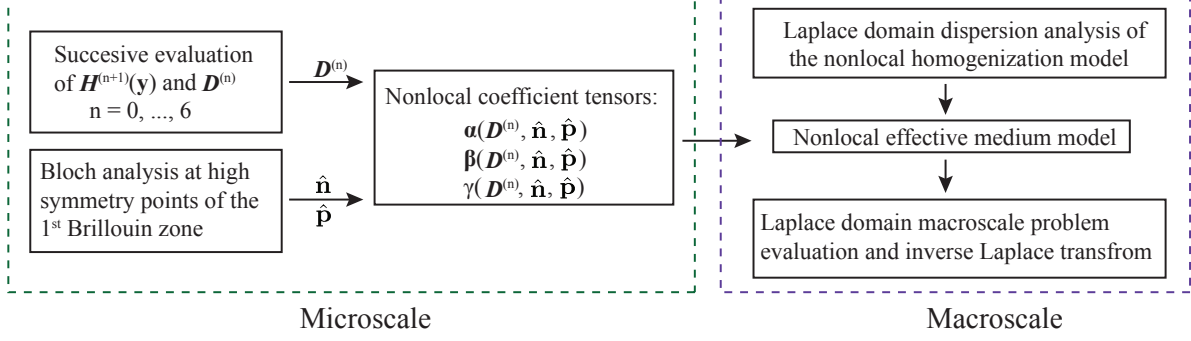


Figure 3: Computational flowchart.

asymptote to 0, the dynamic behavior of the moduli tensor can be interpreted in terms of frequency. In low frequency regime, the nonlocal homogenization model is non-dispersive, therefore the nonlocal effective moduli tensor recovers the local counterpart and the nonlocal effective medium model recovers local homogenization model. As the frequency increases, wave dispersion occurs. The nonlocal effective medium model matches the dispersion of the nonlocal homogenization model in high symmetry directions and approximates wave dispersion in other directions.

4 Model implementation

In this section, we briefly present the implementation procedure of the multiscale system for transient wave propagation simulations. As is shown in Fig. 3, the overall procedure consists of two steps: (1) microscale problem solution and coefficient tensors computation; (2) Laplace domain macroscale problem evaluation and inverse Laplace transform.

Microscale problems defined over the domain of the unit cell constitute evaluation of influence functions and homogenized moduli, as well as Bloch analysis for the characteristic waves. The influence functions $\mathbf{H}^{(n+1)}$ and homogenized moduli $\mathbf{D}^{(n)}$ are evaluated sequentially using the finite element method (FEM). Periodic boundary conditions are applied at the microstructure boundary nodes. $\mathbf{H}^{(n+1)}$ is an order $(n+3)$ tensor and has 6×2^n independent components. Since the sequence of taking the $(n)^{th}$ gradient of $\nabla_x^s \mathbf{U}^{(i-n-1)}(\mathbf{x}, t)$ in Eq. 6 is interchangeable, when the resulting tensor is contracted with the microstructural influence function, only the symmetric part of $\mathbf{H}^{(n+1)}$ in those n dimensions, $\mathbf{H}_{sym,n}^{(n+1)}(\mathbf{y})$, affects the contraction [27]. Therefore, $\mathbf{H}_{sym,n}^{(n+1)}(\mathbf{y})$ is computed instead and the number of independent component is $6(n+1)$. Similarly, only the symmetric part of $\mathbf{D}^{(n)}$ in dimensions contracted with $(\nabla_x)^n \nabla_x^s \mathbf{U}^{(\alpha-n)}$ is computed in Eq. 10. The resulting number of computed independent component of $\mathbf{D}^{(n)}$ is $12(n+1)$.

The Bloch analysis is performed using the same finite element discretization of the microstructure. $\omega(\mathbf{k})$ method is used and the prescribed wavevector is taken as the positions of the high symmetry points of the first Brillouin zone. Three symmetry points are selected to compute $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$ of the characteristic waves, which are then used to compute the projection tensors (Section 3.2) and weighting tensors (Section 3.3). Thus a unique set of coefficient tensors of the spatial-temporal nonlocal homogenization model is determined for each of the selected high symmetry directions.

The single-variable constrained minimization problems in Sections 3.3.1 and 3.3.2 are implemented using the *fminbnd* function in MATLAB, which is based on Brent's method [12] and it combines the golden section search algorithm with successive parabolic interpolation. The microscale problems and computation of the nonlocal coefficient tensors are independent of the macroscale problem, therefore are implemented off-line as a preprocessing step.

The macroscale problem is implemented in the Laplace domain by sampling the Laplace variable. For each sampled Laplace variable, a dispersion analysis is performed for the fourth-order nonlocal homogenization model with projection tensors and weighting tensors computed in high symmetry directions to obtain the physical wavenumbers in these directions. The nonlocal effective medium model is formulated using the physical wavenumbers following the procedure in Section 3.4. The macroscale displacement field is obtained by solving the nonlocal effective medium model using Isogeometric analysis (IGA) [29] with C^1 continuity. Compared to FEM with quadratic Lagrange shape functions, IGA achieves higher convergence rate [30], therefore accurately describes high frequency waves with fewer degrees of freedom. Time domain response is obtained by a numerical inverse Laplace transform algorithm [15]. The evaluation of macroscale problem for each sampled Laplace variable is independent of another, therefore, the implementation for the macroscale problem is easily parallelized.

5 Model verification

We evaluate the proposed approach in three aspects, (1) dispersion relation of the nonlocal homogenization model (Eq. 14), (2) dispersion relation of the nonlocal effective medium model (Eq. 30), and (3) transient wave propagation simulation using the nonlocal effective medium model. Two-dimensional in-plane elastic wave propagation in two types of composites are considered, i.e., bi-material layered and matrix reinforced with a circular inclusion.

5.1 Dispersion of the nonlocal homogenization model

Wave dispersion of two types of unit cells are considered, as shown Fig. 2(b), where the length of the unit cell is $l = 0.02$ m. The density, Young's modulus and Poisson's ratio are

respectively taken as 2700 kg/m³, 68 GPa and 0.3 for Phase 1, and 7900 kg/m³, 210 GPa and 0.3 for Phase 2. The volume fraction of Phase 2 is 0.5 and 0.2 for the layered and matrix-inclusion unit cells respectively. The first Brillouin zone of the unit cells is shown in Fig. 2(c) and only the shaded area is considered due to the symmetry of the unit cells.

5.1.1 Verification of dispersion relation

The dispersion curves using the spatial-temporal nonlocal homogenization model (referred to as STNHM) are computed for each high symmetry direction of the first Brillouin zone, i.e., $\Gamma - X$, $\Gamma - M$, $\Gamma - Y$ in Fig. 2(c), using Eq. 31a by sampling the imaginary part and taking the real part of s as a much smaller number (10^{-6} is used in the current analysis). In this limiting case, the Laplace domain dispersion analysis can be viewed similar to the frequency domain analysis, while the complex-valued wavenumbers can be identified as physical (wavenumbers with positive real part and positive imaginary part) or non-physical (wavenumbers with positive real part and negative imaginary part). Only the physical wavenumbers are studied here since the non-physical wavenumbers lead to unstable waves and are suppressed in transient simulations. Wavenumbers with positive real part and non-negative imaginary part computed from the Bloch wave expansion using the $\mathbf{k}(\omega)$ approach [2, 36] are employed as reference solutions. The stop band is defined as the frequency range that no propagating Bloch wavenumber solutions exist. This not only includes the situations, where real wavenumber solutions do not exist (as is often used for identifying the stop band), but also the situations, where the wavenumber is evanescent [37]. In case of evanescent waves, the wavenumber is complex-valued and wave propagation is exponentially attenuated due to the presence of the imaginary part. The lowest 10 Bloch wavenumbers are computed and sorted. The first optical Bloch wavenumber branch is plotted by mirroring the corresponding one within the first Brillouin zone, so that positive group velocity is obtained as it is in the acoustic branch. The different wave modes (P and SV modes) of STNHM and the reference solutions are classified by projecting respectively the normalized mode shapes (the real part of the eigenvector) and normalized unit cell averaged mode shapes onto the direction of wave propagation. The P and SV modes are identified as the absolute values of this projection being greater and less than $\sqrt{2}/2$, respectively.

