Sub-Bergman Hilbert Spaces

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Introduction

Range space

Let A be a bounded operator on a Hilbert space H. We define the range space $\mathcal{M}(A) = AH$, and endow it with the inner product

$$\langle Af, Ag \rangle_{\mathcal{M}(A)} = \langle f, g \rangle_H, \qquad f, g \in H \ominus \mathrm{Ker}A.$$

$\mathcal{M}(A)$ has a Hilbert space structure that makes A a coisometry on H.

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Toeplitz operators on the Hardy space

Let H^2 be the Hardy space on the unit disk $\mathbb{D}$. The Toeplitz operator on the Hardy space H^2 with symbol f in $L^\infty(\partial\mathbb{D})$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2$. Here P be the orthogonal projections from $L^2(\partial\mathbb{D})$ to H^2.

Sub-Hardy Hilbert Spaces

Let b be a nonconstant function in H_1^∞, the closed unit ball of $H^\infty(\mathbb{D})$.

Definition

The de Branges-Rovnyak space $\mathcal{H}(b)$ is defined to be the range space:

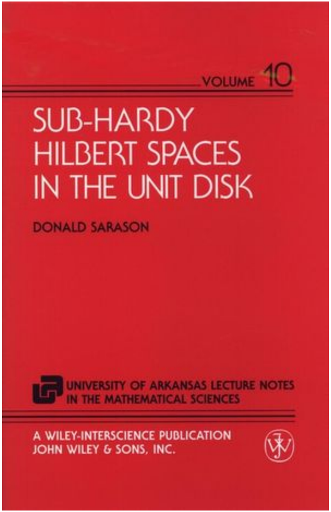
$$\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{1/2} H^2.$$

We also define the space $\mathcal{H}(\bar{b})$ in the same way as $\mathcal{H}(b)$, but with the roles of b and $\bar{b}$ interchanged, i.e.

$$\mathcal{H}(\bar{b}) = (I - T_{\bar{b}} T_b)^{1/2} H^2.$$

Sub-Hardy Hilbert Spaces

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are also called sub-Hardy Hilbert spaces (the terminology comes from the title of Sarason's book).



Sub-Hardy Hilbert Spaces

$$\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{1/2} H^2, \mathcal{H}(\bar{b}) = (I - T_{\bar{b}} T_b)^{1/2} H^2.$$

Basic facts about $\mathcal{H}(b)$:

1. If $||b||_\infty < 1$, then $\mathcal{H}(b) = \mathcal{H}(\bar{b}) = H^2$.
2. If b is an inner function, then $\mathcal{H}(b) = H^2 \ominus bH^2$ is a closed subspace of H^2.
3. In general, $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are contractively contained in H^2, but are not closed subspaces.
4. For $\varphi \in H^\infty$, $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under $T_{\bar{\varphi}}$.

Sub-Hardy Hilbert Spaces

The theory of $\mathcal{H}(b)$-spaces splits into two mutually exclusive cases: whether b is or is not an extreme point of the unit ball of the closed unit ball of H^∞.

Theorem

The following statements are equivalent.

1. b is non-extreme.
2. $\log(1 - |b|^2) \in L^1(\partial\mathbb{D})$.
3. $b \in \mathcal{H}(b)$.
4. $\mathcal{H}(b)$ contains all the polynomials.

The Bergman space A^2 is the space of analytic functions on $\mathbb{D}$ that are square-integrable with respect to the Lebesgue area measure dA.

Toeplitz operators on the Bergman space

The Toeplitz operator on the Bergman space A^2 with symbol f in $L^\infty(\mathbb{D})$ is defined by

$$\tilde{T}_f(h) = \tilde{P}(fh),$$

for $h \in A^2$. Here $\tilde{P}$ be the orthogonal projections from $L^2(\mathbb{D})$ to $A^2(\mathbb{D})$.

Kehe Zhu introduced the sub-Bergman Hilbert spaces in 1996.

Definition

Let b be a function in H_1^∞. Define

$$\mathcal{A}(b) = (I - \widetilde{T}_b \widetilde{T}_{\bar{b}})^{1/2} A^2$$

and

$$\mathcal{A}(\bar{b}) = (I - \widetilde{T}_{\bar{b}} \widetilde{T}_b)^{1/2} A^2.$$

Sub-Bergman Hilbert Spaces

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Sub-Bergman Hilbert Spaces

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Question: What is $\mathcal{A}(b)$ if b is an inner function?

