A property on reinforcing edge-disjoint spanning hypertrees in uniform hypergraphs

Xiaofeng Gu
(University of West Georgia)

joint work with Hong-Jian Lai (WVU)

May 21, 2017
A necessary condition for a union of $k$ edge-disjoint spanning trees: $|E(G)| = k(|V(G)| - 1)$.
Motivation

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- “Dynamic” process
Definition

- $G$ is a simple graph satisfying $|E(G)| = k(|V(G)| - 1)$. 

- $A$ $k$-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(G)$ such that $|X_i| = |V(G)| - 1$ for $1 \leq i \leq k$ is a uniform $k$-partition.

- Let $P_k(G)$ be the collection of uniform $k$-partitions of $E(G)$.

- Define $\varepsilon(\pi) = \sum_{i=1}^{k} c(G(X_i)) - k \varepsilon(G)$, where $\varepsilon(G) = \min_{\pi \in P_k(G)} \varepsilon(\pi)$.

- By definition, $\varepsilon(G) \geq 0$. 

- $\varepsilon(G) = 0$ if and only if for every $1 \leq i \leq k$, $G(X_i)$ is a spanning tree of $G$.

- Thus $\varepsilon(G) = 0$ if and only if $G$ has $k$ edge-disjoint spanning trees.
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- Thus $\varepsilon(G) = 0$ if and only if $G$ has $k$ edge-disjoint spanning trees.
Reinforce edge-disjoint spanning trees

The following theorem was conjectured by Payan in 1986 and proved by Lai, Lai and Payan in 1996.

**Theorem**

If $G$ is a simple graph with $|E(G)| = k(|V(G)| - 1)$ and $\varepsilon(G) > 0$, then there exist $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G - e + e') < \varepsilon(G)$.
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The theorem iteratively defines a finite $\varepsilon$-decreasing sequence of simple graphs $G, G_1, G_2, \cdots, G_m$ such that $G_m$ is a union of $k$ edge-disjoint spanning trees and any two consecutive graphs in the sequence differ by exactly one edge.
A hypergraph $H = (V, E)$

A hyperforest is a hypergraph $H$ such that for every nonempty subset $U \subseteq V(H)$, $|E(H[U])| \leq |U| - 1$.

If in addition, $|E(H)| = |V(H)| - 1$, then $H$ is a hypertree.
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![NOT a hyperforest](image2)
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A hypergraph $H$ is $k$-partition-connected if $e(\pi) \geq k(|\pi| - 1)$ for every partition $\pi$ of $V(H)$, where $e(\pi)$ denotes the number of edges intersecting at least 2 parts of $\pi$. 
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c($H$) denotes the number of maximal partition-connected components in $H$. 
Packing spanning hypertrees

- **Theorem** (Nash-Williams, Tutte, independently, 1961)
  For a graph, $k$-partition-connectedness is equivalent to $k$ edge-disjoint spanning trees.
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- **Theorem** (Frank, Király and Kriesell, 2003)
  A hypergraph $H$ is $k$-partition-connected if and only if $H$ has $k$ edge-disjoint spanning hypertrees.
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We define

\[ \varepsilon(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k, \]

\[ \varepsilon(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi) \]

By definition, \( \varepsilon(H) \geq 0 \).

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Theorem (G. and Lai 2017+)
If $H$ is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $\varepsilon(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H^c)$ such that $\varepsilon(H - e + e') < \varepsilon(H)$. $\varepsilon(H)$ can be considered as a measure of how close $H$ is from a union of $k$ edge-disjoint spanning hypertrees. The theorem iteratively defines a finite $\varepsilon$-decreasing sequence of uniform hypergraphs $H, H_1, H_2, \ldots, H_m$ such that $H_m$ is a union of $k$ edge-disjoint spanning trees and any two consecutive hypergraphs in the sequence differ by exactly one hyperedge.
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that \( H_m \) is a union of \( k \) edge-disjoint spanning trees and any two consecutive hypergraphs in the sequence differ by exactly one hyperedge.
Sketch of the proof

Let \( P'_k(H) \) be the collection of all \( k \)-partitions of \( E(H) \), and define \( \varepsilon'(H) = \min_{\pi \in P'_k(H)} \varepsilon(\pi) \).
Let $P'_{k}(H)$ be the collection of all $k$-partitions of $E(H)$, and define $\varepsilon'(H) = \min_{\pi \in P'_{k}(H)} \varepsilon(\pi)$.

**Lemma** For any uniform hypergraph $H$ with $|E(H)| = k(|V(H)| - 1)$, we have $\varepsilon(H) = \varepsilon'(H)$. 
Sketch of the proof

- Find a sub-hypergraph $S$ of $H$ that contains $k$ edge-disjoint spanning hypertrees, i.e., there is a partition $(Y_1, Y_2, \cdots, Y_k)$ of $E(S)$ such that each part induces a partition-connected sub-hypergraph.
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- The corresponding partition of $E(F)$ from $\pi$ is denoted by $\pi'$. Show $\varepsilon(F) < \varepsilon(H)$. 
Thanks