Figure 4 shows the lowest wavenumber of each mode computed using STNHM and the Bloch wave expansion for the layered microstructure, where the real part is normalized and plotted in the right panel and the imaginary part is plotted in the left panel. Wave dispersion in $\Gamma - X$ direction is shown in Fig. 4(a). The homogenization models with the weighting tensor $\boldsymbol{\nu}^{(3)}$ obtained by minimizing the asymptotic error (Section 3.3.1) and calibrating the stop band width (Section 3.3.2) are denoted as STNHM-1 and STNHM-2, respectively. Both

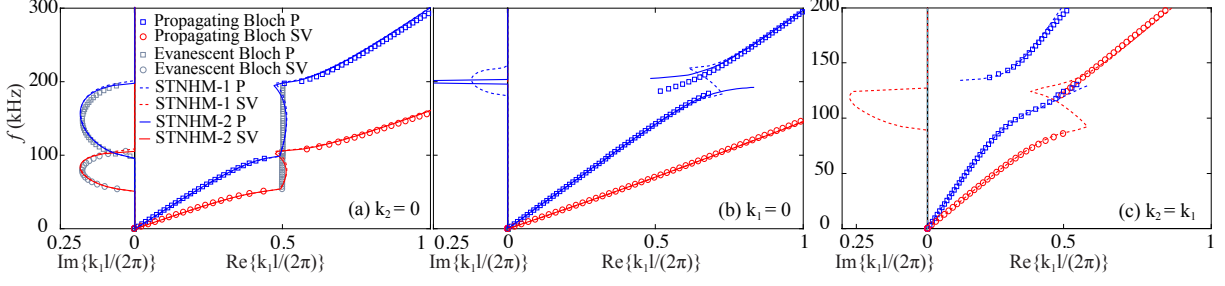


Figure 4: Dispersion curves of the layered microstructure for wave propagation in (a) $\Gamma - X$ direction, (b) $\Gamma - Y$ and (c) $\Gamma - M$ direction.

methods of determining the model parameters result in accurate prediction of the acoustic branch ($\text{Re}\{k_1 l/(2\pi)\} < 0.5$), the first stop band ($\text{Re}\{k_1 l/(2\pi)\} = 0.5$) and the first optical branch ($\text{Re}\{k_1 l/(2\pi)\} > 0.5$). The stop bands of Bloch waves for P and SV modes are featured by the wavenumber solutions that have both positive real and imaginary parts. While wave propagation is supported by the real part of solution, it is exponentially attenuated due to the imaginary part. Within the first stop band, STNHM captures both real part and imaginary part of the lowest evanescent Bloch modes. The stop band of SV mode occurs at a lower frequency and has a smaller size compared to the P mode. The dispersion curves beyond $\text{Re}\{k_1 l/(2\pi)\} = 1$ are not plotted since the wavelength is smaller than the size of microstructure and the asymptotic homogenization does not apply in that regime. Figure 4(b) shows wave dispersion in the $\Gamma - Y$ direction. The SV mode is non-dispersive and both STNHM models predict this behavior. P mode is dispersive only at high frequency, where the lowest eigenvalue switches from one branch to another. This switch is captured by the STNHM models. While Bloch P mode is propagative (eigenvalues are real-valued) for all frequencies, $f \in [0, 300]$ kHz, STNHM P modes become evanescent at the switch. STNHM models capture the propagating mode at higher frequencies. Wave dispersion in $\Gamma - M$ direction is shown in Fig. 4(c). STNHM-1 predicts wave dispersion of the acoustic branch and the first optical branch. The stop band is not observed in both Bloch and STNHM P modes since real-valued eigenvalues exist for all frequencies, $f \in [0, 200]$ kHz. At $\text{Re}\{k_1 l/(2\pi)\} = 0.5$, the SV mode enters the stop band, which is featured by the fact that the Bloch waves do not have a propagating SV mode. STNHM-1 captures the location and size of the stop band. However, instead of having an absence of wavenumber solutions, it predicts evanescent waves that are strongly attenuated by the imaginary part.

Figure 5 shows the dispersion curves of the matrix-inclusion microstructure in directions $\Gamma - X$ and $\Gamma - M$. The response in the $\Gamma - Y$ direction is identical to that of $\Gamma - X$ due to symmetry. In Fig. 5(a), both STNHM-1 and STNHM-2 capture the acoustic branch and the first optical branch of P and SV modes. Compared to determining $\nu^{(3)}$ by minimizing the

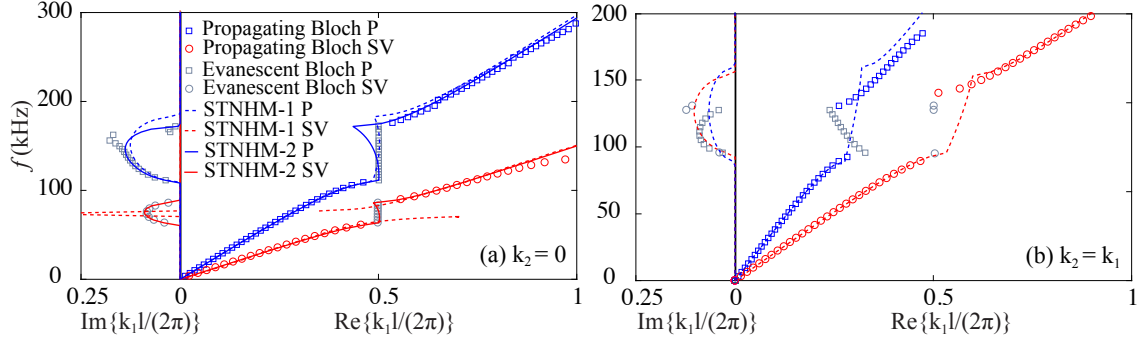


Figure 5: Dispersion curves of the matrix-inclusion microstructure.

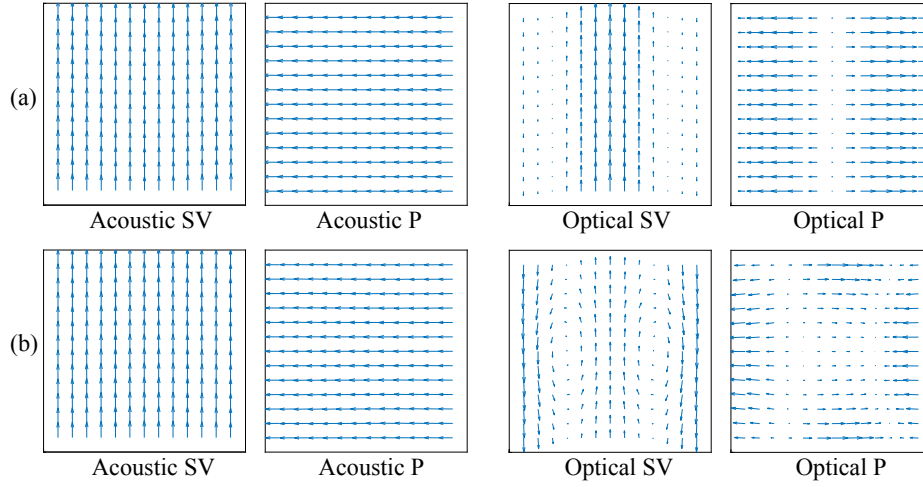


Figure 6: Typical acoustic and optical Bloch mode shapes in the $\Gamma - X$ direction.

asymptotic error, calibrating $\nu^{(3)}$ by minimizing the discrepancy of stop band width prediction between STNHM and Bloch solutions results in more accurate prediction in the width of the stop band and wave attenuation caused by the imaginary part of the wavenumber. This is natural as the model is optimized to capture particularly this behavior. In the $\Gamma - M$ direction, Fig. 5(b), STNHM-1 accurately predicts the acoustic branch of P and SV modes. While STNHM-1 predicts the initiation of the stop band, the error increases in the prediction of the attenuation and end of the stop band that occurs at higher frequency. Wave propagation in the first optical branch is captured well. STNHM-1 predicts the group velocity (slope of the dispersion curves) of both modes, while the phase velocity of P wave is over predicted since the optical P branch is shifted to higher frequency due to the error in the prediction of the onset of the optical pass band.

