Efficient Boltzmann Samplers for Weighted Partitions and Selections

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Weighted Partitions

Def. An *integer partition* of n is a decomposition of n into a nonincreasing sequence of positive integers that sums to n.

- Partitions of 4: (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, $(1, 1, 1, 1)$

Def. A *weighted partition* of n is a partition of n where each summand of size k belongs to one of b_k types. The class $\mathcal{C}$ of weighted partitions has the generating function

$$C(z) \equiv \sum_{n=0}^{\infty} c_n z^n \equiv \prod_{k=1}^{\infty} \left(1 - z^k\right)^{-b_k}$$
$$= \prod_{k=1}^{\infty} \left(1 + z^k + z^{2k} + \dots\right)^{b_k}.$$

- If $(b_k)_{k=1}^{\infty} = (1, 1, 3, 1, 1, \dots)$ the weighted partitions of 4 are: (4), $(3, 1)$, $(3, 1)$, $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, $(1, 1, 1, 1)$.

Weighted Partitions

Example. Bose–Einstein condensation occurs when $b_k = \binom{k+2}{2}$.

- $C(z) = \prod_{k=1}^{\infty}(1 - z^k)^{-\binom{k+2}{2}} = 1 + 3z + 12z^2 + 38z^3 + \dots$

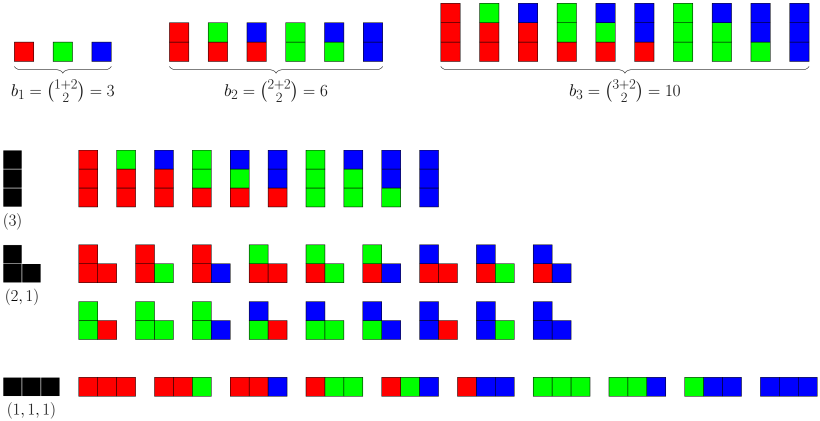

Figure: Young diagrams for $n = 3$.

Weighted Partitions

Problem. Generate a weighted partition of n uniformly at random.

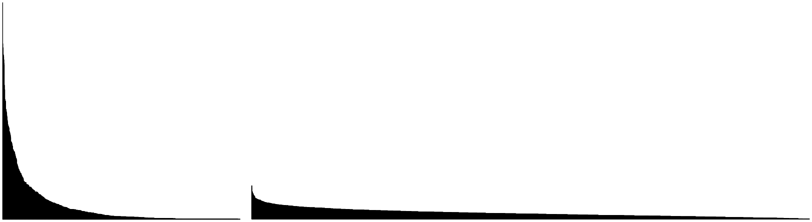

Figure: Random integer partition and weighted partition for $n = 10^5$.

Boltzmann Samplers

Def. A *Boltzmann distribution* over the class $\mathcal{C}$ parameterized by $0 < \lambda < \rho_C$ is the probability distribution, for all $\gamma \in \mathcal{C}$, defined as

$$\mathbb{P}_\lambda(\gamma) = \frac{\lambda^{|\gamma|}}{C(\lambda)},$$

where ρ_C is the radius of convergence of $C(z) = \sum_{n=0}^{\infty} c_n z^n$.

Def. A *Boltzmann sampler* $\Gamma C(\lambda)$ is an algorithm that generates objects from $\mathcal{C}$ according to the Boltzmann distribution parameterized by λ.

- All objects of size n occur with equal probability.

