Asymptotic Bohr Radius for the Polynomials

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Introduction

$\mathbb{D} = \{|z| < 1\}$, $\mathbb{T} = \partial\mathbb{D}$. H^∞ is the Hardy space on $\mathbb{D}$.

Motivation

Let $r \in (0,1)$. Is it possible to find a power series $\sum_{k=0}^{\infty} a_k z^k$ such that

1. $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^\infty$.
2. $|f(z)| \leq 1$, for $z \in \mathbb{D}$.
3. $\sum_{k=0}^{\infty} |a_k| r^k > 1$?

Introduction

Theorem (H. Bohr, M. Riesz, Schur, F. Wiener, 1914)

If

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^{\infty},$$

and $||f||_{\infty} \leq 1$, then

$$\sum_{k=0}^{\infty} |a_k| r^k \leq 1,$$

for $0 \leq r < \frac{1}{3}$. Moreover, the radius $\frac{1}{3}$ is the best possible.

Definition(Bohr radius)

$$R = \sup\{r \in (0,1) | \sum_{k=0}^{\infty} |a_k| r^k \leq || \sum_{k=0}^{\infty} a_k z^k ||_{\infty}, \forall \sum_{k=0}^{\infty} a_k z^k \in H^{\infty}\}.$$

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Definition(Bohr radius for polynomials)

Let P_n be the set of polynomials of degree at most n.

$$R_n = \sup\{r \in (0,1) | \sum_{k=0}^{n} |a_k| r^k \leq || \sum_{k=0}^{n} a_k z^k ||_{\infty}, \forall \sum_{k=0}^{n} a_k z^k \in P_n\}.$$

Question

Obviously, $R_0 = 1$, $R_n \to \frac{1}{3}$, as $n \to \infty$.
What is the asymptotic of R_n?

Introduction

Definition(Bohr radius for polynomials)

$$R_n = \sup\{r \in (0,1) | \sum_{k=0}^{n} |a_k| r^k \leq || \sum_{k=0}^{n} a_k z^k ||_\infty, \forall \sum_{k=0}^{n} a_k z^k \in P_n\}.$$

Theorem (Guadarrama, 2005)

There are constants C_1, C_2 such that

$$\frac{C_1}{3^{n/2}} < R_n - \frac{1}{3} < C_2 \frac{\log n}{n}.$$

In 2008, R. Founier obtained a formula for R_n.

Bounded Preserving Functions

Definition

Given

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k.$$

The Hadamard product is:

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Define the bounded preserving functions:

$$\mathcal{B} = \{F \in Hol(\mathbb{D}) | F(0) = 1, ||F * f||_\infty \leq ||f||_\infty, \forall f \in H^\infty\}.$$

Bounded Preserving Functions

Definition

$$\mathcal{B} = \{F \in Hol(\mathbb{D}) | F(0) = 1, ||F * f||_\infty \le ||f||_\infty, \forall f \in H^\infty\}.$$

Notice that:

$$\sum_{k=0}^{n} |a_k| r^k \le || \sum_{k=0}^{n} a_k z^k ||_\infty, \quad \forall \{a_k\}$$

$\iff$

$$1 + \sum_{k=0}^{n} e^{i\phi_k} r^k z^k \in \mathcal{B}, \quad \forall \{\phi_k\} \subset \mathbb{R}.$$

Theorem (Sheil-Small, 1973)

$f \in \mathcal{B}$ if and only if

$$\operatorname{Re} f \ge \frac{1}{2}.$$

Bounded Preserving Functions

Theorem (Sheil-Small, 1973)

$f \in \mathcal{B}$ if and only if

$$\operatorname{Re} f \geq \frac{1}{2}.$$

Then apply the classical theorem of Caratheodory and Szegő.

Theorem (Caratheodory and Szegő)

Let $f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in Hol(\mathbb{D})$.

Then $\operatorname{Re} f \geq \frac{1}{2}$ if and only if

$$\begin{pmatrix} 1 & c_1 & \cdots & c_n \\ \bar{c_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ \bar{c_n} & \cdots & \bar{c_1} & 1 \end{pmatrix} \geq 0, \quad \forall n \geq 0.$$

A Formula for R_n

Theorem (Fournier, 2008)

For each $n \geq 1$, let $T_n(r)$ be the following $(n+1) \times (n+1)$ symmetric Toeplitz matrix

$$\begin{pmatrix} 1 & r & -r^2 & r^3 & \cdots & (-1)^{n-1} r^n \\ r & 1 & r & -r^2 & \cdots & (-1)^{n-2} r^{n-1} \\ -r^2 & r & 1 & r & & \\ r^3 & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-1} r^n & & \cdots & & r & 1 \end{pmatrix}.$$

Then R_n is equal to the smallest root in $(0,1)$ of the equation

$$\det T_n(r) = 0.$$

Main Theorem

Based on numerical evidence, Fournier conjectured that

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + \frac{3\pi^4}{4n^4} + \dots$$

Theorem (C.)