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Theorem (Zhu, 2002)

If b is a finite Blaschke product, then $\mathcal{A}(b) = H^2$.

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Theorem (Zhu, 2002)

If b is a finite Blaschke product, then $\mathcal{A}(b) = H^2$.

We present a new proof using the theory of reproducing kernels and get a stronger result.

Theorem (C.)

Let $b \in H_1^\infty$. Then

1. $H^2 \subset \mathcal{A}(b)$.
2. $\mathcal{A}(b) = H^2$ if and only if b is a finite Blaschke product.

Reproducing kernels

Definition

Let $X \subset \mathbb{C}^d$. We say a function $K : X \times X \to \mathbb{C}$ is positive semi-definite ($K \geq 0$) if $K(x, y) = \overline{K(y, x)}$, and for all finite sets $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset X$, the matrix $(K(\lambda_i, \lambda_j))_{i,j=1}^m \geq 0$.

Definition (RKHS)

A reproducing kernel Hilbert space $\mathcal{H}$ on X is a Hilbert space of complex valued functions such that for every $w \in X$, there exists $K_w \in \mathcal{H}$ satisfying

$$\langle f, K_w \rangle_{\mathcal{H}} = f(w), \text{for every } f \in \mathcal{H}.$$

Let $K(z, w) = K_w(z)$. Then $K \geq 0$.

Reproducing kernels

Moore's Theorem

Let $X \subset \mathbb{C}^d$ and let $K : X \times X \to \mathbb{C}$ be positive semi-definite. Then there exists a unique reproducing kernel Hilbert space with reproducing kernel K.

Spaces and kernel functions

- H^2: $K^S(z, w) = \frac{1}{1-\bar{w}z}$
- $\mathcal{H}(b)$: $K_{\mathcal{H}(b)}(z, w) = \frac{1-\overline{b(w)}b(z)}{1-\bar{w}z}$
- A^2: $K_{A^2}(z, w) = \frac{1}{(1-\bar{w}z)^2}$
- $\mathcal{A}(b)$: $K_{\mathcal{A}(b)}(z, w) = \frac{1-\overline{b(w)}b(z)}{(1-\bar{w}z)^2}$

Reproducing kernels

Proposition

Let K_1, K_2 be positive semi-definite. Then

1. $K_1 + K_2 \geq 0$.
2. $K_1 \cdot K_2 \geq 0$.

Multiplier Criterion

Let $\mathcal{H}$ be a RKHS with kernel K. Then f is a multiplier of $\mathcal{H}$ with multiplier norm at most δ if and only if

$$(\delta^2 - f(z)\overline{f(w)}) \cdot K(z, w) \geq 0.$$

Multiplier Criterion

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Theorem (Zhu)

Let $b \in H_1^\infty$. Every function in H^∞ is a multipliers of $\mathcal{A}(b)$.

Proof. A function f is a contractive multiplier of $\mathcal{A}(b)$ if and only if

$$(1 - \overline{f(w)}f(z)) \cdot \frac{1 - \overline{b(w)}b(z)}{(1-\bar{w}z)^2} \geq 0.$$

And

$$(1 - \overline{f(w)}f(z)) \cdot \frac{1 - \overline{b(w)}b(z)}{(1-\bar{w}z)^2} = \frac{1 - \overline{f(w)}f(z)}{1-\bar{w}z} \cdot \frac{1 - \overline{b(w)}b(z)}{1-\bar{w}z} \geq 0.$$

Theorem (Zhu)

Let $b \in H_1^\infty$. Every function in H^∞ is a multipliers of $\mathcal{A}(b)$.

Using the identity

$$||f||^2_{\mathcal{A}(\bar{b})} = ||f||^2_{A_2} + ||\widetilde{T}_{\bar{b}} f||^2_{\mathcal{A}(b)},$$

one can get

Corollary

Norm equivalence: $\mathcal{A}(b) = \mathcal{A}(\bar{b})$.

