Biased Graphs and Gain Graphs

by Daniel Slilaty

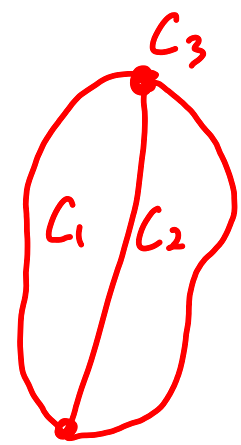
joint work with

N. Neudauer, D. Funk,
D. Chun, T. Moss, and X. Zhou,
V. Sivaraman.

Def: A <u>biased graph</u> is a pair (G, B)
(Zaslavsky '86)

graph

A collection of cycles in G such that every <u>Theta subgraph</u> has 0, 1, or 3 cycles in B, ie, not exactly 2 cycles in B.



called "<u>Balanced</u> cycles"

Simple Examples

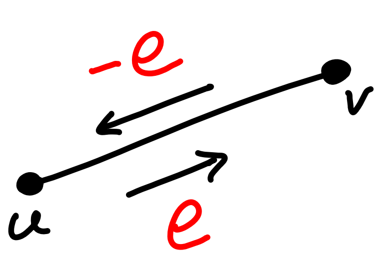
1. $(G, \text{all cycles})$

2. $(G, \emptyset)$ "Contrabalanced"

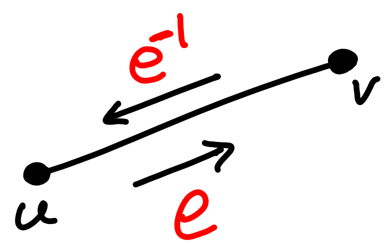
3. (G, B) is a biased graph when B is any collection of Hamilton cycles in G and G is simple.

The Canonical Example of a Biased Graph

Each edge e in a graph G has two orientations



additive



multiplicative

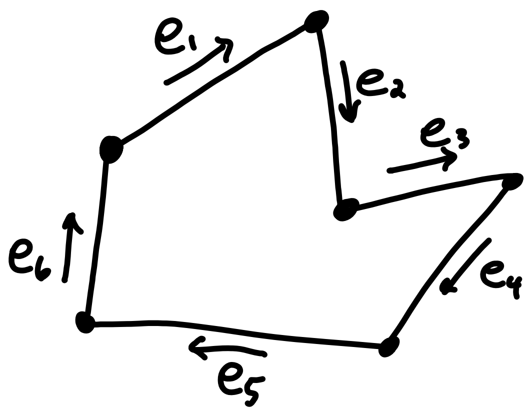
Given a group Γ, a Γ-gain function or a Γ-voltage function is

$$\varphi : \vec{E}(G) \longrightarrow \Gamma$$ such that

$\varphi(e^{-1}) = \varphi(e)^{-1}$ if Γ is multiplicative

$\varphi(-e) = -\varphi(e)$ if Γ is additive

* Now let B_φ be the collection of cycles such that



$$\varphi(e_1) + \cdots + \varphi(e_n) = 0 \quad \text{(additive)}$$

or

$$\varphi(e_1) \cdots \varphi(e_n) = 1 \quad \text{(multiplicative)}$$

<u>Proposition</u> (Zaslavsky)

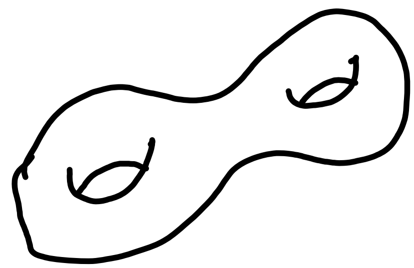
(G, B_φ) is a biased graph.

<u>Theorem</u> (Chen, Funk, Pivotto)

There are infinitely many minor-minimal biased graphs not obtainable as gain graphs.

Topological Example

Embed graph G in surface K



More generally, let G be the 1-skeleton of some 2-dimensional cellular complex K.

Homology Bias $\mathcal{B}_K$ = collection of cycles that separate K.

Γ = 1st Homology group of K.