Let R_n be the smallest root in $(0,1)$ of the equation $detT_n(r) = 0$, where

$$T_n(r) = \begin{pmatrix} 1 & r & -r^2 & r^3 & \cdots & (-1)^{n-1}r^n \\ r & 1 & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ -r^2 & r & 1 & r & & \\ r^3 & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-1}r^n & & \cdots & & r & 1 \end{pmatrix}$$

Then

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + o(\frac{1}{n^2}), \text{ as } n \to \infty.$$

Main Theorem

Proof: Fix $r \in (0,1)$, we consider the eigenvalues of the matrix $T_n(r)$.

Eigenvalues of Toeplitz Matrix

Let $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$ be a real-valued continuous function on $\mathbb{T}$.
Denote the eigenvalues of the associated Toeplitz matrix

$$T_n[f] = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_{-n} & \cdots & c_{-1} & c_0 \end{pmatrix}$$

as $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_{n+1}^{(n)}$.
If $m \le f \le M$, then $m \le \lambda_1^{(n)} \le \lambda_{n+1}^{(n)} \le M$.
Hence, for every v,

$$\lambda_v^{(n)} = f(x_v^{(n)}), \quad \textit{for some } x_v^{(n)} \in \mathbb{T}.$$

Main Theorem

Theorem (Szegő, 1915)

Let $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$ be a real-valued function on $\mathbb{T}$ and $m \leq f \leq M$.

Let $\{\lambda_v^{(n)}\}_{v=1}^{n+1}$ be the eigenvalues of the associated Toeplitz matrix

$$T_n[f] = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_{-n} & \cdots & c_{-1} & c_0 \end{pmatrix}$$

Then for every continuous function F on $[m, M]$,

$$\lim_{n\to\infty} \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \cdots + F(\lambda_{n+1}^{(n)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} F(f(x))dx.$$

Hence the sequences $\{\lambda_v^{(n)}\}_{v=1}^{n+1}$ and $\{f(\frac{2v\pi}{n+2})\}_{v=1}^{n+1}$ are "equally distributed".

$$T_n(r) = \begin{pmatrix} 1 & r & -r^2 & r^3 & \cdots & (-1)^{n-1}r^n \\ r & 1 & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ -r^2 & r & 1 & r & & \\ r^3 & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-1}r^n & & \cdots & & r & 1 \end{pmatrix}$$

The function associated with the Toeplitz matrix $T_n(r)$ is

$$f(x) = 1 + \sum_{|n|>0} (-1)^{n-1} r^n e^{inx} = \frac{3r^2 + 4r\cos x + 1}{r^2 + 2r\cos x + 1}.$$

Let

$$\Delta_n(\lambda) = \det(T_n(r) - \lambda I).$$

Main Theorem

We have a linear recurrence relation:

$$\Delta_n(\lambda) = [(3-\lambda)r^2 + 1 - \lambda]\Delta_{n-1}(\lambda) - (2-\lambda)^2\Delta_{n-2}(\lambda), \quad \forall n \geq 1,$$

with $\Delta_{-1}(\lambda) = 1$ and $\Delta_0(\lambda) = 1 - \lambda$.

Use the substitution

$$\lambda = f(x) = \frac{3r^2 + 4r\cos x + 1}{r^2 + 2r\cos x + 1}, \quad x \in [0, \pi],$$

we get

$$\Delta_n(\lambda) = \frac{[(\lambda-2)r]^{n+1}}{1-r^2}\left(\frac{\sin(n+2)x}{\sin x} + 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}\right).$$

Main Theorem

Let

$$p_n(x) = \frac{\sin(n+2)x}{\sin x} + 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}.$$

p_n is a polynomial of degree $n+1$ in $\cos x$, so it has $n+1$ zeros on $[0,\pi]$:

$$0 \le x_1^{(n)} \le x_2^{(n)} \le \cdots \le x_{n+1}^{(n)} \le \pi.$$

Since

$$f(x) = \frac{3r^2 + 4r\cos x + 1}{r^2 + 2r\cos x + 1}$$

is decreasing on $[0,\pi]$, $\lambda_{min}^{(n)} = f(x_{n+1}^{(n)})$ is the smallest eigenvalue of $T_n(r)$.

