Planar graphs with girth at least 5 are (3, 4)-colorable

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A proper $r$-coloring of graph is a coloring of the graph with $r$ colors so that each color class forms an independent set.
A proper \textit{r-coloring} of graph is a coloring of the graph with \textit{r} colors so that each color class forms an independent set.

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We may relax the requirement by allowing some edges in each color class.
A graph $G$ is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into $r$ nonempty subsets so that the subgraph induced by the $i$th part has maximum degree at most $d_i$ for each $i \in \{1, \ldots, r\}$, where $d_i$s are non-negative integers.
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Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. (See Montassier and Ochem, Near-colorings: non-colorable graphs and NP-completeness, the electronic journal of combinatorics 22(1) (2015), #P1.57)
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In 1986, Cowen, Cowen, and Woodall proved that planar graphs are $(2, 2, 2)$-colorable. In 1999, Eaton and Hull, Škrekovski, separately, proved that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each $k$. Thus, the problem is completely solved when $r \geq 3$. 

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Planar graphs with girth at least 5 are $(3, 4)$-colorable
The girth of a graph $G$ is the length of a shortest cycle. Let $G_g$ denote the class of planar graphs with girth at least $g$. 

There are non-$(d_1, d_2)$-colorable planar graphs in $G_4$ for any $d_1$, $d_2$. (Montassier and Ochem, 2015)

There are non-$(0, k)$-colorable planar graphs in $G_6$ for any $k$. (Borodin, Ivanova, Montassier, Ochem and Raspaud, 2010)

There are non-$(2, 0)$-colorable planar graphs in $G_7$. (Montassier and Ochem, 2015)

There are non-$(3, 1)$-colorable planar graphs in $G_5$. (Montassier and Ochem, 2015)

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Some known results on \((d_1, d_2)\)-colorable graphs in \(G_5\)

- Planar graphs in \(G_5\) are \((1, 10)\)-colorable. (Choi, Choi, Jeong and Suh 2016)

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A summary on $(d_1, d_2)$-coloring

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- **Claim.** \(G\) must be connected and there are no 1-vertices in \(G\).

- **Lemma 2** There is no 3-vertex in \(G\).

![Diagram](image-url)
Let the initial charge of each element $x \in V \cup F$ be $\mu(x) = d(x) - 4$. Then by Euler formula,

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Clearly, by Lemma 2, each face and each vertex has a non-negative initial charge except 2-vertices.
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Proof–4

- Three special vertices: $5p$-vertex $x$, $5s$-vertex $y$ and $6p$-vertex $z$.

\[\mu'(x) = 5 - 4 - 4 = -1.\]
\[\mu'(y) = 5 - 4 - 3 = -\frac{1}{2}.\]
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Considering these three special vertices and some special faces, we design the discharging rules.

By the discharging rules, there is $\mu^*(x) \geq 0$ for each $x \in V \cup F$. So we have

$$\sum_{x \in V \cup F} \mu^*(x) \geq 0,$$

a contradiction.
Some problems

Problem 1. Given a pair \((d_1, d_2)\), determine the minimum \(g = g(d_1, d_2)\) such that every planar graph with girth \(g\) is \((d_1, d_2)\)-colorable.

Problem 2. Given a pair \((g, d_1)\), determine the minimum \(d_2 = d_2(g, d_1)\) such that every planar graph with girth \(g\) is \((d_1, d_2)\)-colorable.

Problem 3. What is the minimum \(d\) where graphs with girth 5 are \((3, d)\)-colorable in \{2, 3, 4\}?

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Thank you for your attention!