It is observed that the propagating Bloch wavenumbers captured by STNHM are the ones of the lowest rank of P and SV modes, i.e., the acoustic and first optical branch. In fact, there are infinitely many wavenumber solutions for any given frequency, which correspond to

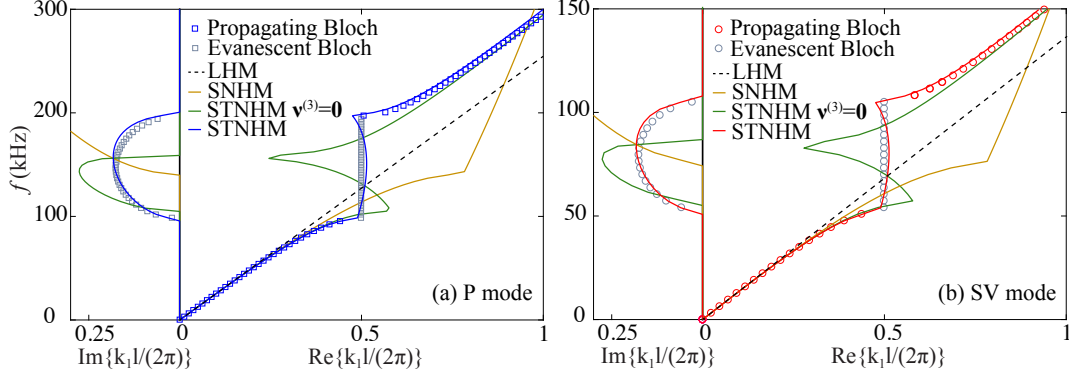


Figure 7: Dispersion curves of different asymptotic homogenization models.

wavenumbers of different multiplicity of $2\pi/l$. The typical Bloch mode shapes of the acoustic and first optical branch are shown in Fig. 6. Figures 6(a) and (b) illustrate the displacement field of the unit cell when the wave is propagating in the $\Gamma - X$ direction for the layered and matrix-inclusion unit cells, respectively. The acoustic and optical mode shapes are plotted at $\text{Re}\{k_1 l/(2\pi)\} = 0.25$ and $\text{Re}\{k_1 l/(2\pi)\} = 0.75$. For both cases, the entire unit cell move uniformly in the acoustic regime, parallel and perpendicular to the direction of wave propagation for P and SV waves respectively. The optical modes are featured by the out-of-phase displacement field. The material points change direction of motion within a distance of about half of the unit cell in the direction of wave propagation. Higher optical modes that are not captured correspond to the displacement field varies more rapidly (e.g., the displacement field changes direction of motion multiple times within the unit cell) in the direction of wave propagation, or varies not only in the direction of wave propagation, but also along the transverse direction.

5.1.2 Effects of the nonlocal terms

The nonlocal terms in the homogenization model, Eq. 14, contribute to its capability in capturing wave dispersion. By setting the weighting tensors, $\boldsymbol{\nu}^{(1)}$, $\boldsymbol{\nu}^{(2)}$, $\boldsymbol{\nu}^{(3)}$, all equal to $\mathbf{0}$, nonlocal terms of the spatial-temporal nonlocal homogenization model vanish and it recovers the local homogenization model (LHM) with asymptotic residual of $O(\zeta^2)$. The spatial nonlocal homogenization model proposed in Refs. [21, 32] (SNHM) is recovered by setting $\boldsymbol{\nu}^{(1)} = \mathbf{I}$, $\boldsymbol{\nu}^{(2)} = \boldsymbol{\nu}^{(3)} = \mathbf{0}$, which is of order $O(\zeta^4)$. An alternative spatial-temporal nonlocal homogenization model is obtained by using $\boldsymbol{\nu}^{(1)} = \boldsymbol{\nu}^{(2)} = \mathbf{I}$, $\boldsymbol{\nu}^{(3)} = \mathbf{0}$. This model can be derived by employing macroscale balance equations (Eq. 10) up to $O(\zeta^4)$ therefore has asymptotic accuracy of $O(\zeta^6)$. Figure 7 compares the dispersion behaviors predicted by these models with the STNHM model with $\boldsymbol{\nu}^{(3)}$ computed by minimizing the $O(\zeta^6)$ asymptotic residual in the $\Gamma - X$ direction.

For both P and SV modes, STNHM with minimized asymptotic residual achieves the

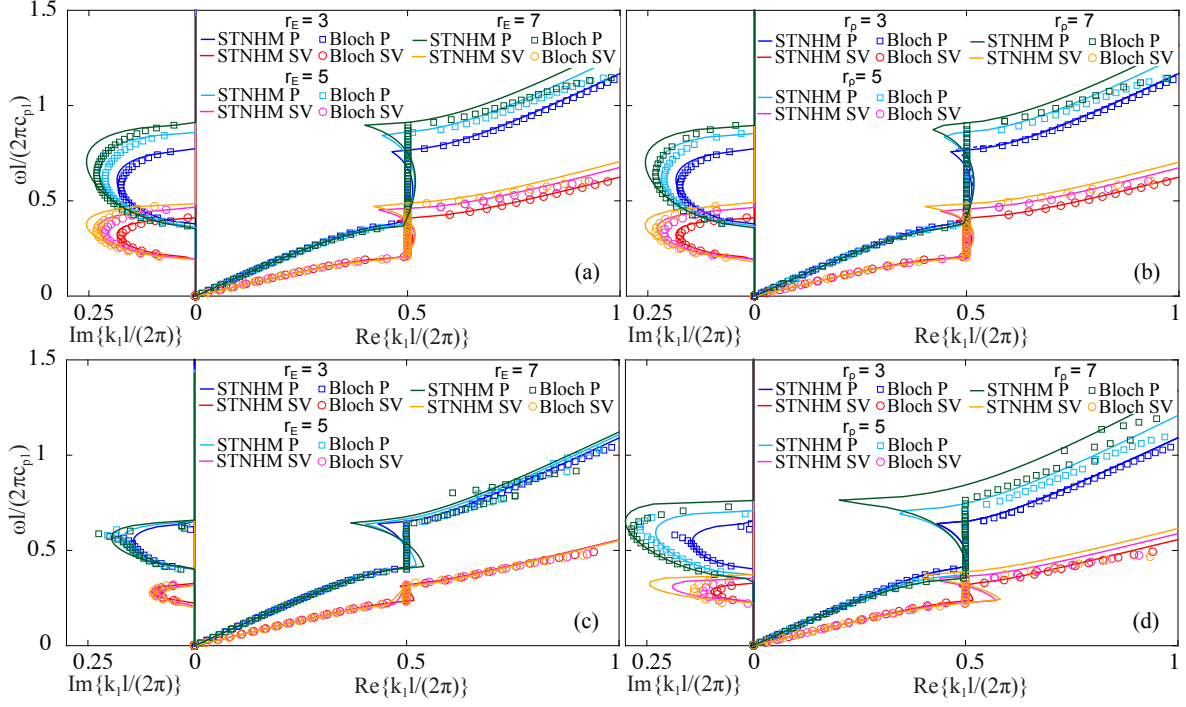


Figure 8: Dispersion curves of wave propagation in the $\Gamma - X$ direction. (a) Layered microstructure with $r_\rho = 3$. (b) Layered microstructure with $r_E = 3$. (c) matrix-inclusion microstructure with $r_\rho = 3$. (d) matrix-inclusion microstructure with $r_E = 3$.

best accuracy. LHM predicts wave propagation without dispersion which is valid up to $\text{Re}\{k_1l/(2\pi)\} = 0.2$. The spatial nonlocal term enables SNHM to predict wave dispersion in the long wavelength regime, $\text{Re}\{k_1l/(2\pi)\} < 0.4$. Although it introduces wave attenuation, the initiation of the stop band is not well predicted. Moreover, its dispersion relation behaves as a low-pass filter, which artificially attenuates all the high frequency waves. The significance of the temporal nonlocal term and mixed spatial-temporal nonlocal term is that they restrict the stop band to a finite size, which allows STNHM to capture the optical branch, as is demonstrated by STNHM with $\boldsymbol{\nu}^{(3)} = \mathbf{0}$. Furthermore, the increased asymptotic accuracy results in more accurate prediction in the acoustic regime, $\text{Re}\{k_1l/(2\pi)\} < 0.5$. By minimizing the asymptotic error, STNHM improves the accuracy in the prediction of dispersion of shorter waves, $0.5 < \text{Re}\{k_1l/(2\pi)\} < 1$. However, due to the limit of separation of scales, STNHM cannot be applied to situations where the wavelength is shorter than the size of microstructure.

