MATCHING EXTENSION IN PRISM GRAPHS

Michael D. Plummer
Department of Mathematics
Vanderbilt University

Joint work with:

Robert Aldred (Otago University)

Matching Extension in general as well as in the plane (and other surfaces) has been extensively studied.

Let G be a graph with at least $2m + 2n + 2$ vertices which contains a perfect matching.

Def: G satisfies property $E(m, n)$ if given any two matchings M and N with $|M| = m$ and $|N| = n$ and such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$.

The property $E(m, n)$ generalizes the older concept of n-extendability of a graph in that

G is n-extendable iff it is $E(n, 0)$.

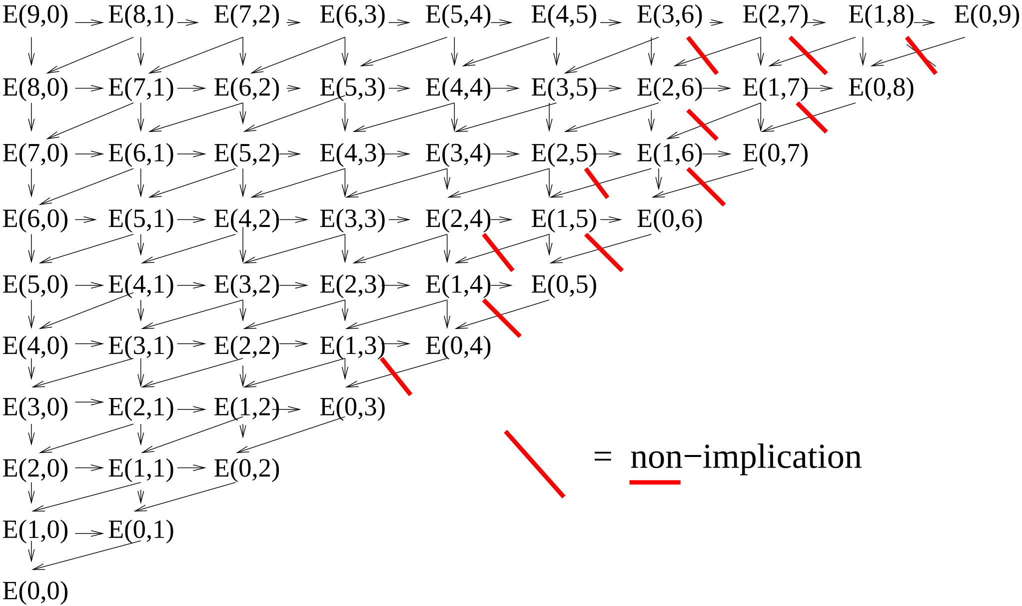
n-extendability, in turn, generalizes 1-extendability (or matching-covered) which historically arose in dealing with counting perfect matchings in a graph.

Organic Chemistry

(1) topological resonance energy

(2) benzenoid compounds

Some implications and *non*-implications among the $E(m, n)$ properties are shown below:


E(9,0) E(8,1) E(7,2) E(6,3) E(5,4) E(4,5) E(3,6) E(2,7) E(1,8) E(0,9)
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E(3,0) E(2,1) E(1,2) E(0,3)
E(2,0) E(1,1) E(0,2)
E(1,0) E(0,1)
E(0,0)
= non−implication

Two basic results to keep in mind are:

Theorem (P-1980): If $m \geq 2$, then:

(i) if G is m-extendable, then G is $(m-1)$-extendable

and

(ii) if G is m-extendable, then G is $(m+1)$-connected.

Two other basic matching concepts are:

Def.: A graph G is said to be bicritical if $G - u - v$ contains a perfect matching, for every choice of two distinct vertices u and v.

