Biased Graphs and Gain Graphs

by Daniel Slilaty

joint work with

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Def: A **biased graph** is a pair $(G, B)$ (Zaslavsky 86)

A **collection of cycles in $G$ such that every theta subgraph has 0, 1, or 3 cycles in $B$, i.e., not exactly 2 cycles in $B$**

called **"Balanced cycles"**

**Simple Examples**

1. $(G, \text{all cycles})$

2. $(G, \emptyset)$ **"Contrabalanced"**

3. $(G, B)$ is a biased graph when $B$ is any collection of Hamilton cycles in $G$ and $G$ is simple.
The Canonical Example of a Biased Graph

Each edge $e$ in a graph $G$ has two orientations

$\begin{align*}
\begin{array}{c}
\text{additive} \\
\end{array}
\begin{array}{c}
\text{multiplicative} \\
\end{array}
\end{align*}$

Given a group $\Gamma$, a $\Gamma$-gain function or a $\Gamma$-voltage function is

$$\varphi : E(G) \to \Gamma$$

such that

$$\begin{align*}
\varphi(e^{-1}) &= \varphi(e)^{-1} \\
\varphi(-e) &= -\varphi(e)
\end{align*}$$

if $\Gamma$ is multiplicative

if $\Gamma$ is additive
* Now let $B_\phi$ be the collection of cycles such that

\[ \phi(e_1) + \cdots + \phi(e_n) = 0 \quad \text{(additive)} \]

or

\[ \phi(e_1) \cdots \phi(e_n) = 1 \quad \text{(multiplicative)} \]

**Proposition (Zaslavsky)**

$(G, B_\phi)$ is a biased graph.

**Theorem (Chen, Funk, Pivotto)**

There are infinitely many minor-minimal biased graphs not obtainable as gain graphs.
Topological Example

Embed graph $G$ in surface $K$

More generally, let $G$ be the 1-skeleton of some 2-dimensional cellular complex $K$.

Homology Bias $B_k =$ collection of cycles that separate $K$.

$\Gamma = 1^{st}$ Homology group of $K$.

Homotopy Bias $B_k =$ collection of cycles that are contractible on $K$.

$\Gamma =$ Fundamental group of $K$. 
Group Realizability

Def Given a group $\Gamma$, a biased graph $(G, B)$ is said to be $\Gamma$-Realizable when $B = B\varphi$ for some $\varphi$.

Def Two $\Gamma$-realizations $\varphi$ and $\psi$ of $(G, B)$ are switching equivalent if there is $\eta: V(G) \to \Gamma$ such that $\varphi^n = \psi$

\[
\begin{array}{c}
\eta: V(G) \to \Gamma \\
n \mapsto \eta(n) \end{array}
\]

\[
\varphi^n(e) = \eta(n) \varphi(e) \eta(n)^{-1}
\]

Def Two $\Gamma$-realizations are equivalent when there is some automorphism $\alpha$ of $\Gamma$ and some $\Gamma$ such that $\alpha \varphi^n = \psi$
Theorem (Neudaur, me) If $\Gamma$ is a finite group, then there is $N(\Gamma)$ such that any 3-connected $(G, B)$ has at most $N(\Gamma)$ $\Gamma$-realizations up to equivalence.

$N(\Gamma)$ depends on $\Gamma$ only, not on $(G, B)$!!

3-connectivity is necessary.

$(2C_k, \emptyset)$ has at least
\[
\frac{2^k}{|\text{Aut}(\Gamma)|}
\]
$\Gamma$-realizations when $\Gamma$ has a proper subgroup $\Lambda$ of order at least 3.

$\alpha, \beta \notin \Lambda$

However, $(2C_k, \emptyset)$ is in essence the only problem.
**Theorem** (Neudauer, me) If $\Gamma$ is a finite group and $k \geq 3$, then there is $N(\Gamma, k)$ such that any 2-connected $(G, B)$ with no $(2C_k, \phi)$-minor has at most $N(\Gamma, k)$ $\Gamma$-realizations, up to equivalence.

**Corollary** (Neudauer, me) Given a prime $p$, there is $N(p)$ such that any 2-connected $(G, B)$ has at most $N(p)$ $\mathbb{Z}_p$-realizations.

**Idea of the proof**

1. Show, $(G, B)$ has a minor $(H, \phi)$ which has the same connectivity and bounds the $\Gamma$-realizations of $(G, B)$.

2. Show, there are only finitely many $(H, \phi)$ that are $\Gamma$-realizable.
\( \Gamma \)-realizations of Contrabalanced Biased graphs

**Theorem** (D. Chun, Moss, Zhou, me)

Let \( \Gamma \) be a finite group.
There are finitely many 3-connected 
\( (G, \phi) \) that are \( \Gamma \)-realizable.

**Theorem** (Neudauer, me) Let

\( \Gamma \) be a finite group.
There are finitely many 2-connected 
\( (G, \phi) \) without a \( (2C_k, \phi) \)-minor and
having minimum degree 3 that are \( \Gamma \)-realizable.

* So for small \( \Gamma \), can we completely characterize which graphs \( G \) have \( (G, \phi) \) \( \Gamma \)-realizable?
$Z_2$(Zaslavsky) Let $(G, \phi)$ be 2-connected. Then $(G, \phi)$ is $Z_2$-realizable if and only if $G$ is a cycle.