5.1.3 Effects of material property contrast

The accuracy of the proposed nonlocal homogenization model, when the weighting tensor $\boldsymbol{\nu}^{(3)}$ is determined by minimizing the asymptotic residual, depends on the contrast between the properties of constituent materials. It has been demonstrated that wave dispersion up

to the initiation of the stop band is not affected significantly with the property contrast, the error in the prediction of the size of the stop band increases as the material property contrast increases in the context of scalar-field waves [26, 27]. The alternative approach of computing $\nu^{(3)}$, i.e., calibrating the width of the stop band (STNHM-2) naturally guarantees the accuracy in predicting the width of the stop band independent of the material property contrast.

Figure 8 presents a parametric study of the accuracy of STNHM-2 against the Bloch wave solution, in predicting wave dispersion in the $\Gamma - X$ direction for the unit cells in Fig. 2(b). The vertical axis $\omega l / (2\pi c_{P1})$ is the normalized frequency, where c_{P1} is the P wave velocity of the homogenization model in the quasi-static condition, i.e., $c_{P1} = \sqrt{D_{1111}^{(0)} / \rho_0}$. The parametric study is performed by fixing the properties of Phase 1 and varying the Young's modulus and density of Phase 2. The contrast is measured by ratio between Phase 2 and Phase 1, e.g., $r_E = E_2 / E_1$. The investigated material properties remain in the low contrast regime. For all studied cases, the general trend is that the prediction of dispersion within the acoustic branch, and the initiation and size of the stop band is accurate and not sensitive to the contrast in Young's modulus and density. Wave attenuation, due to the imaginary part of the wavenumber, within the stop band in the high frequency regime and the wave dispersion within the optical branch are affected, i.e., the prediction error increases as the material property contrast increases. The proposed approach captures the group velocity in the lower optical branch. As the wavelength decreases in higher frequency regime and approaches the limit of separation of scales, the model becomes non-dispersive, featured by the constant group velocity. While this behavior matches with Bloch waves for unit cells with low material property contrast, i.e., $r_E = 3$ and $r_\rho = 3$. The discrepancy becomes significant as the material property contrast increases.

5.2 Dispersion of the nonlocal effective medium model

In this section, we investigate the dispersion behavior predicted by the nonlocal effective medium (NEM) model for the microstructures shown in Fig. 2(b) with material properties used in Section 5.1.1. The NEM model is formulated based on the procedure described in Section 3.4, where the spatial-temporal nonlocal homogenization models with projection tensors and weighting tensors evaluated in $\Gamma - X$, $\Gamma - Y$ and $\Gamma - M$ directions are employed. The homogenization models with $\nu^{(3)}$ determined by the band gap size matching approach are used in $\Gamma - X$ and $\Gamma - Y$ directions, and the asymptotic residual minimization approach is used in the $\Gamma - M$ direction. Wavenumbers computed by Eq. 30 at two frequencies, $f = 50$ kHz and $f = 80$ kHz, in all directions within the $[x_1, x_2]$ plane are compared with the Bloch waves. Polar representation is employed to reveal the different dispersion behavior in different directions of wave propagation.

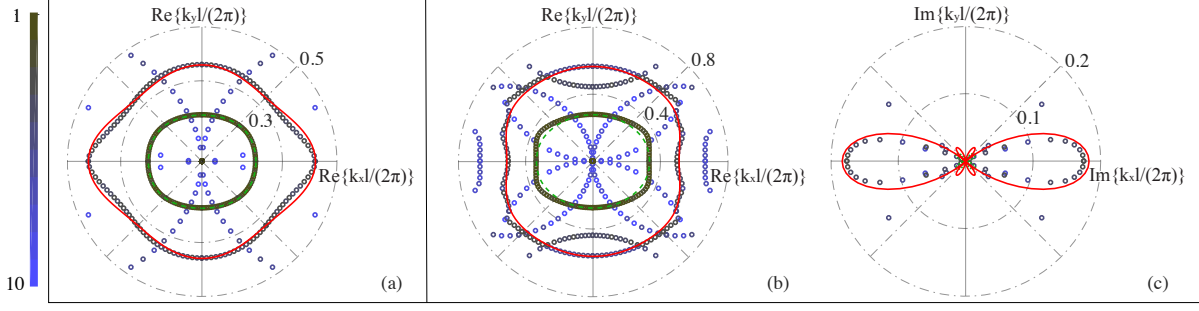


Figure 9: Wavenumbers in polar coordinate for the layered microstructure. (a) Real part of the wavenumber at $f = 50$ kHz. (b) Real part of the wavenumber at $f = 80$ kHz. (c) Imaginary part of the wavenumber at $f = 80$ kHz. Solid line, '—', SV mode; Dashed line, '- -', P mode; Circles, 'o', Bloch waves. The rank of Bloch eigenvalues increases as the color scale changes from dark to light.

Figure 9 shows the dispersion relation for the layered microstructure. It is observed from Fig. 4 that P wave is non-dispersive at $f = 50$ kHz in $\Gamma - X$, $\Gamma - Y$ and $\Gamma - M$ directions. As a result, the predicted P wavenumber is non-dispersive and matches with the lowest Bloch eigenvalue exactly in all directions in Fig. 9(a). SV wave has slower wave speed and larger wavenumber at this frequency. It enters the dispersive acoustic regime in $\Gamma - X$ and $\Gamma - M$ directions. NEM accurately captures the wavenumber in these directions. The prediction error increases as the direction of wave propagation migrates away from the high symmetry directions, e.g., the noticeable discrepancy in 25° . As wave frequency is increased to $f = 80$ kHz, SV mode within $(-30^\circ, 30^\circ)$ enters the stop band as is indicated by the imaginary part of the wavenumber shown in Fig. 9(c). P mode becomes dispersive which is revealed in the change of the shape of wavesurface (Fig. 9(b)) compared to Fig. 9(a)). The proposed model predicts the wavesurfaces of both P and SV modes. It captures the SV imaginary wavenumber in the directions that reside in the stop band. We observe that spurious imaginary SV and P wavenumbers are introduced in the directions of 50° and 40° , respectively. This is linked to the formulation of the nonlocal effective moduli tensor of Eq. 30 based on selected directions. When the SV mode in the $\Gamma - X$ direction enters the stop band, the computed value of $D_{2112}^{(e)}$ is complex, thus introducing imaginary part to wavenumbers of SV and P modes in all directions, not only within $(-30^\circ, 30^\circ)$ where Bloch wavenumber is complex-valued, but also the directions that no wave attenuation occurs (e.g, 50°). Nevertheless, they have small magnitudes in the studied cases, therefore, do not result in significant artificial wave attenuation.