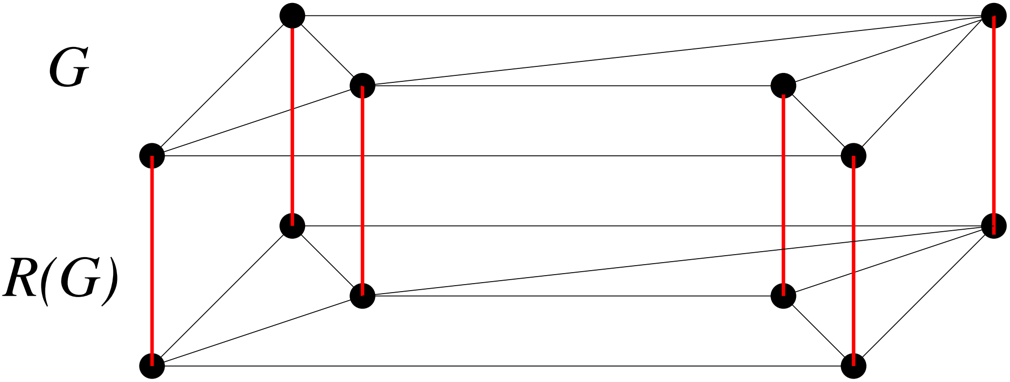
Def.: A graph G is said to be factor-critical if $G - v$ contains a perfect matching, for every choice of a vertex $v \in V(G)$.

Let G and H be any two graphs with vertex sets $V(G)$ and $V(H)$ respectively.

Def.: The *Cartesian product* $G \times H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{(u, v)(x, y) | u = x, vy \in E(H), \text{ or } ux \in E(G), v = y\}$.

We shall focus on the special case when $H = K_2$.

Def.: The graph $P(G) = G \times K_2$ is called the *prism over* G.



G
R(G)

Unsettled Conjecture: (Rosenfeld & Barnette 1973; Kaiser et al. 2007)

Every 3-connected planar graph is prism-hamiltonian.

Theorem (Ellingham & Biebighauser (2007)):

(a) Every triangulation of the plane, projective plane, torus or Klein bottle is prism-hamiltonian.

(b) Every 4-connected triangulation of a surface of sufficiently large face-width is prism-hamiltonian.

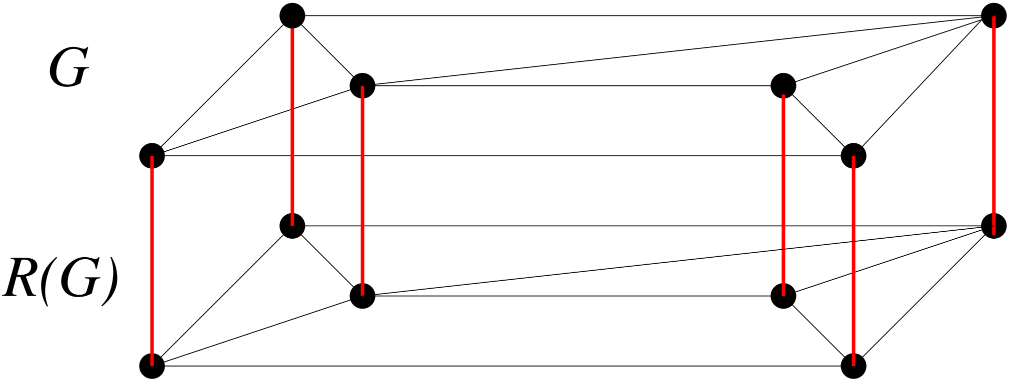
For other related prism-hamiltonian results, see Kaiser et al. (2007).

Basic motivation for our studies will be the following corollary to a more general result due to Györi and the presenter and, independently, to Liu and Yu.

Theorem: If G is a k-extendable graph, then $G \times K_2$ is $(k+1)$-extendable.

Some Basic Results for Prism Graphs

The prism graph $P(G)$ consists of two copies G and G' of G joined by a perfect matching.


G
$R(G)$

We call this perfect matching the set of *vertical edges* in $P(G)$.

Each vertical edge joins a vertex in G to its *reflection* in G'.

Theorem: If G is any connected graph, then $P(G)$ is 1-extendable.

It might not be 2-extendable!

For example, if G has a bridge, then $P(G)$ is not even $E(1,1)$!

In fact, there are *arbitrarily highly connected* graphs G for which $P(G)$ is *not* $E(1,1)$!