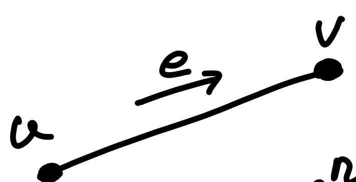
Homotopy Bias $\mathcal{B}_K$ = collection of cycles that are contractible on K.

Γ = Fundamental group of K.

Group Realizability

Def Given a group Γ, a biased graph (G, B) is said to be **Γ-Realizable** when $B = B_\varphi$ for some φ.

Def Two Γ-realizations φ and ψ of (G, B) are **switching equivalent** if there is $\eta : V(G) \to \Gamma$ such that $\varphi^\eta = \psi$



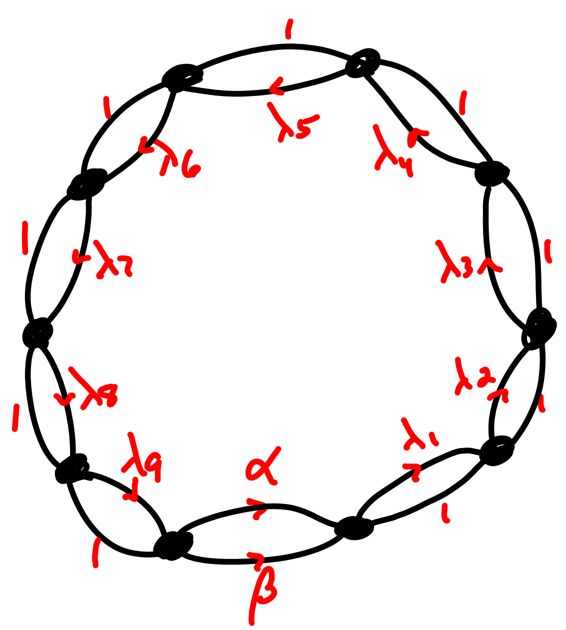
$$\varphi^\eta(e) = \eta(u)\varphi(e)\eta(v)^{-1}$$

Def Two Γ-realizations are **equivalent** when there is some automorphism α of Γ and some Γ such that $\alpha\varphi^\eta = \psi$

Theorem (Neudauer, me) If Γ is a finite group, then there is $N(\Gamma)$ such that any 3-connected (G,B) has at most $N(\Gamma)$ Γ-realizations up to equivalence.

$N(\Gamma)$ depends on Γ only, Not on (G,B) !!

3-connectivity is necessary.



$\alpha, \beta \notin \Lambda$

$(2C_k, \emptyset)$ has at least $\dfrac{2^k}{|Aut(\Gamma)|}$ Γ-realizations when Γ has a proper subgroup Λ of order at least 3.

However, $(2C_k, \emptyset)$ is in essence the only problem.

Theorem (Neudaur, me) If Γ is a finite group and $k \geq 3$, then there is $N(\Gamma, k)$ such that any 2-connected (G, B) with no $(2C_k, \phi)$-minor has at most $N(\Gamma, k)$ Γ-realizations, up to equivalence.

Corollary (Neudauer, me) Given a prime p, there is $N(p)$ such that any 2-connected (G, B) has at most $N(p)$ $\mathbb{Z}_p$-realizations.

Idea of the proof

1. Show, (G, B) has a minor (H, ϕ) which has the same connectivity and bounds the Γ-realizations of (G, B).

2. Show, there are only finitely many (H, ϕ) that are Γ-realizable.

Γ-realizations of Contrabalanced Biased graphs

Theorem (D. Chun, Moss, Zhou, me)

Let Γ be a finite group.
There are finitely many 3-connected (G, ϕ) that are Γ-realizable.

Theorem (Neudauer, me) Let

Let Γ be a finite group.
There are finitely many 2-connected (G, ϕ) without a $(2C_k, \phi)$-minor and having minimum degree 3 that are Γ-realizable.

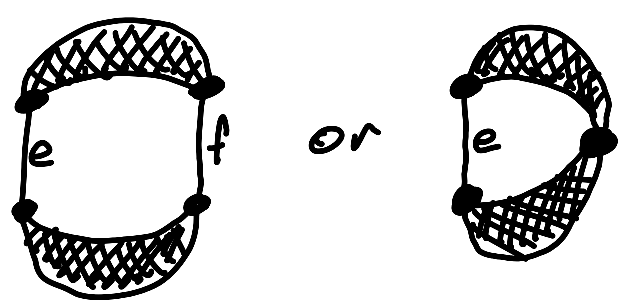
* So for small Γ, can we completely charcterize which graphs G have (G, ϕ) Γ-realizable?