$Z_3$(Sivaraman) Let $(G, \phi)$ be 2-connected. Then $(G, \phi)$ is $\Gamma$-realizable if and only if $G$ is a theta graph.

* For $|\Gamma|>3$ we need an irreducibility property stronger than minimum degree 3.

Def A 2-connected graph is 2-bond irreducible if it doesn't look like

\[
\begin{array}{c}
\text{e} & \text{f} & \text{or} & \text{e} \\
\end{array}
\]
**Z_4 (Chun, Moss, Zhou, me)** Let \((G, \phi)\) be 2-connected and 2-band irreducible. Then \((G, \phi)\) is \(Z_4\)-realizable if and only if \(G\) is \((2C_n, \phi)\) or \(K_4\).

**Z_2 \times Z_2 (Sivaraman, me)** Let \((G, \phi)\) be 2-connected and 2-band irreducible. Then \((G, \phi)\) is \((Z_2 \times Z_2)\)-realizable if and only if \(G\) is \((2C_n, \phi)\).

**Z_5 (Sivaraman, me)** Let \((G, \phi)\) be 2-connected and 2-band irreducible. Then \((G, \phi)\) is \(Z_5\)-realizable if and only if \(G\) is a minor of \(\begin{array}{c} \text{ minor of } \end{array}\) or \(\begin{array}{c} \text{ minor of } \end{array}\)

*note: There are only finitely many.*
Let \((G, \phi)\) be 2-connected and 2-band irreducible. Then \((G, \phi)\) is \(\mathbb{Z}_6\)-realizable if and only if \(G\) is a minor of one of

\[
\begin{align*}
\mathbb{Z}_6 & \\
\mathbb{Z}_6 & \\
\end{align*}
\]

A much smaller subset than for \(\mathbb{Z}_6\).
Matroids

“So... what is a matroid? Really....”

- G.C. Rota to his student Joseph Kung.

Def A matroid $M$ is a set $E$ along with a collection $C$ of subsets of $E$ called circuits such that:

1. $\emptyset \in C$

2. If $C_1, C_2 \in C$ then $C_1 \neq C_2$.

3. If $C_1, C_2 \in C$ and $e \in C_1 \cap C_2$ then there is $C_3 \subseteq (C_1 \cup C_2) - e$. 
Theorem (H. Whitney, T. Nakasawa)

If $A$ is a matrix over field $\mathbb{F}$ with columns $E$ and $C$ is the minimal linearly dependent subsets of $E$, then this is a matroid.

Proposition If $A$ is an $r \times c$ matrix over $\mathbb{F}$, $P$ is $r \times r$ and invertible, and $D$ is $c \times c$ diagonal and invertible, then $A$ and $PAD$ represent the same matroid. Denoted $M(A)$.

Def We say that matrices $A$ and $B$ are projectively equivalent when $B = PAD$. 
Theorem (Zaslavsky) If $(G, B)$ is a biased graph with edge set $E$ and $C$ is the collection of edge sets of subgraphs of the form

Then this is a matroid.

Def This is called a Frame Matroid $F(G, B)$. 
Theorem (Kahn, Kung)

If $V$ is a non-degenerate "variety" of matroids, then one of the following holds.

1. $V$ is the set of matroids from matrices over $GF(q)$

2. $V$ is the set of frame matroids from $Γ$-gain graphs for some fixed finite group $Γ$. (Dowling Geometries)

The centrality of frame matroids within the class of all matroids goes much deeper as explained in the work of the "Matroid Minors Project" by Geelen, Gerards, Whittle.
Incidence Matrices

Let $\mathbb{F}$ be any field,

- $\mathbb{F}^*$: The multiplicative group of $\mathbb{F}$
- $\mathbb{F}^+$: The additive group of $\mathbb{F}$

Frame Matrix

Let $\varphi$ be a $\mathbb{F}^*$-gain function on $G$.

Define the columns of $A(G, \varphi)$ by

\[
\begin{bmatrix}
e \\
1 \\
\varphi(e) \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\]
Lift Matrix

Let $\tau$ be a $\mathbb{F}^+$-gain function on $G$.

Define the columns of $A(G,\tau)$ by

$$
\begin{bmatrix}
  v_0 \\
  v_1 \\
  \vdots \\
  v_n
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  e \\
  \tau(e) \\
  \vdots \\
  0
\end{bmatrix}
$$

Theorem (Zaslavsky)

1. The matroid of $A(G,\varphi)$ is $F(G, B_\varphi)$

2. If $(G, B_G)$ has no two vertex-disjoint unbalanced cycles, then the matroid of $A(G, \tau)$ is $F(G, B_\tau)$. 
Theorem (Funk, me)

1. Let \((G, B)\) be a 2-connected biased graph without a balancing vertex and let \(\phi_1\) and \(\phi_2\) be \(F^*\)-realizations of \((G, B)\). Then

   \[ A(G, \phi_1) \text{ and } A(G, \phi_2) \]

   are projectively equivalent if and only if \(\phi_1\) and \(\phi_2\) are switching equivalent.

2. Same for \(A(G, \tau_1)\) and \(A(G, \tau_2)\)
Theorem (Funk, me)

Let $A$ be a matrix whose matroid is $F(6, B)$ where $(6, B)$ is 2-connected and has no balancing vertex, then

$A$ is projectively equivalent to a unique $A(6, 0)$ or $A(6, 1)$.