Sub-Bergman Hilbert Spaces

Theorem (Zhu, 2002)

If b is a finite Blaschke product, then $\mathcal{A}(b) = H^2$.

Zhu's proof uses an integral representation formula for $\mathcal{A}(\bar{b})$ and shows that $\mathcal{A}(\bar{b}) = H^2$.

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Another proof is found by S. Sultanic in 2006.

First show $\mathcal{A}(b) = H^2$ if b is a single Blaschke product by comparing the reproducing kernels of $\mathcal{A}(b)$ and H^2.

Then use an induction argument to prove the theorem holds for any finite Blaschke product.

Sub-Bergman Hilbert Spaces

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Let $b \in H_1^\infty$. Then

1. $H^2 \subset \mathcal{A}(b)$.
2. $\mathcal{A}(b) = H^2$ if and only if b is a finite Blaschke product.

We use the following result

Theorem (Aronszajn)

Let $\mathcal{H}(K_1), \mathcal{H}(K_2)$ be two RKHS on X with kernel functions $\mathcal{H}(K_1), \mathcal{H}(K_2)$, respectively. Then

$$\mathcal{H}(K_1) \subset \mathcal{H}(K_2)$$

if and only if

$$K_1 \leq cK_2, \quad \text{i.e.} \quad cK_2 - K_1 \geq 0,$$

for some constant $c > 0$.

$H^2 \subset \mathcal{A}(b)$

Theorem (Aronszajn)

$\mathcal{H}(K_1) \subset \mathcal{H}(K_2)$ if and only if $K_1 \leq cK_2$.

To show $H^2 \subset \mathcal{A}(b)$, we need to find c so that

$$\frac{1}{1-\bar{w}z} \leq c \cdot \frac{1-\overline{b(w)}b(z)}{(1-\bar{w}z)^2} = c \cdot \frac{1-\overline{b(w)}b(z)}{1-\bar{w}z} \cdot \frac{1}{1-\bar{w}z}.$$

We can find a function f s.t.

$$\overline{f(w)}f(z) \leq c_1 \cdot \frac{1-\overline{b(w)}b(z)}{1-\bar{w}z}$$

and

$$\overline{f(w)}f(z)\frac{1}{1-\bar{w}z} \geq c_2 \cdot \frac{1}{1-\bar{w}z}.$$

$\mathcal{A}(b) = H^2 \iff b$ is a finite Blaschke product

If b is a finite Blaschke product, then $\mathcal{H}(b)$ has finite dimension N. Let $\{f_n\}_{n=0}^{N-1}$ be an orthonormal basis for $\mathcal{H}(b)$. Then

$$\frac{1 - \overline{b(w)}b(z)}{1 - \bar{w}z} = \sum_{n=0}^{N-1} \overline{f_n(w)} f_n(z).$$

And for each n,

$$\overline{f_n(w)} f_n(z) \frac{1}{1 - \bar{w}z} \leq C \cdot \frac{1}{1 - \bar{w}z}.$$

Then

$$\frac{1 - \overline{b(w)}b(z)}{(1 - \bar{w}z)^2} \leq CN \cdot \frac{1}{1 - \bar{w}z}.$$

Density of polynomials in $\mathcal{A}(b)$

Theorem

$\mathcal{H}(b)$ contains all the polynomials if and only if b is non-extreme. In the non-extreme case, the polynomials are dense in $\mathcal{H}(b)$.

If b is non-extreme, then there is a unique outer function a such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.
This function is a basic tool in the theory of $\mathcal{H}(b)$ in the non-extreme case.

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Since $H^2 \subset \mathcal{A}(b)$, $\mathcal{A}(b)$ contains all the polynomials.

Question: Are the polynomials dense in $\mathcal{A}(b)$?

Density of polynomials in $\mathcal{A}(b)$

Theorem (C.)

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Lemma

Let L_b^2 denote the space $L^2(\mathbb{D}, (1-|b|^2)dA)$. Let A_b^2 be the closure of polynomials in L_b^2. Define the operator S_b by

$$S_b g = \widetilde{P}((1-|b|^2)g).$$

Then S_b is an isometry from A_b^2 onto $\mathcal{A}(\bar{b})$.

The End