Theorem: If G is connected with $\delta(G) \geq n$, then $P(G)$ is $E(0, k)$, for all $0 \leq k \leq n$.

It is known that if a graph is $E(m, 0)$, it is $(m + 1)$-connected.

However, if a graph G is $E(m, 1)$, the minimum required connectivity remains at $m + 1$.

That is, there are graphs which are $E(m, 1)$, but only $(m + 1)$-connected.

To see this, consider the graph

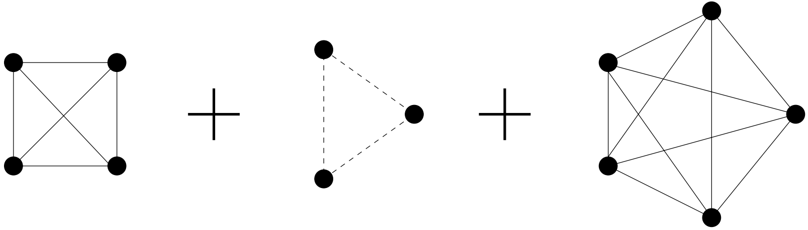
$$G_{5m+1} = \overline{K_{m+1}} + (K_{2m} \cup K_{2m})$$

on $5m + 1$ vertices, when m is odd, and the graph

$$G_{5m+2} = \overline{K_{m+1}} + (K_{2m} \cup K_{2m+1})$$

on $5m + 2$ vertices, when m is even.

For example, let $m = 2$ and consider G_{12}:



G_{12} is $E(2,1)$, but only 3-connected.

In general, these graphs are $E(m, 1)$, but only $(m+1)$-connected.

Theorem: Let G be connected and $k \geq 1$. Then if $P(G)$ is k-extendable, G is k-connected.

Remark: Why $k \geq 1$ here?

Note that if G is disconnected, so is $P(G)$ and hence extendability for $P(G)$ is not defined!

Theorem (Sabidussi 1957): If $k \geq 1$ and G is k-connected, then $P(G)$ is $(k+1)$-connected.

$E(m, n)$ in G versus $P(G)$

Theorem: If G is $E(0, 0)$, then $P(G)$ is $E(0, n)$ for $0 \leq n \leq |V(G)|$.

Recall that if G is $E(m,0)$, then $P(G)$ is $E(m+1,0)$.

However, it is *not* necessarily true that if G is a graph possessing the property $E(m,1)$, $P(G)$ then has the property $E(m+1,1)$.

For all $m \geq 1$, a counterexample is provided by the graph $K_{2m+2} + \overline{K_2}$.

Theorem: If $m \geq 0$ and G is $E(m, 1)$, then $P(G)$ is $E(m + 1, 0)$.

Note that it follows from the lattice and the above Theorem that $P(G)$ is $E(m, 1)$.

But more can be said!

Theorem:*** If $m \geq 0$ and G is $E(m,1)$, then $P(G)$ is $E(m,2)$.

Corollary: If $m \geq 1$ and G is $E(m,1)$, then $P(G)$ is also $E(m,1)$.

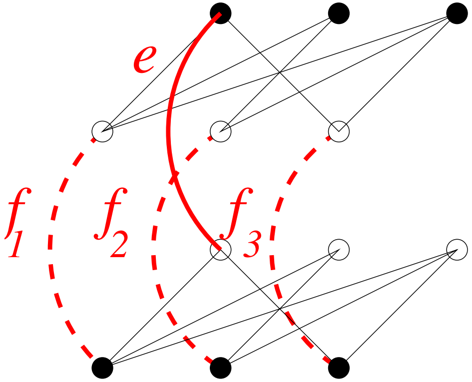
Remark: The conclusion of the preceding theorem is best possible in the sense that, for each $m \geq 1$, there are graphs G which are $E(m,1)$, but $P(G)$ is *not* $E(m,3)$.

As an example, let G be the balanced complete bipartite graph $K_{m+2,m+2}$ with partite sets $A = \{a_1, \ldots, a_{m+2}\}$ and $B = \{b_1, \ldots, b_{m+2}\}$.

