A Spectral Area Estimate of Some Toeplitz Operators

Cheng Chu

Washington University in St. Louis

(Joint work with Dmitry Khavinson)

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Introduction

Definition

Let $L^2 = L^2(\partial\mathbb{D})$ and let H^2 be the Hardy space on $\mathbb{D}$. Given a symbol function $\varphi \in L^\infty(\partial\mathbb{D})$, define the Toeplitz operator T_φ and the Hankel operator H_φ as

$$T_\varphi : H^2 \to H^2, \quad T_\varphi h = P(\varphi h),$$

and

$$H_\varphi : H^2 \to H^2, \quad H_\varphi h = V(I - P)(\varphi h).$$

where $P : L^2 \to H^2$ is the orthogonal projection,
$Vf(z) = \bar{z} f(\bar{z})$ is a unitary operator on L^2.

Some results about the spectrum of Toeplitz operators

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- If φ is analytic, $sp(T_\varphi) = \overline{\varphi(\mathbb{D})}$.

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- If φ is analytic, $sp(T_\varphi) = \overline{\varphi(\mathbb{D})}$.
- If φ is continuous, $sp(T_\varphi) = \text{Ran}(\varphi) \cup \{\lambda \in \mathbb{C} | i_t(\varphi, \lambda) \neq 0\}$. Here $i_t(\varphi, \lambda)$ is the winding number of the curve determined by φ with respect to λ.

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Theorem (H. Widom, 1964)

For $\varphi \in L^\infty$, $sp(T_\varphi)$ is connected.

Hyponormal Operators

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For a bounded linear operator T on a Hilbert space, let $[T^*, T]$ denote the self-commutator of T, i.e.

$$[T^*, T] = T^*T - TT^*.$$

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Putnam's Inequality (1970)

If T is hyponormal , then

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi}.$$

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Proof: For $f \in H^\infty$ and $p \in H^2$,

$$\begin{aligned}
\langle [T_f^*, T_f]p, p\rangle &= \langle T_f p, T_f p\rangle - \langle T_f^* p, T_f^* p\rangle \\
&= ||fp||^2 - ||T_{\bar{f}} p||^2 \\
&= ||\bar{f} p||^2 - ||T_{\bar{f}} p||^2 \geq 0
\end{aligned}$$

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Theorem (Khavinson, 1984)

Suppose G is a finitely connected bounded domain in $\mathbb{C}$ with a piecewise-smooth boundary, φ is analytic in a neighborhood of $\bar{G}$. Then

$$\|[T_\varphi^*, T_\varphi]\| \geq \frac{4\, Area^2(sp(T_\varphi))}{||\varphi'||_{L^2(ds)}^2 \cdot Peri(G)}.$$

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Combining the two inequalities together, we obtained

$$Area(sp(T_\varphi)) \leq \frac{||\varphi'||_{L^2(ds)}^2 \cdot Peri(G)}{4\pi}.$$

Alexander's Spectral Area Estimate (Lower bound)

For a uniform algebra A, let M_A denote the maximal ideal space of A, the space of nonzero multiplicative linear functionals of A.

For any $m \in M_A$, there exists a representing measure μ_m on M_A such that $m(f) = \int f d\mu_m$, for all $f \in A$.

Let $sp(f) = \{\lambda \in \mathbb{C} \,|\, f - \lambda \text{ is not invertible in } A\}$.

Theorem (H. Alexander, 1978)

Suppose A is a uniform algebra. Let $m \in M_A$ and μ be its representing measure. Then

$$Area(sp(f)) \geq \pi \int |f - m(f)|^2 d\mu.$$

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Take $A = H^\infty(\mathbb{D})$, we identify $\mathbb{D}$ in the usual way as a subset of $M(H^\infty)$.

Let $m = 0$, then $\mu = \frac{d\theta}{2\pi}$.

And $sp(f) = sp(T_f)$.

Corollary 1

Let $f \in H^\infty$. Then

$$Area(sp(T_f)) \geq \pi ||f - f(0)||_2^2.$$

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Let $m = z \in \mathbb{D}$, then $\mu = \mu_z$, the Poisson measure.

Corollary 2

Let $f \in H^\infty$. Then

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Question: What about hyponormal Toeplitz operators?

Hyponormal Toeplitz Operator

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T is called hyponormal if $[T^*, T] \geq 0$.

Characterization of hyponormal Toeplitz operators.

Theorem (C.Cowen, 1988)

Let $\varphi \in L^\infty$, where $\varphi = f + \bar{g}$ for f and g in H^2. Then T_φ is hyponormal if and only if

$$g = c + T_{\bar{h}} f,$$

for some constant c and $h \in H^\infty$ with $\|h\|_\infty \leq 1$.

Main Result

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Theorem (Khavinson, C.)

Suppose $\varphi \in L^\infty$ and

$$\varphi = f + \overline{T_{\bar{h}} f},$$

for $f, h \in H^\infty$, $\|h\|_\infty \leq 1$ and $h(0) = 0$. Then

$$\|[T_\varphi^*, T_\varphi]\| \geq \int |f - f(0)|^2 \frac{d\theta}{2\pi} = ||P(\varphi) - \varphi(0)||_2^2.$$

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Proof:

$$||[T_\varphi^*, T_\varphi]|| = \sup_{\substack{||p||=1 \\ p \in H^2}} |\langle [T_\varphi^*, T_\varphi]p, p\rangle|.$$

We can compute that

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle = ||H_{\bar{f}} p||^2 - ||T_{\bar{h}} H_{\bar{f}} p||^2.$$

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T_φ is normal if and only if h is a unimodular constant.
We added the assumption that $h(0) = 0$ to avoid these trivial cases.

Approximate h by inner functions

$$||[T_\varphi^*, T_\varphi]|| = \sup_{\substack{||p||=1 \\ p\in H^2}} |\langle [T_\varphi^*, T_\varphi]p, p\rangle|.$$

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- If h is inner, it is easy to show that

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- If h is inner, it is easy to show that

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle \ge |H_{\bar{f}}p(0)|^2.$$

- Approximate h by convex linear combinations of inner functions. The above estimate remains valid for all $h \in H^\infty$, $\|h\|_\infty \le 1$.

Main Result

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle \geq |H_{\bar{f}}p(0)|^2.$$

$$H_\varphi h = V(I-P)(\varphi h), \quad Vf(z) = \bar{z}f(\bar{z}).$$

A duality argument shows that

$$\sup_{\substack{||p||=1 \\ p\in H^2}} |H_{\bar{f}}p(0)| = \sup_{\substack{||p||=1 \\ p\in H^2}} |\langle p\bar{f}, \bar{z}\rangle| = \text{dist}(\bar{f}, H^2) = ||f - f(0)||_2.$$

Theorem (Khavinson, C.)

Suppose $\varphi \in L^\infty$ and

$$\varphi = f + \overline{T_{\bar{h}}f},$$

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We have the spectral area estimate

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Remark

Corollary 1 of Alexander's Spectral Area Estimate

Let $f \in H^\infty$. Then

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Corollary 2 of Alexander's Spectral Area Estimate

Let $f \in H^\infty$. Then

$$Area(sp(T_f)) \geq \pi \sup_{z \in \mathbb{D}} \{ \int |f - f(z)|^2 d\mu_z \} \approx ||f||_{BMO}^2$$

Open question: What can we say about the BMO norm?

The End