Compact Product of Hankel and Toeplitz Operators on the Hardy space

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Introduction

Definition

Let $L^2 = L^2(\partial\mathbb{D})$ and let H^2 be the Hardy space on $\mathbb{D}$. Given a symbol function $f \in L^\infty$, define the Toeplitz operator T_f and the Hankel operator $\mathcal{H}_f$ as:

$$T_f : H^2 \to H^2, T_f h = P(fh),$$

and

$$\mathcal{H}_f : H^2 \to (H^2)^\perp, \mathcal{H}_f h = (I - P)(fh),$$

where $P : L^2 \to H^2$ is the orthogonal projection.

Introduction

$$\mathcal{H}_f : H^2 \to (H^2)^\perp, \mathcal{H}_f h = (I - P)(fh).$$

For convenience, we will use an alternative definition for Hankel operators.

Definition

Let $Vf(z) = \bar{z}f(\bar{z})$. Then V is unitary on L^2. Define:

$$H_f : H^2 \to H^2, H_f h = PV(fh).$$

Then

$$H_f = V\mathcal{H}_f.$$

Introduction

Question

When is $H_f T_g$ compact?

It is known that:

1. T_f is compact iff $f = 0$.
2. $H_f = 0$ iff $f \in H^\infty$.
3. (Hartman's Criterion) H_f is compact iff $f \in H^\infty + C$, where C denotes the space of continuous functions $\partial\mathbb{D}$.

Relations between Toeplitz and Hankel operators

Consider the multiplication operator M_f on L^2 for $f \in L^\infty$, defined by $M_f h = fh$. M_f can be expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus (H^2)^\perp$ as the following:

$$M_f = \begin{pmatrix} T_f & H_{\tilde{f}} V \\ VH_f & VT_{\tilde{f}} V \end{pmatrix}$$

For $f, g \in L^\infty$, $M_{fg} = M_f M_g$, so multiplying the matrices and comparing the entries, we get:

1. $T_{fg} = T_f T_g + H_{\tilde{f}} H_g$.
2. $H_{fg} = H_f T_g + T_{\tilde{f}} H_g$.

Here $\tilde{f}(z) = f(\bar{z})$, $Vf(z) = \bar{z} f(\bar{z})$.

A theorem of Axler, Chang, Sarason and Volberg

Problem

When is $T_{fg} - T_f T_g = H_{\tilde{f}} H_g$ compact?

Theorem (Brown, Halmos, 1963)

$H_{\tilde{f}} H_g = 0$ if and only if $\bar{f} \in H^\infty$ or $g \in H^\infty$.

Theorem (Axler, Chang, Sararson, 1978; Volberg, 1982)

$H_{\tilde{f}} H_g$ is compact if and only if

$$H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C. \quad \text{(Algebraic Condition)}$$

Here $H^\infty[f]$ denotes the closed subalgebra of L^∞ generated by H^∞ and f.

A theorem of Axler, Chang, Sarason and Volberg

For a uniform algebra B, let $M(B)$ denote the maximal ideal space of B, the space of nonzero multiplicative linear functionals of B. We identify $\mathbb{D}$ in the usual way as a subset of $M(H^\infty)$.

By Carleson's Corona Theorem, $\mathbb{D}$ is dense in $M(H^\infty)$. Moreover, $M(H^\infty + C) = M(H^\infty)\backslash\mathbb{D}$.

For any m in $M(H^\infty)$, there exists a representing measure μ_m such that $m(f) = \int f d\mu_m$, for all $f \in H^\infty$.

Definition

A subset S of $M(H^\infty)$ is called a **support set** if it is the support of a representing measure for a functional in $M(H^\infty + C)$.

A theorem of Axler, Chang, Sarason and Volberg

Definition

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Theorem (Another Version)

$H_{\tilde{f}} H_g$ is compact if and only if for each support set S, either $\bar{f}|_S \in H^\infty|_S$ or $g|_S \in H^\infty|_S$. (Local Condition)

This condition localized the condition when $H_{\tilde{f}} H_g = 0$.

Theorem (Brown, Halmos, 1963)

$H_{\tilde{f}} H_g = 0$ if and only if $\bar{f} \in H^\infty$ or $g \in H^\infty$.

An Elementary Condition for the Compactness of $H_{\tilde{f}} H_g$

Theorem (Zheng, 1996)

$H_{\tilde{f}} H_g$ is compact if and only if

$$\lim_{|z| \to 1^-} ||H_{\bar{f}} k_z||_2 \cdot ||H_g k_z||_2 = 0. \quad \text{(Elementary Condition)}$$

Here k_z denotes the normalized reproducing kernel at z:

$$k_z(w) = \frac{\sqrt{1 - |z|^2}}{1 - \bar{z} w}.$$

An Elementary Condition for the Compactness of $H_{\tilde{f}} H_g$

By the Corona Theorem, the elementary condition

$$\lim_{|z| \to 1^-} ||H_{\bar{f}} k_z|| \cdot ||H_g k_z|| = 0$$

can be rephrased as:
For all $m \in M(H^\infty + C)$,

$$\lim_{z \to m} ||H_{\bar{f}} k_z|| \cdot ||H_g k_z|| = 0.$$

An Elementary Condition for the Compactness of $H_{\tilde{f}} H_g$

The next theorem links the elementary condition and the local condition.

Theorem (Gorkin, Zheng)

Let $m \in M(H^\infty + C)$, and let S be the support set of m. Then the following are equivalent:

1. $f|_S \in H^\infty|_S$.
2. $\underline{\lim}_{z \to m} ||H_f k_z|| = 0$.
3. $\lim_{z \to m} ||H_f k_z|| = 0$.