Boltzmann Samplers

The size of an object generated by $\Gamma C(\lambda)$ is a random variable denoted by U with the probability distribution

$$\mathbb{P}_\lambda(U = n) = \frac{c_n \lambda^n}{C(\lambda)}.$$

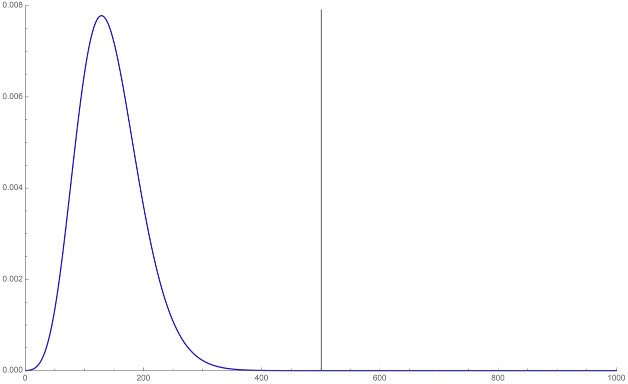

Figure: Distribution for integer partitions ($n = 500, \lambda = 0.90$).

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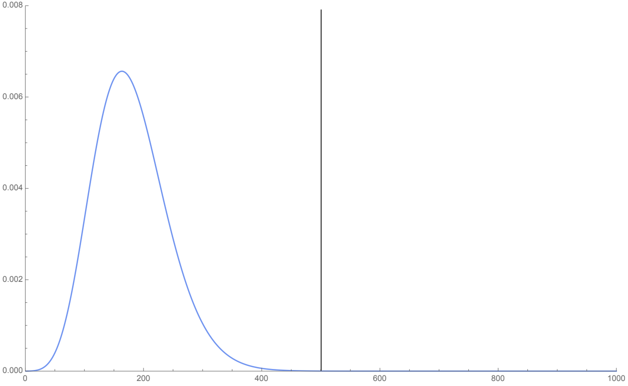

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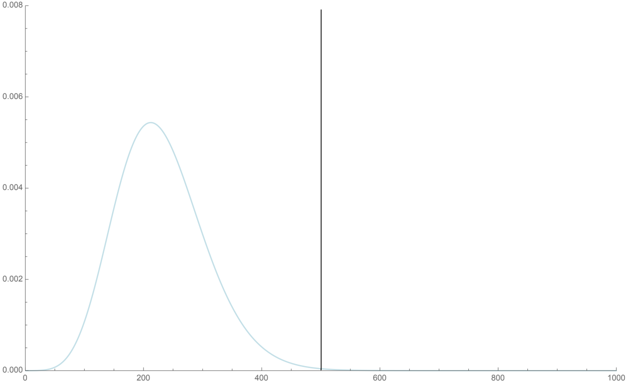

Figure: Distribution for integer partitions ($n = 500, \lambda = 0.92$).

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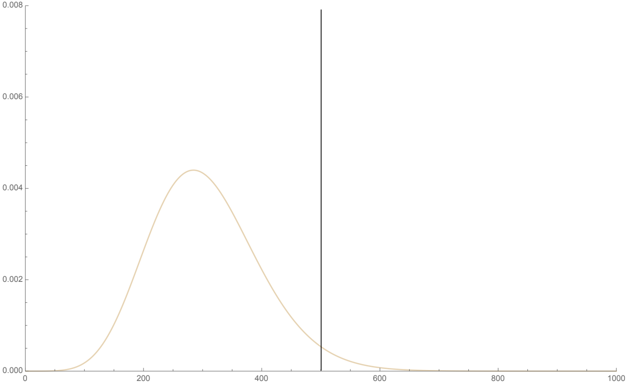

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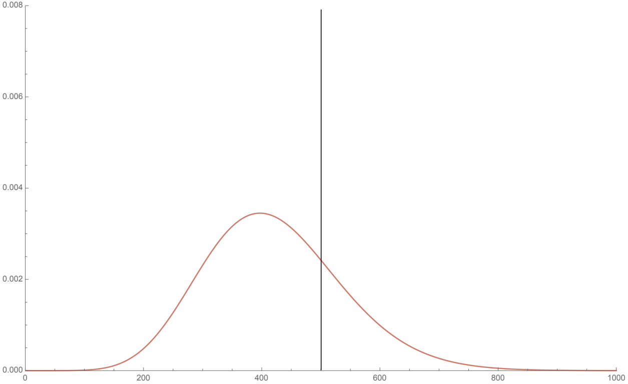

Figure: Distribution for integer partitions ($n = 500, \lambda = 0.94$).