$\underline{\mathbb{Z}_2 \text{ (Zaslavsky)}}$ Let (G, ϕ) be 2-connected. Then (G, ϕ) is $\mathbb{Z}_2$-realizable if and only G is a cycle.

$\underline{\mathbb{Z}_3 \text{ (Sivaraman)}}$ Let (G, ϕ) be 2-connected. Then (G, ϕ) is Γ-realizable if and only if G is a theta graph.

* For $|\Gamma| > 3$ we need an irreducibility property stronger than minimum degree 3.

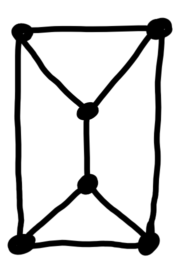
<u>Def</u> A 2-connected graph is <u>2-bond irreducible</u> if it doesnt look like

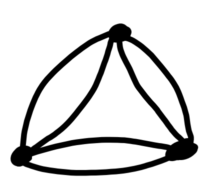


$\mathbb{Z}_4$ (Chun, Moss, Zhou, me) Let (G, ϕ) be 2-connected and 2-band irreducible. Then (G, ϕ) is $\mathbb{Z}_4$-realizable if and only if G is $(2C_n, \phi)$ or K_4.

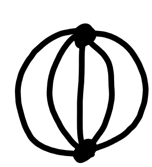
$\mathbb{Z}_2 \times \mathbb{Z}_2$ (Sivaraman, me) Let (G, ϕ) be 2-connected and 2-band irreducible. Then (G, ϕ) is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-realizable if and only if G is $(2C_n, \phi)$.

$\mathbb{Z}_5$ (Sivaraman, me) Let (G, ϕ) be 2-connected and 2-band irreducible. Then (G, ϕ) is $\mathbb{Z}_5$-realizable if and only if G is a minor of