Figure 10 shows the dispersion relation for the matrix-inclusion microstructure at the same frequencies as Fig. 9. At $f = 50$ kHz, the non-dispersive P wavesurface is circular-shaped, indicating isotropic macroscopic wave propagation. The SV wavesurface is distorted due to wave dispersion, which has larger magnitude in 0° and 90° compared to 45° . NEM captures

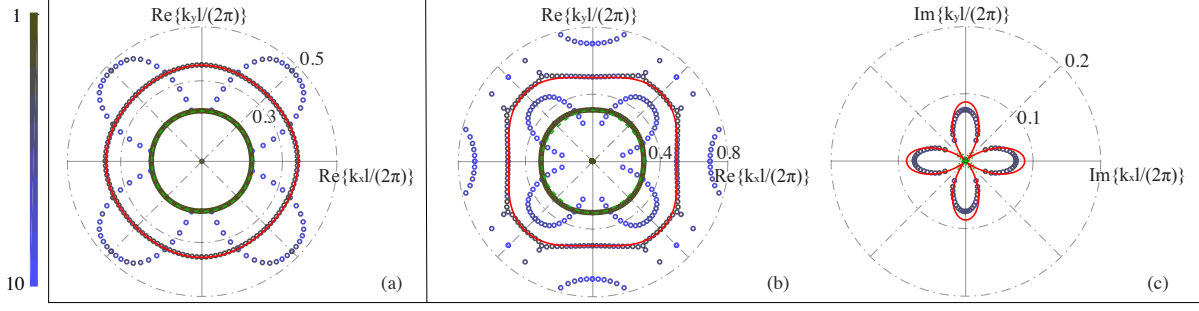


Figure 10: Wavenumbers in polar coordinate for the matrix-inclusion microstructure. (a) Real part of the wavenumber at $f = 50$ kHz. (b) Real part of the wavenumber at $f = 80$ kHz. (c) Imaginary part of the wavenumber at $f = 80$ kHz.

both P and SV wavesurfaces very accurately. At $f = 80$ kHz, the SV mode within $(-30^\circ, 30^\circ)$ and $(60^\circ, 120^\circ)$ enters the stop band. The proposed model predicts the wavenumbers of SV and P modes. In Figs. 9 and 10, the proposed model matches with the dark-colored Bloch eigenvalues only. The light-colored Bloch eigenvalues correspond to evanescent waves that are of higher rank. Although their real parts fall inside the first Brillouin zone due to the periodicity of the wave vector in the Bloch theory, they are usually subject to strong attenuation [37] and do not contribute to wave propagation in the investigated frequency regime.

5.3 Transient elastic wave propagation

In this section, we investigate the transient elastic wave propagation in composites made of periodic layered microstructures. The volume fraction, size and material properties of the unit cell are identical to those in Section 5.1.1. Two examples are provided to evaluate the performance of the proposed model in predicting wave dispersion and attenuation, i.e., (1) transient uni-directional elastic wave propagation in the layered composite, (2) transient two-dimensional wave propagation in an elastic waveguide. Direct simulations of the heterogeneous structure are employed as the reference. IGA with sufficient refinement is used to discretize the structure for both direct simulation and the nonlocal effective medium model. Time integration of the direct simulation is implemented using the Newmark-beta implicit method.

5.3.1 Transient uni-directional elastic wave propagation

We consider transient wave propagation in a composite structure made of a row of 20 layered microstructures as shown in Fig. 11. The structure is fixed at the right edge and periodic boundary conditions are applied on the top and bottom edges. Two types of sinusoidal in-plane displacement load are applied at the left edge, i.e., $\tilde{u}_1(t) = M \sin(2\pi ft)$ and $\tilde{u}_2(t) = M \sin(2\pi ft)$, which respectively generate P and SV waves.

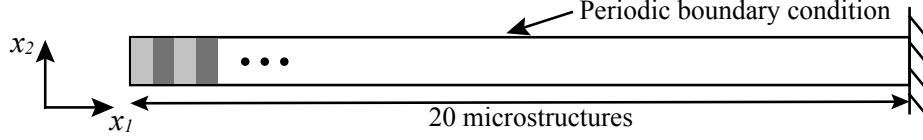


Figure 11: Periodic layered composite and boundary conditions.

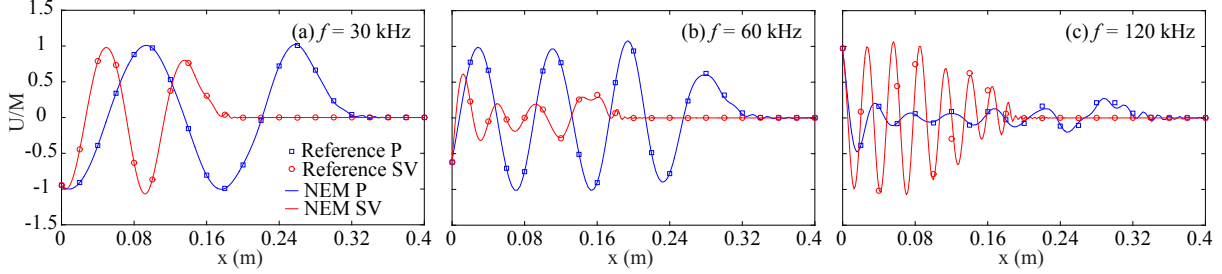


Figure 12: Displacement snapshots along the bottom edge of the structure.

Figure 12 shows the displacement snapshots measured along the bottom edge of the structure at $t = 0.06$ ms. The vertical axis U/M is the displacement normalized by the amplitude of applied load. Since only the displacement at the corner points of the unit cell allows one-to-one comparison between the direct simulation and the homogenization model [27], the displacement of direct simulation at these points are plotted as the reference. In Fig. 12(a), both SV and P waves are within the acoustic regime and dispersion occurs for the SV wave as manifested by the distorted wave front. As the frequency is increased to 60 kHz, SV wave enters the stop band and P wave becomes dispersive. Within the stop band, SV wave amplitude is significantly reduced. Further increase in the loading frequency results in shorter waves. At $f = 120$ kHz, SV wave enters the first optical branch while P wave falls inside the stop band. At all three frequencies, NEM accurately predicts the displacement of the reference model.

5.3.2 Elastic waveguide

Figure 13 shows the two-dimensional elastic waveguide and the boundary conditions for the simulation. The waveguide ($2l \leq x_1 \leq 14l$) is made by inserting Phase 2 layers into the base material, Phase 1, within the region labeled as Heterogeneous Medium. The structure is fixed on the right edge. Sinusoidal displacement load, $\tilde{u}_1(t) = M \sin(2\pi ft)$, is applied at the center of the left edge within $5l$. The rest of edges are traction free.

Figure 14 shows the displacement fields of NEM (the Heterogeneous Medium is modeled using the NEM model) compared with the direct simulation at $f = 60$ kHz. The total simulation time is $T = 0.1$ ms and the displacement fields are taken at $t = 0.3T$, $0.6T$ and T . At this frequency, the overall wave field predicted by NEM, Eq. 30, matches with the direct simulation. Propagation of macroscopic wave through the layered composite is allowed, the

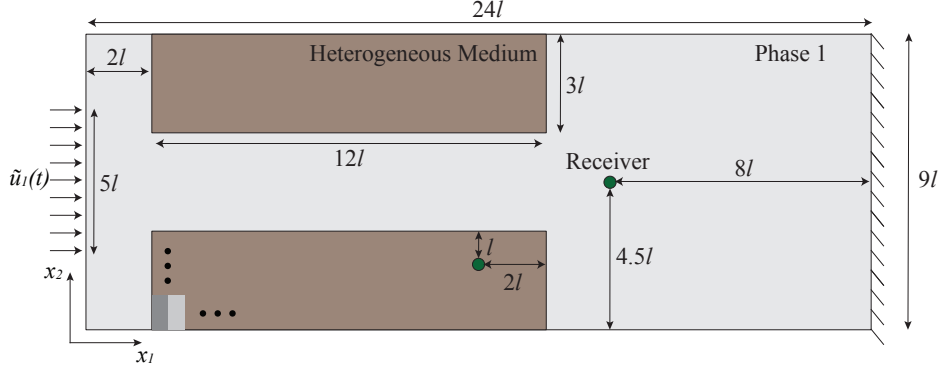


Figure 13: Sketch of the elastic waveguide and boundary conditions.