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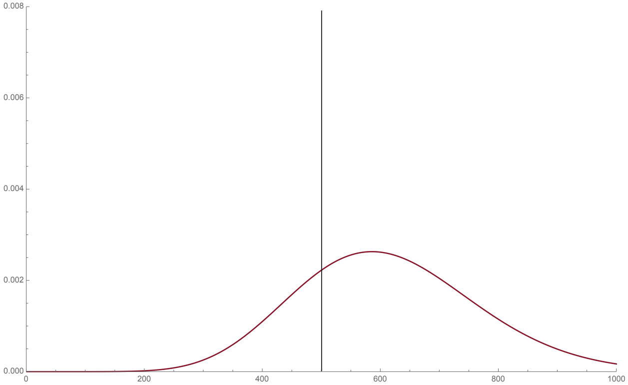

Figure: Distribution for integer partitions ($n = 500, \lambda = 0.95$).

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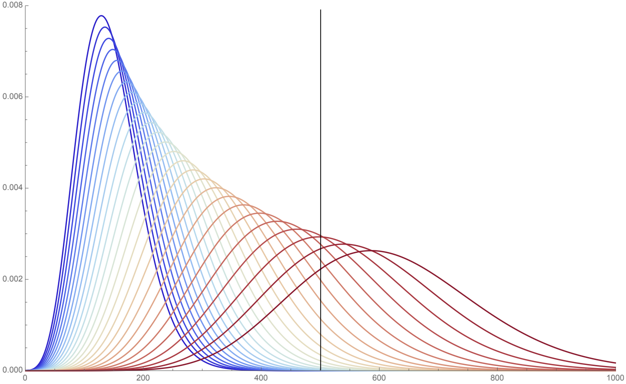

Figure: Distribution for integer partitions ($n = 500, \lambda \in [0.90, 0.95]$).

Sampling Algorithm for Weighted Partitions

Def. Let $(b_k)_{k=1}^{\infty}$ be a sequence such that $b_k = p(k)$ for some polynomial $p(x) = a_0 + a_1 x + \cdots + a_r x^r \in \mathbb{R}[x]$ with $\deg(p) = r$.

- $\binom{k+2}{2} = \frac{1}{2}(k+1)(k+2)$

Algorithm 1 Sampling algorithm for weighted partitions.

1: **procedure** $\textsc{RandomWeightedPartition}(n)$
2: $\lambda_n \leftarrow$ Solution to $\sum_{k=1}^{n} k b_k \lambda^k / (1 - \lambda^k) = n$ ▷ Tuning
3: **repeat** ▷ Rejection sampling
4: $\gamma \leftarrow \Gamma C_n(\lambda_n)$
5: **until** $|\gamma| = n$
6: **return** γ

Theorem. Algorithm 1 runs in expected $O(n^{r+1}((r+2)n)^{3/4})$ time and uses $O(n)$ space.

Sampling Algorithm for Weighted Partitions

Observation.

$$\mathbb{P}_\lambda(\gamma) = \frac{\lambda^{|\gamma|}}{\prod_{k=1}^{n}(1-\lambda^k)^{-b_k}} = \prod_{k=1}^{n}\prod_{j=1}^{b_k} \frac{\lambda^{(\#\text{ of columns of type } b_{k,j})k}}{(1-\lambda^k)^{-1}}$$

Algorithm 2 Boltzmann sampler for weighted partitions.

procedure $\Gamma C_n(\lambda)$
    $\gamma \leftarrow$ Empty associative array
    for $k = 1$ **to** n **do**
        for $j = 1$ **to** b_k **do**
            $m \leftarrow$ Geometric$(1-\lambda^k)$
            if $m \geq 1$ **then**
                $\gamma[(k,j)] \leftarrow m$
    return γ

- The PDF of Geometric$(1-\lambda^k)$ is $\mathbb{P}_{1-\lambda^k}(m) = \lambda^{mk}(1-\lambda^k)$.