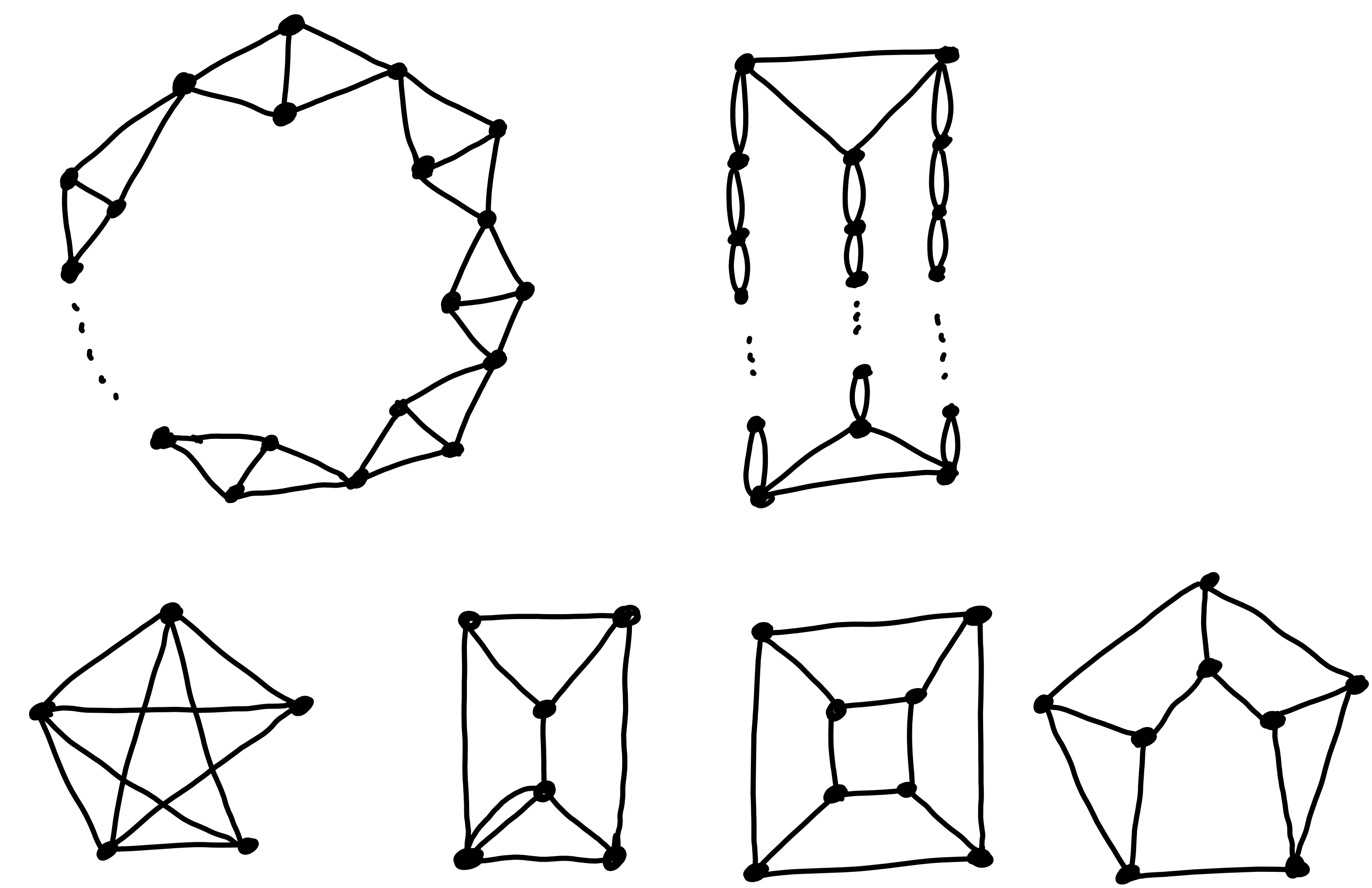


or



note
There are only finitely many.

$\mathbb{Z}_5$ (Sivaraman, me) Let (G, ϕ) be 2-connected and 2-band irreducible. Then (G, ϕ) is $\mathbb{Z}_6$-realizable if and only if G is a minor of one of



S_3 (Sivaraman, me) A much smaller subset than for $\mathbb{Z}_6$.

Matroids

"So... what is a matroid? Really...."

— G.C. Rota to his student Joseph Kung.

Def A matroid M is a set E along with a collection $\mathcal{C}$ of subsets of E called circuits such that:

1. $\phi \notin \mathcal{C}$
2. If $C_1, C_2 \in \mathcal{C}$ then $C_1 \not\subseteq C_2$.
3. If $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$ then there is $C_3 \subseteq (C_1 \cup C_2) - e$.

Theorem (H. Whitney, T. Nakasawa)

If A is a matrix over field $\mathbb{F}$ with columns E and $\mathcal{C}$ is the minimal linearly dependent subsets of E, then this is a matroid.

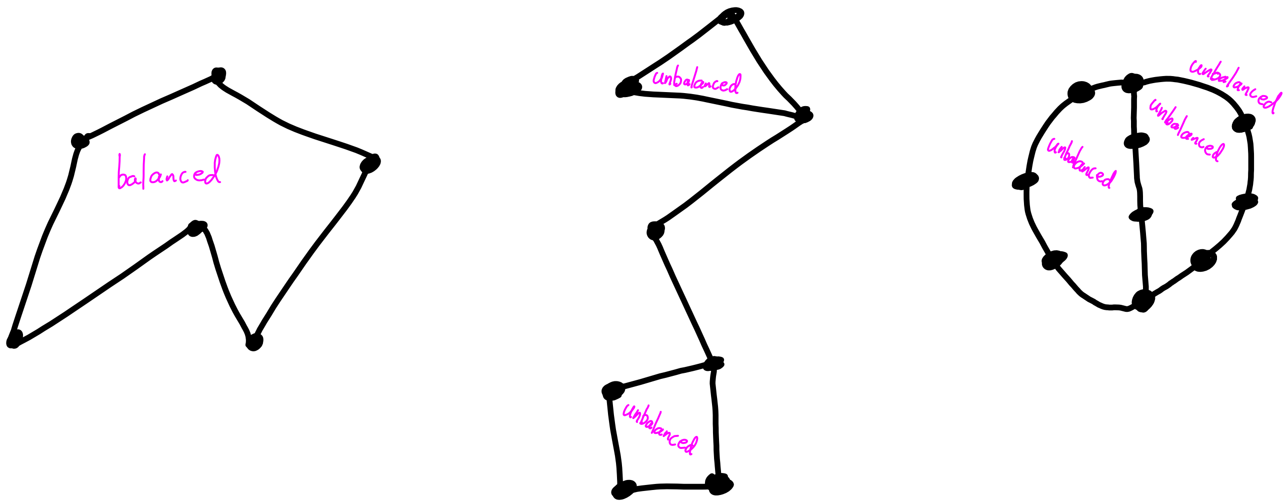
Proposition If A is an $r \times c$ matrix over $\mathbb{F}$, P is $r \times r$ and invertible, and D is $c \times c$ diagonal and invertible, then