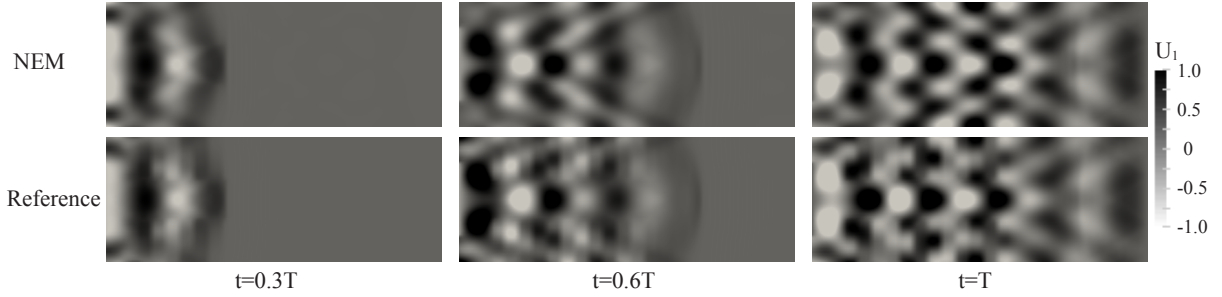


Figure 14: Displacement fields of wave propagation at $f = 60$ kHz.

propagating wave and reflected wave at the traction free boundary superimpose, resulting in the short wavelength pattern in x_2 direction (right columns of figures), not only in the base material but also in the layered composite. When the loading frequency is increased to $f = 150$ kHz as shown in Fig. 15, wave propagation is highly confined in the homogeneous base material as the wave passes the waveguide. Much smaller wave amplitude is observed within the layered composite compared to the base material. NEM captures the wave fields within the waveguide and after the wave exits the waveguide. Comparing Fig. 15 with 14, the predicted wave fields differs from the direct simulations at the waveguide entrance ($x_1 \leq 2l$) at high frequency. The cause of this discrepancy is that the homogenization model is formulated for the heterogeneous domain without considering its interface with other domains. It could be addressed by introducing a boundary layer between the domain of homogenized medium and adjacent domains, where energy flux balance and continuity conditions between these domains are enforced [65].

In order to quantitatively compare the results of NEM to the reference at different frequencies, the maximum transmitted wave amplitude U_t is recorded at two locations, $(12l, 2l)$ and $(16l, 4.5l)$ in Fig. 13, while sweeping the loading frequency within the range $[0, 150]$ kHz. Figure 16 shows the normalized transmitted wave amplitude spectrum. NEM captures the overall wave transmission pattern and the prediction error increases at high frequencies. At

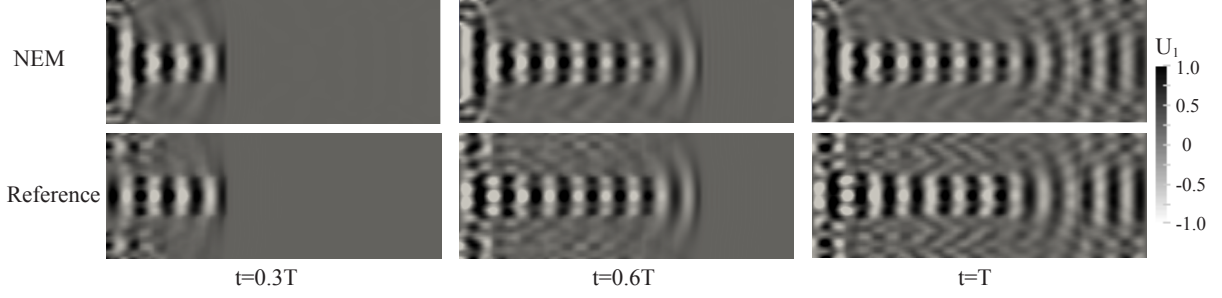


Figure 15: Displacement fields of wave propagation at $f = 150$ kHz.

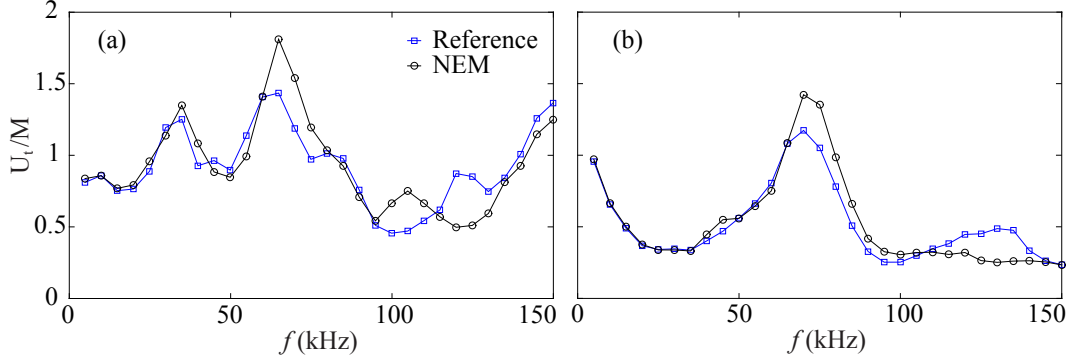


Figure 16: Transmitted wave amplitude spectrum at locations (a) $(12l, 2l)$ and (b) $(16l, 4.5l)$.

both locations, good accuracy is observed for $f \in [0, 60]$ kHz. Increased error is observed for $f \in [65, 75]$ kHz and $f \in [100, 130]$ at the Receiver located within the base material, and $f \in [70, 85]$ kHz and $f \in [125, 135]$ at the Receiver located within the layered composite. These errors result from three factors. First, the current homogenization model does not account for the presence of interface between the homogenized domain and the homogeneous domain, which results in prediction error at high frequency, as is observed in Fig. 15. Second, NEM is formulated based on the high symmetry directions of the Brillouin zone. However, for transient wave propagation in a two-dimensional domain, wave modes in all directions occur. Although these modes may be approximated by the nonlocal effective medium (Fig. 9), the approximation becomes less accurate at high frequency. Third, the fundamental assumption of separation of scales restricts the capability of the nonlocal effective medium model from predicting wave propagation of wavelength shorter than the size of unit cell and the accuracy of NEM decreases in high frequency regime. For the present in-plane wave propagation problem, this limit is imposed by the SV wave which propagates in shorter length compared to P wave at the same frequency.

6 Conclusion

This manuscript developed a spatial-temporal nonlocal homogenization model and a non-local effective medium model for in-plane wave propagation in periodic composites, accounting for wave dispersion and attenuation due to the Bragg scattering. In order to derive the nonlocal homogenization model and extend the applicability of asymptotic homogenization to short wavelength regime, asymptotic expansions of up to the eighth order are employed. The resulting homogenized momentum balance equation has higher order gradient terms, which are critical in capturing the stop band and optical branch of the dispersion curves.

A nonlocal effective medium model is formulated for transient elastic wave propagation based on the nonlocal homogenization model with model parameters computed from high symmetry directions of the first Brillouin zone. The effective medium model is a second order PDE, which shares the same structure as the classical local homogenization model. However, the nonlocal effective moduli is frequency dependent and carries the nonlocal characteristics of the fourth-order nonlocal homogenization model. Transient in-plane wave propagation is simulated using the nonlocal effective medium model.

The proposed model is verified for in-plane elastic wave propagation in two-dimensional composite configurations. It is shown that the nonlocal homogenization model captures the acoustic branch, the stop band and the first optical branch of the dispersion curves in the direction of high symmetry points of the first Brillouin zone. A general trend is that the accuracy decreases as the frequency increases beyond the acoustic regime and as the material property contrast increases. The nonlocal effective medium model matches the dispersion behavior of the spatial-temporal nonlocal homogenization model in the high symmetry directions and approximates it in other directions with reasonable accuracy. It accurately predicts uni-directional wave propagation and the overall wave dispersion and attenuation behavior of wave propagation in a two-dimensional waveguide.