Compactness of $H_{\tilde{f}} H_g$

Theorem

The following are equivalent:

1. $H_{\tilde{f}} H_g$ *is compact.*
2. $$H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C. \quad \textit{(Algebraic Condition)}$$
3. *For each support set* S,
$$\textit{either} \quad \bar{f}|_S \in H^\infty|_S \quad \textit{or} \quad g|_S \in H^\infty|_S. \quad \textit{(Local Condition)}$$
4. $$\lim_{|z|\to 1^-} ||H_{\bar{f}} k_z|| \cdot ||H_g k_z|| = 0. \quad \textit{(Elementary Condition)}$$

The compactness of $H_f T_g$

Turning to the operator $H_f T_g$. It is not hard to obtain the zero condition.

Lemma

$H_f T_g = 0$ if and only if one of the following holds:

1. $f \in H^\infty$.
2. $g \in H^\infty$ *and* $fg \in H^\infty$.

Localize these conditions, we want to prove:

Theorem (C.)

Let $f, g \in L^\infty$. The $H_f T_g$ is compact if and only if for each support set S, one of the following holds:

1. $f|_S \in H^\infty|_S$.
2. $g|_S \in H^\infty|_S$ and $(fg)|_S \in H^\infty|_S$.

The compactness of $H_f T_g$

Theorem (C.)

Let $f, g \in L^\infty$. The $H_f T_g$ is compact if and only if for each support set S, one of the following holds:

1. $f|_S \in H^\infty|_S$.
2. $g|_S \in H^\infty|_S$ and $(fg)|_S \in H^\infty|_S$.

Proof: "$\Longrightarrow$". Suppose $H_f T_g$ is compact. Apply the lemma:

Lemma (Zheng)

If $K : H^2 \to H^2$ is a compact operator, then

$$\lim_{|z| \to 1^-} ||K - T^*_{\phi_z} K T_{\bar{\phi}_z}|| = 0,$$

where

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

The compactness of $H_f T_g$

We get:

$$\lim_{z \to m} ||H_f k_z|| \cdot ||H_g k_z|| = 0,$$

for every $m \in M(H^\infty + C)$.

Translate to the local conditions, we have:
either $f|_S \in H^\infty|_S$, or $g|_S \in H^\infty|_S$, where S is the support set of m.

For the second case, use the identity:

$$H_{fg} = H_f T_g + T_{\tilde{f}} H_g,$$

then:

$$H_{fg} k_z = H_f T_g k_z + T_{\tilde{f}} H_g k_z.$$

The compactness of $H_f T_g$

$$H_{fg} k_z = H_f T_g k_z + T_{\tilde{f}} H_g k_z,$$

and

$$\lim_{z \to m} ||H_g k_z|| = 0.$$

Notice that: $H_f T_g$ is compact and $k_z \to 0$ weakly. Thus

$$\lim_{z \to m} ||H_f T_g k_z|| = 0.$$

So

$$\lim_{z \to m} ||H_{fg} k_z|| = 0.$$

This means $(fg)|_S \in H^\infty|_S$.

The compactness of $H_f T_g$

Theorem

Let $f, g \in L^\infty$. The $H_f T_g$ is compact if and only if for each support set S, one of the following holds:

1. $f|_S \in H^\infty|_S$.
2. $g|_S \in H^\infty|_S$ and $(fg)|_S \in H^\infty|_S$.

" $\Longleftarrow$." Is the converse of

Lemma

If $K : H^2 \to H^2$ is a compact operator, then

$$\lim_{|z| \to 1^-} ||K - T^*_{\phi_z} K T_{\bar{\phi}_z}|| = 0.$$

still true?

The compactness of $H_f T_g$

Theorem (Guo, Zheng, 2001)

If K is a finite sum of finite products of Toeplitz operators, and if

$$\lim_{|z|\to 1^-} ||K - T_{\phi_z}^* K T_{\phi_z}|| = 0.$$

Then $K =$ Toeplitz operator + Compact operator.

$H_f T_g$ is not necessarily a finite sum of finite products of Toeplitz operators, but

$$(H_f T_g)^*(H_f T_g) = T_{\bar{g}}(T_{\bar{f}g} - T_{\bar{f}} T_g) T_g.$$

The compactness of $H_f T_g$

Consider the symbol map $\sigma : T_\phi \to \phi$.

The symbol map is a $*$-homomorphism on the C^*-algebra generated by Toeplitz operators.

Theorem (Barría, Halmos, 1982)

σ can be extended to a $*$-homomorphism on the C^*-algebra generated by both Toeplitz and Hankel operators s.t.
$\sigma(Hankel) = 0$ and $\sigma(Compact) = 0$.

Notice that

$$\sigma((H_f T_g)^*(H_f T_g)) = 0.$$

Then
$(H_f T_g)^*(H_f T_g) =$ Toeplitz operator $+$ Compact operator
if and only if $(H_f T_g)^*(H_f T_g)$ is compact.

The compactness of $H_f T_g$

Thus we have the following corollary:

Corollary

$K = H_f T_g$ is compact if and only if

$$\lim_{|z|\to 1^-} ||K^*K - T^*_{\phi_z} K^* K T_{\phi_z}|| = 0.$$

Theorem (C.)

Let $f, g \in L^\infty$. The $H_f T_g$ is compact if and only if for each support set S, one of the following holds:

1. $f|_S \in H^\infty|_S$.
2. $g|_S \in H^\infty|_S$ and $(fg)|_S \in H^\infty|_S$.

Now we can verify that each of the two conditions implies the compactness of $H_f T_g$.

The End