Bounded Composition Operators on the Bidisk

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Introduction

Definition

For a bounded convex domain $\Omega \subset \mathbb{C}^d$ ($\Omega = \mathbb{D}$ or $\mathbb{D}^2$), the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$, for an analytic self-map φ of Ω.

Let $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$ be the Hardy spaces.

Littlewood's Subordination Principle

Every composition operator is bounded on $H^2(\mathbb{D})$.

Question: What about composition operators on $H^2(\mathbb{D}^2)$?

Introduction

For a point $z \in \mathbb{D}^2$, we use $z = (z_1, z_2)$ to denote the coordinates of z.

Example

If $B(z_1, z_2) = (z_1, z_1)$, then C_B is not bounded on $H^2(\mathbb{D}^2)$.

Jury reproved the boundedness of composition operators on $H^2(\mathbb{D})$ using only reproducing kernels. We adapt this approach to study the two-variable case and find nontrivial bounded composition operators on $H^2(\mathbb{D}^2)$.

Reproducing kernels

Definition

Let $X \subset \mathbb{C}^d$. We say a function $K : X \times X \to \mathbb{C}$ is a positive semi-definite ($K \geq 0$) if $K(x, y) = \overline{K(y, x)}$, and for all finite sets $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset X$, the matrix $(K(\lambda_i, \lambda_j))_{i,j=1}^m \geq 0$.

Definition (RKHS)

A reproducing kernel Hilbert space $\mathcal{H}$ on X is a Hilbert space of complex valued functions such that for every $w \in X$, there exists $K_w \in \mathcal{H}$ satisfying

$$\langle f, K_w \rangle_{\mathcal{H}} = f(w), \text{for every } f \in \mathcal{H}.$$

Let $K(z, w) = K_w(z)$. Then $K \geq 0$.

Reproducing kernels

Moore's Theorem

Let $X \subset \mathbb{C}^d$ and let $K : X \times X \to \mathbb{C}$ be positive semi-definite. Then there exists a unique reproducing kernel Hilbert space with reproducing kernel K.

Example

- $K_{L^2_a(\mathbb{D})}(z, w) = \frac{1}{(1-\bar{w}z)^2}$
- $K_{H^2(\mathbb{D})}(z, w) = \frac{1}{1-\bar{w}z}$
- $K_{H^2(\mathbb{D}^2)}(z, w) = \frac{1}{(1-\bar{w}_1 z_1)(1-\bar{w}_2 z_2)}$

Reproducing kernels

Proposition

Let K_1, K_2 be positive semi-definite. Then

1. $K_1 + K_2 \geq 0$.
2. $K_1 \cdot K_2 \geq 0$.

Multiplier Criterion

Let $\mathcal{H}$ be a RKHS with kernel K. Then f is a multiplier of $\mathcal{H}$ with multiplier norm at most δ if and only if

$$(\delta^2 - f(z)\overline{f(w)}) \cdot K(z,w) \geq 0.$$

If $\delta \leq 1$, then f is called a contractive multiplier of $\mathcal{H}$.

Reproducing kernels

Multiplier Criterion

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de Branges-Rovnyak space

Let $b \in H^\infty(\mathbb{D})$ and $||b||_\infty \leq 1$.
$\mathcal{H}(b)$ is the Hilbert space with reproducing kernel:

$$\frac{1 - \overline{b(w)}b(z)}{1 - \bar{w}z}.$$

It is invariant under S^*, the backward shift operator.

Composition operators on $H^2(\mathbb{D}^2)$

We use the following theorem to find bounded composition operators.

Theorem

Let $B = (\phi, \psi)$ be an analytic map from $\mathbb{D}^2$ to $\mathbb{D}^2$. Define a function R as

$$R(z,w) = \frac{1 - \overline{\phi(w)}\phi(z)}{1 - \bar{w}_1 z_1} \cdot \frac{1 - \overline{\psi(w)}\psi(z)}{1 - \bar{w}_2 z_2}.$$

If $R \geq 0$, then C_B is a bounded composition operator on $H^2(\mathbb{D}^2)$ and

$$||C_B|| \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{\frac{1}{2}} \cdot \left(\frac{1 + |\psi(0)|}{1 - |\psi(0)|}\right)^{\frac{1}{2}}.$$

Sub-Hardy Hilbert Space of the Bidisk

$$R(z,w) = (1 - \overline{\psi(w)}\psi(z)) \cdot \frac{1 - \overline{\phi(w)}\phi(z)}{(1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)}.$$

Multiplier Criterion

Let $\mathcal{H}$ be a RKHS with kernel K. Then f is a contractive multiplier of $\mathcal{H}$ if and only if

$$(1 - f(z)\overline{f(w)}) \cdot K(z,w) \geq 0.$$

$$\frac{1 - \overline{\phi(w)}\phi(z)}{(1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)} \geq 0.$$

The Hilbert space with the above reproducing kernel, denoted by $\mathcal{H}^2(\phi)$, is called a sub-Hardy Hilbert space of the bidisk.

It is invariant under the two backward shift operators S_1^*, S_2^* on $H^2(\mathbb{D}^2)$.

Composition operators on $H^2(\mathbb{D}^2)$

$$\phi, \psi : \mathbb{D}^2 \to \mathbb{D},\ R(z,w) = (1 - \overline{\psi(w)}\psi(z)) \cdot \frac{1 - \overline{\phi(w)}\phi(z)}{(1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)}.$$

Proposition

$R \geq 0$ if and only if ψ is a contractive multiplier of $\mathcal{H}^2(\phi)$.