$$A \quad \text{and} \quad PAD$$

represent the same matroid. Denoted $M(A)$.

Def We say that matrices A and B are projectively equivalent when $B = PAD$.

Theorem (Zaslavsky) If $(G, \mathcal{B})$ is a biased graph with edge set E and $\mathcal{C}$ is the collection of edge sets of subgraphs of the form



Then this is a matroid.

Def This is called a Frame Matroid $F(G, \mathcal{B})$.

Theorem (Kahn, Kung)

If $\mathcal{V}$ is a non-degenerate "variety" of matroids, Then one of The following holds.

1. $\mathcal{V}$ is The set of matroids from matrices over $GF(q)$

2. $\mathcal{V}$ is The set of frame matroids from Γ-gain graphs for some fixed finite group Γ. (Dowling Geometries)

The centrality of frame matroids within The class of all matroids goes much deeper as explained in The work of The "Matroid Minors Project" by Geelen, Gerards, Whittle.

Incidence Matrices

Let $\mathbb{F}$ be any field,

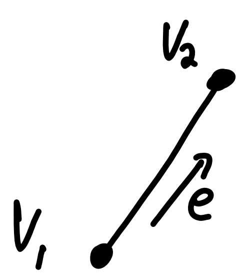
$\mathbb{F}^*$ The multiplicative group of $\mathbb{F}$

$\mathbb{F}^+$ The additive group of $\mathbb{F}$.

Frame Matrix

Let φ be a $\mathbb{F}^*$-gain function on G.

Define the columns of $A(G, \varphi)$ by

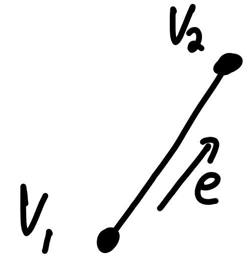


$$\longrightarrow \quad \begin{array}{c c} & e \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ \vdots \\ v_n \end{array} & \begin{bmatrix} 1 \\ -\varphi(e) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}$$

Lift Matrix

Let γ be a $\mathbb{F}^+$-gain function on G.

Define the columns of $A(G,\gamma)$ by



$$\longrightarrow \quad \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} e \\ \begin{bmatrix} \gamma(e) \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}$$

Theorem (Zaslavsky)

1. The matroid of $A(G,\varphi)$ is $F(G,B_\varphi)$
2. If (G,B_γ) has no two vertex-disjoint unbalanced cycles, Then The matroid of $A(G,\gamma)$ is $F(G,B_\gamma)$.

Theorem (Funk, me)

1. Let $(G, \mathcal{B})$ be a 2-connected biased graph without a balancing vertex and let φ_1 and φ_2 be $\mathbb{F}^*$-realizations of $(G, \mathcal{B})$. Then

$$A(G, \varphi_1) \text{ and } A(G, \varphi_2)$$

are projectively equivalent if and only if φ_1 and φ_2 are switching equivalent.

2. Same for $A(G, \tau_1)$ and $A(G, \tau_2)$

Theorem (Funk, me)

Geelen, Gerards, Whittle have a weaker but analogous result.

Let A be a matrix whose matroid is $F(G,B)$ where (G,B) is 2-connected and has no balancing vertex, Then

A is projectively equivalent to a unique $A(G,\varphi)$ or $A(G,\tau)$.