The following challenges are to be addressed in the future in order to improve the proposed framework. First, the current model is based on the assumption of low material property contrast. In order to apply the proposed model to composites with highly contrasted constituents, e.g., acoustic metamaterial [40], a scaling parameter of material properties may need to be introduced, e.g., the double porosity-type scaling [60, 7]. Second, techniques of appropriate transition at the interface between the homogenized domain and other domains, e.g., the boundary layers [65], should be developed for more accurate transient simulations. Third, the homogenization model for viscoelastic composites will be developed based on the current framework. The viscoelastic dissipation induced heating [33, 28] may be an interesting topic to investigate as it introduces another mechanism in controlling wave dispersion and attenuation. It is extremely challenging, if not impossible, to extend the asymptotic approach beyond the

limit of separation of scales. In this regime, multiscale methods that does not rely on separation of scales, e.g., computational continua approach [22, 20] and variational multiscale enrichment method [54, 70, 71], can be applied to model transient dynamics of periodic composites.

Acknowledgment

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Appendix A Asymptotic analysis procedure

The derivation of equilibrium equations for microscale influence functions and macroscale balance equations are provided in this appendix.

Substituting Eqs. 2, 3, 4 into Eq. 1, the balance equations at each order is obtained:

$$O(\zeta^{-1}) : \quad \nabla_y \cdot \boldsymbol{\sigma}^{(0)}(\mathbf{x}, \mathbf{y}, t) = 0 \quad (\text{A.1a})$$

$$O(\zeta^\alpha) : \quad \nabla_x \cdot \boldsymbol{\sigma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t) + \nabla_y \cdot \boldsymbol{\sigma}^{(\alpha+1)}(\mathbf{x}, \mathbf{y}, t) = \rho(\mathbf{y}) \ddot{\mathbf{u}}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t) \quad (\text{A.1b})$$

where,

$$\boldsymbol{\sigma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t) = \mathbf{C}(\mathbf{y}) \cdot \left[\nabla_x^s \mathbf{u}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t) + \nabla_y^s \mathbf{u}^{(\alpha+1)}(\mathbf{x}, \mathbf{y}, t) \right] \quad (\text{A.2})$$

Substituting Eqs. 5 and 6 into Eq. A.2, $\boldsymbol{\sigma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t)$ is written as:

$$\boldsymbol{\sigma}^{(\alpha)}(\mathbf{x}, \mathbf{y}, t) = \sum_{k=0}^{\alpha} \mathbf{C}^{(k)}(\mathbf{y}) \cdot (\cdot \nabla_x)^k \nabla_x^s \mathbf{U}^{(\alpha-k)}(\mathbf{x}, t) \quad (\text{A.3})$$

where, the expression for $\mathbf{C}^{(k)}(\mathbf{y})$ is given in Eq. 8. Substituting Eqs. A.3 and 8 with $\alpha = 0$ into Eq. A.1a, the equilibrium equation for $\mathbf{H}^{(1)}(\mathbf{y})$ is obtained:

$$\nabla_y \cdot \left\{ \mathbf{C}(\mathbf{y}) \cdot \left[\mathbf{I} + \nabla_y \mathbf{H}^{(1)}(\mathbf{y}) \right] \right\} = 0 \quad (\text{A.4})$$

The boundary value problem is defined on the microscale and depends on the microstructure and material properties only. It is evaluated following the procedure in Ref. [32]. Applying the unit cell averaging operator to Eq. A.1b with $\alpha = 0$, the macroscale balance equation at $O(1)$ is obtained:

$$\nabla_x \cdot \left(\mathbf{D}^{(0)} \cdot \nabla_x^s \mathbf{U}^{(0)} \right) = \rho_0 \ddot{\mathbf{U}}^{(0)} \quad (\text{A.5})$$

Generalization of this procedure for higher orders results in the equilibrium equations for microscale influence functions (Eq. 7), the expressions for higher order homogenized moduli (Eq. 9) and the macroscale balance equations at higher orders (Eq. 10). At order $O(\zeta^\alpha)$, the equilibrium equation for the influence function, $\mathbf{H}^{(\alpha+2)}(\mathbf{y})$, is obtained through substituting Eqs. 5, 6, A.3, 8, equilibrium equations for influence functions of lower orders and macroscale balance equations up to order $O(\zeta^\alpha)$ into Eq. A.1b. Employing these equations and applying the averaging operator to Eq. A.1b at $O(\zeta^{\alpha+1})$, the macroscale balance equation is obtained.

Appendix B Derivation of the spatial-temporal non-local governing equations

This appendix first derives the spatial-temporal nonlocal macroscale balance equations at $O(\zeta^6)$, $O(\zeta^4)$ and $O(\zeta^2)$. The spatial-temporal nonlocal homogenized momentum balance equation, Eq. 14, is then derived by employing the macroscale balance equations at $O(\zeta^\alpha)$ with $\alpha = 0, 2, 4, 6$.

Rewriting Eq. 12 with $\alpha = 6$ in indicial notation:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(6)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(6)})_{,x_j} &= \nu_{iq}^{(1)} D_{qjprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + \\ \nu_{iq}^{(2)} D_{qjprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} &+ \nu_{iq}^{(3)} D_{qjprstuvmn}^{(6)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} + \\ (\delta_{iq} - \nu_{iq}^{(1)}) D_{qjprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} &+ (\delta_{iq} - \nu_{iq}^{(2)}) D_{qjprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} + \\ (\delta_{iq} - \nu_{iq}^{(3)}) D_{qjprstuvmn}^{(6)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} \end{aligned} \quad (\text{B.1})$$

Substituting Eq. 13a into the third term on the right side of Eq. B.1, taking two spatial derivatives and inserting Eq. 10 with $\alpha = 4$ into the resulting expression of Eq. B.1 and considering Eqs. 13b and 13c:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(6)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(6)})_{,x_j} &= \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} \\ &+ \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) D_{kjprsl}^{(2)} D_{ltmn}^{(0)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} + \\ \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjmn}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(4)})_{,x_j} &+ (\delta_{iq} - \nu_{iq}^{(1)}) D_{qjprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + \\ (\delta_{iq} - \nu_{iq}^{(2)}) D_{qjprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} &+ (\delta_{iq} - \nu_{iq}^{(3)}) D_{qjprstuvmn}^{(6)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} \end{aligned} \quad (\text{B.2})$$

Taking four spatial derivatives and substituting Eq. 10 with $\alpha = 2$ into the second term on the right side of Eq. B.2:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(6)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(6)})_{,x_j} &= \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + \\ \rho_0 \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) D_{kjprmn}^{(2)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_{rpj}} &+ \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjmn}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(4)})_{,x_j} + \\ (\delta_{iq} - \nu_{iq}^{(1)}) D_{qjprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} &+ (\delta_{iq} - \nu_{iq}^{(2)}) D_{qjprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} + \\ \left[(\delta_{iq} - \nu_{iq}^{(3)}) D_{qjprstuvmn}^{(6)} - \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) D_{kjprsl}^{(2)} D_{ltuvmn}^{(2)} \right] &e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} \end{aligned} \quad (\text{B.3})$$

Taking two time derivatives of Eq. 10 with $\alpha = 4$, substituting the resulting expression into the second term on the right side of Eq. B.3 and using Eq. 10 with $\alpha = 0$, we arrive at the

spatial-temporal nonlocal momentum balance equation at $O(\zeta^6)$:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(6)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(6)})_{,x_j} &= \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + \\ \rho_0 \left[\nu_{iq}^{(3)} A_{qw}^{(1)} \left(\delta_{wk} + A_{wk}^{(2)} \right) - \nu_{iq}^{(2)} A_{qk}^{(3)} \right] D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(4)})_{,x_j} &+ \\ \rho_0^2 \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) \ddot{U}_k^{(4)} + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + \\ E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} + E_{ijprstuvmn}^{(6)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} \end{aligned} \quad (\text{B.4})$$

where $\mathbf{E}^{(2)}$, $\mathbf{E}^{(4)}$ and $\mathbf{E}^{(6)}$ are:

$$E_{ijprmn}^{(2)} = (\delta_{iq} - \nu_{iq}^{(1)}) D_{qjprmn}^{(2)} \quad (\text{B.5a})$$

$$E_{ijprstmn}^{(4)} = (\delta_{iq} - \nu_{iq}^{(2)}) D_{qjprstmn}^{(4)} \quad (\text{B.5b})$$

$$\begin{aligned} E_{ijprstuvmn}^{(6)} &= - \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) \left(D_{kjprsl}^{(2)} D_{ltuvmn}^{(2)} + D_{kjprstul}^{(4)} D_{lvmn}^{(0)} \right) \\ &\quad + (\delta_{iq} - \nu_{iq}^{(3)}) D_{qjprstuvmn}^{(6)} \end{aligned} \quad (\text{B.5c})$$