We can easily show that $R \geq 0$ for some special cases.

1. If one of ϕ, ψ is a constant.
2. If ϕ, ψ are one-variable functions in z_1 and z_2, respectively ($\phi = \phi(z_1)$, $\psi = \psi(z_2)$), then $\frac{1-\overline{\phi(w_1)}\phi(z_1)}{1-\bar{w}_1 z_1} \geq 0$ and $\frac{1-\overline{\psi(w_2)}\psi(z_2)}{1-\bar{w}_2 z_2} \geq 0$, so the product R is positive semi-definite.

Multipliers of $\mathcal{H}^2(\phi)$

Theorem

If ψ is a contractive multiplier of $\mathcal{H}^2(\phi)$, then C_B is bounded on $H^2(\mathbb{D}^2)$, with $B = (\phi, \psi)$.

Multipliers of $\mathcal{H}^2(\phi)$

Theorem

If ψ is a contractive multiplier of $\mathcal{H}^2(\phi)$, then C_B is bounded on $H^2(\mathbb{D}^2)$, with $B = (\phi, \psi)$.

If B maps $\mathbb{D}^2$ into a compact subset of $\mathbb{D}^2$, then C_B is bounded on $H^2(\mathbb{D}^2)$. So we assume $||\phi||_\infty = 1$.

If ϕ is inner, then we get no nontrivial bounded composition operators.

Theorem

If ϕ is a nonconstant inner function, ψ is a nonconstant multiplier of $\mathcal{H}^2(\phi)$. Then one of the functions ϕ, ψ is a one-variable function in z_1 and the other is a one-variable function in z_2.

Multipliers of de Branges-Rovnyak spaces

The study of $\mathcal{H}(b)$ bifurcates into two cases:

1. b is nonextreme: $log(1 - |b|^2) \in L^1(\mathbb{T})$
   (b is not an extreme point of the closed unit ball of H^∞)
2. b is extreme: $log(1 - |b|^2) \notin L^1(\mathbb{T})$

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When b is nonextreme, there exists an outer function a ($a(0) \neq 0$) s.t. $|a|^2 + |b|^2 = 1$, a.e. on $\mathbb{T}$.

$$|a|^2 + |b|^2 = 1 \Rightarrow T_{\bar{a}} T_a = I - T_{\bar{b}} T_b.$$

Here T_b is the Toeplitz operator defined by $T_b(f) = P(bf)$.

"Nonextreme" case in $\mathbb{D}^2$

For $\mathcal{H}^2(\phi)$, we assume there is a nonconstant function $a \in H^\infty(\mathbb{D}^2)$ such that

$$|a|^2 + |\phi|^2 = 1$$

a.e. on $\mathbb{T}^2$. We call a the Pythagorean mate for ϕ.

However, the condition $log(1 - |\phi|^2) \in L^1(\mathbb{T}^2)$ is only necessary for ϕ to have a Pythagorean mate.
Also ϕ may have a Pythagorean mate vanishing at 0.

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Example

1. $\phi(z) = \frac{1+z_1}{2}$, $a(z) = \frac{1-z_1}{2}$.
2. $\phi(z) = \frac{z_1+z_2}{2}$, $a(z) = \frac{z_1-z_2}{2}$.

Back to $\mathcal{H}(b)$

Theorem (Lotto, Sarason)

If b is nonextreme, then every function that is continues on $\overline{\mathbb{D}}$ is a multiplier of $\mathcal{H}(b)$.

The proof relies on the following

1. $b \in \mathcal{H}(b)$
2. For any $h \in \mathcal{H}(b)$, $X^*h = Sh - \langle h, S^*b\rangle_{\mathcal{H}(b)} b$, where $X = S^*|_{\mathcal{H}(b)}$.

Then $S|_{\mathcal{H}(b)}$ =Contraction+ Compact operator, by Fredholm theory, $\sigma(S|_{\mathcal{H}(b)}) \subset \overline{\mathbb{D}}$. If f is continuous on $\overline{\mathbb{D}}$, then $f(S|_{\mathcal{H}(b)})$ is defined and coincides with the multiplication operator M_f.

Multipliers of $\mathcal{H}^2(\phi)$

Let $X = S^*|_{\mathcal{H}(b)}$

$$X^* h = Sh - \langle h, S^* b\rangle_{\mathcal{H}(b)} b,$$

for any $h \in \mathcal{H}(b)$.

Suppose ϕ and has a Pythagorean mate a.
For $\mathcal{H}^2(\phi)$, let $X_1 = S_1^*|_{\mathcal{H}^2(\phi)}$.
Then we still have $\phi \in \mathcal{H}^2(\phi)$, but for every $h \in \mathcal{H}^2(\phi)$

$$X_1^* h(w) = w_1 h(w) - \left\langle h(z), \frac{1}{1-\bar{w}_2 z_2}(S_1^*\phi)(z)\right\rangle_{\mathcal{H}^2(\phi)} \phi(w).$$

Multipliers of $\mathcal{H}^2(\phi)$

Theorem

Suppose ϕ and has a Pythagorean mate a. Then every polynomial is a multiplier of $\mathcal{H}^2(\phi)$.

Question

1. If ψ is continuous on $\overline{\mathbb{D}^2}$, is ψ a multiplier of $\mathcal{H}^2(\phi)$
2. What about the "extreme case" in $\mathbb{D}^2$?

The End