Now let $M = \{a_1a'_1, \ldots, a_ma'_m\}$ and $F = \{b_1b'_1, b_2b'_2, b_3b'_3\}$.

Clearly, then, there is no perfect matching M_P in $P(G)$ containing M, but none of the three edges in F.

For example, let $m = 1$ and hence $G = K_{3,3}$:



Bipartite Graphs and their Prisms

If the graph G is *bipartite*, can we expect more?

Theorem:* ** Suppose $k \geq 1$ and G be k-connected and bipartite. Then $P(G)$ is $E(k, 0)$.

Moreover, we cannot drop the bipartite hypothesis in the above theorem and achieve the same conclusion given the (*non*-bipartite) graphs created in the Construction earlier which are arbitrarily highly connected and have prism graphs which are not $E(1,1)$ and hence not $E(2,0)$.

Finally, we cannot decrease the connectivity hypothesis in the above Theorem either.

Example: For $k \geq 2$, the bipartite graph $K_{k-1,k}$ is $(k-1)$-connected, but $P(K_{k-1,k})$ is NOT k-extendable.

Bicritical and Factor-critical Graphs and their Prisms

Two important families of graphs in matching theory are the bicritical graphs and the factor-critical graphs. A 3-connected bicritical graph is often called a *brick*.

An old result due to the second author is the following.

Theorem: If G is any 2-extendable graph then either it is bipartite or a brick.

(Of course G cannot be both.)

Combining this result with the fact that $E(2,0)$ implies $E(1,1)$ in general, we have that a non-bipartite 2-extendable graph is both $E(1,1)$ and bicritical.

The Petersen graph serves to show that the converse is not true, since this graph is both $E(1,1)$ and bicritical, but not 2-extendable.

Remark: Neither of the properties bicritical and $E(1,1)$ necessarily implies the other.

Although a bicritical graph (or brick) need not be $E(1,1)$, one could conjecture that they must be $E(0,2)$.

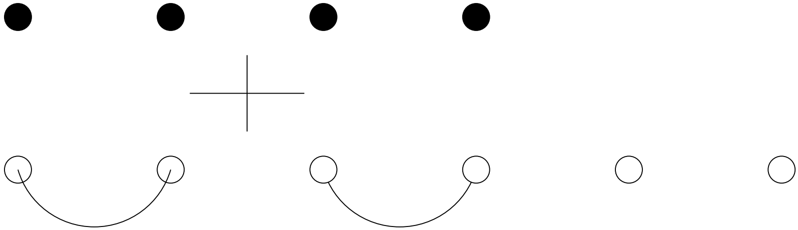
Kothari and Murty proved that all cubic bricks are $E(0,2)$.

The following is an easy generalization of their result.

Theorem: If $k \geq 3$ and G is a k-regular k-connected bicritical graph, then G is $E(0,2)$.

However, it is FALSE that every brick is $E(0,2)$!

Consider the following:



Theorem: If G is bicritical, $P(G)$ is 2-extendable and bicritical and hence a brick.

Although a bicritical graph need not be $E(1,1)$,

(think of the triangular prism)

the following is true.

Corollary: If G is bicritical, then $P(G)$ is $E(1,1)$ and $E(0,n)$, for all $n, 0 \leq n \leq |V(G)|$.

Remark: It is false that if G is bicritical (or even a brick), then $P(G)$ is necessarily $E(1,2)$.

Remark: It then follows that if G is a brick, $P(G)$ is not necessarily $E(2,1)$ or $E(3,0)$.

Let us now consider *factor-critical* graphs and their prisms.

Theorem: If G is factor-critical, then $P(G)$ is bicritical.

Remark: If G is factor-critical, it is *not* necessarily true that $P(G)$ is $E(1,1)$, even if we demand high minimum degree or high connectivity.

On the other hand, we do have the following result.

Theorem: If G is a factor-critical graph, then $P(G)$ is $E(0,n)$, for all $1 \leq n \leq |V(G)| - 1$.

Note that the above result is best possible in that no perfect matching in $P(G)$ can avoid the set of all $|V(G)|$ vertical edges in $P(G)$, since G is factor-critical and hence $|V(G)|$ is odd.