Rewritting Eq. 12 with $\alpha = 4$:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(4)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(4)})_{,x_j} &= \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\ + \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} &+ \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\ - \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} &+ \nu_{iq}^{(2)} D_{qjprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} \\ + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} &+ E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}} \end{aligned} \quad (\text{B.6})$$

Taking two time derivatives of Eq. 10 with $\alpha = 2$, multiplying the resulting equation with ρ_0

and employing Eq. 10 with $\alpha = 0$:

$$\begin{aligned} \rho_0^2 \ddot{U}_i^{(2)} - \rho_0 D_{ijmn}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} &= \rho_0 D_{ijprmn}^{(2)} e_{xnm}(\ddot{\mathbf{U}}^{(0)})_{,x_{rpj}} = \\ D_{ijprsl}^{(2)} D_{ltmn}^{(0)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}} \end{aligned} \quad (\text{B.7})$$

Using the approximation Eq. 13c and substituting Eq. B.7 into Eq. B.6:

$$\begin{aligned} \rho_0 \ddot{U}_i^{(4)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(4)})_{,x_j} &= \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\ + \rho_0 \left(\nu_{iq}^{(3)} A_{qk}^{(1)} - \nu_{iq}^{(2)} A_{qk}^{(3)} \right) D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} &+ \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\ - \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} &+ \rho_0^2 \nu_{iq}^{(2)} A_{qk}^{(3)} \ddot{U}_k^{(2)} \\ + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} &+ E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}} \end{aligned} \quad (\text{B.8})$$

Taking two spatial derivatives of Eq. 10 with $\alpha = 2$, substituting the resulting expression into

$\nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} - \rho_0 \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j}$ and recalling the approxi-

mation Eq. 13b:

$$\begin{aligned}
& \rho_0 \ddot{U}_i^{(4)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(4)})_{,x_j} = \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\
& + \rho_0 \left(\nu_{iq}^{(3)} A_{qw}^{(1)} - \nu_{iq}^{(2)} A_{qk}^{(3)} \right) D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} + \rho_0^2 \nu_{iq}^{(2)} A_{qk}^{(3)} \ddot{U}_k^{(2)} \\
& - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} D_{kjprsl}^{(2)} D_{ltmn}^{(0)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}} \\
& + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} + E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}}
\end{aligned} \tag{B.9}$$

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Taking four spatial derivatives of Eq. 10 with $\alpha = 0$, substituting the resulting expression into the fourth term on the right side of Eq. B.9 and employing Eq.10 with $\alpha = 2$ taken two time derivatives, we obtain the spatial-temporal nonlocal momentum balance equation at $O(\zeta^4)$:

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$$\begin{aligned}
& \rho_0 \ddot{U}_i^{(4)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(4)})_{,x_j} = \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} \\
& + \rho_0 \left[\nu_{iq}^{(3)} A_{qw}^{(1)} \left(\delta_{wk} + A_{wk}^{(2)} \right) - \nu_{iq}^{(2)} A_{qk}^{(3)} \right] D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(2)})_{,x_j} \\
& + \rho_0^2 \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) \ddot{U}_k^{(2)} \\
& + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} + E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}}
\end{aligned} \tag{B.10}$$

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The spatial-temporal nonlocal momentum balance equation at $O(\zeta^2)$ is obtained by rewriting Eq. 12 with $\alpha = 2$ while considering the momentum balance of Eq. 10 with $\alpha = 0$:

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$$\begin{aligned}
& \rho_0 \ddot{U}_i^{(2)} - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U}^{(2)})_{,x_j} = \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U}^{(0)})_{,x_{rpj}} \\
& + \rho_0 \left[\nu_{iq}^{(3)} A_{qw}^{(1)} \left(\delta_{wk} + A_{wk}^{(2)} \right) - \nu_{iq}^{(2)} A_{qk}^{(3)} \right] D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}}^{(0)})_{,x_j} \\
& + \rho_0^2 \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) \ddot{U}_k^{(0)} + E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{rpj}}
\end{aligned} \tag{B.11}$$

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Employing Eqs B.4, B.10, B.11 and 10 with $\alpha = 0$, and using the summation of macroscale displacement from all orders, Eq. 11, the spatial-temporal nonlocal homogenized momentum balance equation is obtained as:

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$$\begin{aligned}
& \rho_0 \ddot{U}_i - D_{ijmn}^{(0)} e_{xnm}(\mathbf{U})_{,x_j} = \left(\nu_{iq}^{(1)} D_{qjprmn}^{(2)} - \nu_{iq}^{(3)} A_{qk}^{(1)} D_{kjpl}^{(0)} D_{lrnm}^{(0)} \right) e_{xnm}(\mathbf{U})_{,x_{rpj}} + \\
& \rho_0 \left[\nu_{iq}^{(3)} A_{qw}^{(1)} \left(\delta_{wk} + A_{wk}^{(2)} \right) - \nu_{iq}^{(2)} A_{qk}^{(3)} \right] D_{kjm n}^{(0)} e_{xnm}(\ddot{\mathbf{U}})_{,x_j} + \\
& \rho_0^2 \left(\nu_{iq}^{(2)} A_{qk}^{(3)} - \nu_{iq}^{(3)} A_{qw}^{(1)} A_{wk}^{(2)} \right) \ddot{U}_k + \zeta^2 E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{rpj}} + \\
& \zeta^4 \left(E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{rpj}} + E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{tsrpj}} \right) + \\
& \zeta^6 \left(E_{ijprmn}^{(2)} e_{xnm}(\mathbf{U}^{(4)})_{,x_{rpj}} + E_{ijprstmn}^{(4)} e_{xnm}(\mathbf{U}^{(2)})_{,x_{tsrpj}} + E_{ijprstuvmn}^{(6)} e_{xnm}(\mathbf{U}^{(0)})_{,x_{vutsrpj}} \right)
\end{aligned} \tag{B.12}$$

Appendix C Expressions of a_i , b_i , c_i , $\tilde{b}_i$ and $\tilde{c}_i$

$$a_i = \left(\beta_{[i]jmn} \beta_{[i]pqr} - 4\alpha_{[i]jpqmn} \gamma_{[i]r} \right) \hat{n}_j \hat{n}_m \hat{n}_p \hat{n}_q \hat{p}_n \hat{p}_r \quad (\text{C.1a})$$

$$b_i = -2 \left(\beta_{[i]jmn} D_{[i]pqr}^{(0)} + 2\rho_0 \alpha_{[i]jpqmn} \delta_{[i]r} \right) \hat{n}_j \hat{n}_m \hat{n}_p \hat{n}_q \hat{p}_n \hat{p}_r \quad (\text{C.1b})$$

$$c_i = D_{[i]jmn}^{(0)} D_{[i]pqr}^{(0)} \hat{n}_j \hat{n}_m \hat{n}_p \hat{n}_q \hat{p}_n \hat{p}_r \quad (\text{C.1c})$$

$$\tilde{b}_i = 2 \left(\beta_{[i]jmn} \delta_{[i]r} \rho_0 + 2D_{[i]jmn}^{(0)} \gamma_{[i]r} \right) \hat{n}_j \hat{n}_m \hat{p}_n \hat{p}_r \quad (\text{C.1d})$$

$$\tilde{c}_i = \rho_0^2 \delta_{[i]n} \delta_{[i]r} \hat{p}_n \hat{p}_r \quad (\text{C.1e})$$

For a selected pair of vectors $(\hat{\mathbf{n}}, \hat{\mathbf{p}})$, $c_i > 0$ and $\tilde{c}_i > 0$.

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