Tuning the Boltzmann Samplers

The random variable for the size of an object produced is

$$U_n = \sum_{k=1}^{n} \sum_{j=1}^{b_k} k Y_{k,j},$$

where $Y_{k,j} \sim \text{Geometric}\big(1 - \lambda^k\big)$.

Lemma. We have

$$\mathbb{E}_\lambda[U_n] = \sum_{k=1}^{n} k b_k \left(\frac{\lambda^k}{1 - \lambda^k} \right).$$

- Follows from the linearity of expectation and the mean of geometric random variable
- Strictly increasing so binary search to solve $\mathbb{E}_\lambda[U_n] = n$

Bounding the Rejection Rates

Lemma. The *Dirichlet generating series* for weighted partitions is

$$D(s) = \sum_{k=1}^{\infty} \frac{b_k}{k^s} = \sum_{k=0}^{r} a_k \zeta(s-k),$$

and has at most $r+1$ simple poles on the positive real axis at positions $\rho_k = k+1$ with residue $A_k = a_k$ if and only if $a_k \neq 0$.

Proof. The Riemann zeta function converges uniformly and is analytic on $\mathbb{C} \setminus \{1\}$, so

$$D(s) = \sum_{k=1}^{\infty} \frac{b_k}{k^s} = \sum_{k=1}^{\infty} \frac{a_0 + a_1 k + \cdots + a_r k^r}{k^s} = \sum_{k=0}^{r} a_k \zeta(s-k).$$

Since $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ has a simple pole at $s=1$ with residue 1, the claim about the poles of $D(s)$ follows.

Bounding the Rejection Rates

Theorem [Granovsky and Stark, 2012]. For n sufficiently large,

$$\mathbb{P}_{\lambda_n}(U_n = n) \geq \frac{1}{10}\left(\frac{A_r}{2}\Gamma(\rho_r+2)\zeta(\rho_r+1)\right)^{\frac{1}{2(r+2)}}((r+2)n)^{-3/4}.$$

Lemma. For any positive integer sequence of degree r,

$$\frac{A_r}{2}\Gamma(\rho_r+2)\zeta(\rho_r+1) \geq 1.$$

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Proof. Let

$$p(k) = \sum_{j=0}^{r} \Delta^j p(0)\binom{k}{j}.$$

[Stanley, EC1, Cor 1.9.3] $\implies$ $\Delta^r p(0) \in \mathbb{Z}$ $\implies$ $a_r = A_r \geq \frac{1}{r!}$.
Since $\rho_r = r+1$,

$$\frac{A_r}{2}\Gamma(\rho_r+2)\zeta(\rho_r+1) \;\geq\; \frac{1}{2r!}(r+2)! \;\geq\; 1.$$

Boltzmann Sampler for Bose–Einstein Condensation

Lemma. If $b_k = \binom{k+d-1}{d-1}$, for $d \geq 1$, then

$$C(z) = \prod_{k=1}^{\infty} \left(1 - z^k\right)^{-\binom{k+d-1}{d-1}} = \prod_{k=1}^{\infty} \exp\left(\frac{1}{k}\left[\left(1 - z^k\right)^{-d} - 1\right]\right).$$

Theorem. We can uniformly sample objects of size n in $\mathcal{C}$ in expected $O(n((d+1)n)^{3/4})$ time while using $O(n)$ space.

- Runtime reduced from $O(n^{3.75})$ to $O(n^{1.75})$
- Columns are sampled from the zero-truncated negative binomial distribution
- Geometric random variable is a weighted sum of independent Poisson random variables

References

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3. Boris L. Granovsky and Dudley Stark. A Meinardus theorem with multiple singularities. *Communications in Mathematical Physics*, 314(2):329–350, 2012.

4. Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.