Main Theorem

Theorem (Szegő, 1915)

Let $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$ be a real-valued function on $\mathbb{T}$ and $m \leq f \leq M$.

Let $\{\lambda_v^{(n)}\}_{v=1}^{n+1}$ be the eigenvalues of the associated Toeplitz matrix

$$T_n[f] = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ c_{-n} & \cdots & c_{-1} & c_0 \end{pmatrix}$$

Then for every continuous function F on $[m, M]$,

$$\lim_{n\to\infty} \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \cdots + F(\lambda_{n+1}^{(n)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} F(f(x))dx.$$

Hence the sequences $\{\lambda_v^{(n)}\}_{v=1}^{n+1}$ and $\{f(\frac{2v\pi}{n+2})\}_{v=1}^{n+1}$ are "equally distributed".

Main Theorem

$$p_n(x) = \frac{\sin(n+2)x}{\sin x} + 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}.$$

Let

$$t_\nu^{(n)} = \frac{\nu\pi}{n+2}, \quad \nu = 1, 2, \cdots, n+1.$$

Direct computation shows that

$$p_n(t_\nu^{(n)}) = (-1)^{\nu+1}2r(1 + r\cos\nu),$$

and

$$\lim_{x\to\pi^-} p_n(x) = 2(-1)^{n+1}(1-r)^2.$$

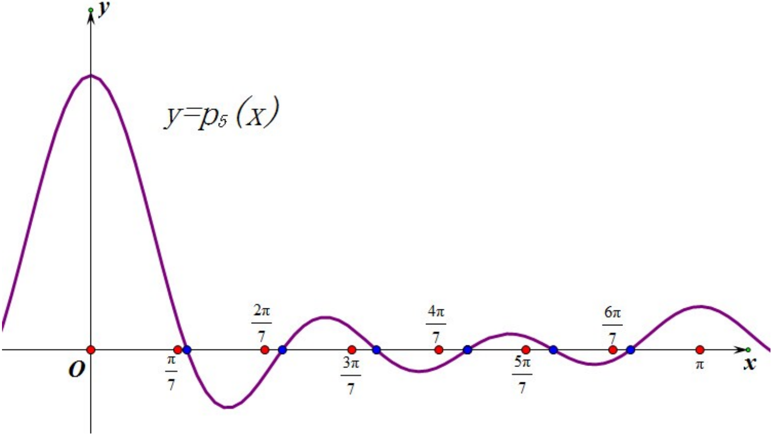
So the zeros $\{x_v^{(n)}\}_{v=1}^{n+1}$ satisfies

$$0 < t_1^{(n)} < x_1^{(n)} < t_2^{(n)} < x_2^{(n)} < \cdots < t_{n+1}^{(n)} < x_{n+1}^{(n)} < \pi.$$

Main Theorem

$$t_\nu^{(n)} = \frac{\nu\pi}{n+2}, \quad \nu = 1, 2, \cdots, n+1.$$

$$0 < t_1^{(n)} < x_1^{(n)} < t_2^{(n)} < x_2^{(n)} < \cdots < t_{n+1}^{(n)} < x_{n+1}^{(n)} < \pi.$$



Main Theorem

$$\lim_{n\to\infty}(-1)^{n+1}\frac{p_n(\pi-\frac{z}{n+2})}{n+2} = (1-r)^2\frac{\sin z}{z}$$

$$\Longrightarrow \quad x_{n+1}^{(n)} = \pi - \frac{\pi}{n} + o(\frac{1}{n}), \quad \text{as } n\to\infty.$$

$$f(x) = \frac{3r^2+4r\cos x+1}{r^2+2r\cos x+1}.$$

$r = R_n$ is the root in $(0,1)$ of the equation

$$\begin{aligned}
\lambda_{min}^{(n)} = f(x_{n+1}^{(n)}) = 0 \quad &\Longrightarrow \quad 3R_n^2 + 4R_n\cos x_{n+1}^{(n)} + 1 = 0 \\
&\Longrightarrow \quad R_n = \frac{1}{3}(-2\cos x_{n+1}^{(n)} - \sqrt{4\cos^2 x_{n+1}^{(n)} - 3}) \\
&\Longrightarrow \quad R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + o(\frac{1}{n^2}). \quad \square
\end{aligned}$$

The End