Conditional Connectivity in Networks

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Interconnection networks play an important role in parallel and distributed computing/communication systems and data centers. An interconnection network can be modeled by a graph $G = (V, E)$, where $V$ is the set of processors and $E$ is the set of communication links in the network.

**Figure:** Topological structure of some simple networks
Sunway TaihuLight Supercomputer (Top 1)

Architecture and Performance
- Computer nodes 40,960
- Number of core 10,649,600
- Total CPU plus coprocessor memory 1.31 PB
- Total peak performance 93 petaflops.
Tianhe-2 Supercomputer (Top 2)

Architecture and Performance
- Computer nodes 16,000
- Number of core 3,120,000
- Total CPU plus coprocessor memory 1,375 TB
- Total peak performance 33.9 petaflops.
Titan supercomputer (Top 3)

Architecture and Performance

- Computer nodes 18,688
- Number of core 299,008
- Total CPU plus coprocessor memory 710 TB
- Total peak performance 20 petaflops.
Introduction

Main Results

Further Problems

Background

Fault Tolerance of graphs

Cayley graph

Characteristics of interconnection networks

Extendability

It should be possible to build a network of any given size, or at least to build arbitrarily large versions of the network. Furthermore, it would be easy to construct large networks from small ones.

Symmetry

Regularity and some symmetric properties on the graph.
Classical connectivity

Connectivity, Edge connectivity

A subset $S \subset V(G)$ ($S \subset E(G)$) of a connected graph $G$ is called a cut (edge-cut) if $G - S$ is disconnected. The connectivity (edge-connectivity) $\kappa(G)$ ($\lambda(G)$) of $G$ is defined as the minimum cardinality over all cuts (edge-cuts) of $G$, that is

$$\kappa(G) = \min\{|S| : \text{$S$ is a cut of $G$}\},$$

$$\lambda(G) = \min\{|S| : \text{$S$ is an edge-cut of $G$}\}.$$
Flaws

When computing these parameters, one implicitly assumes that all links incident with the same processor may fail simultaneously. Consequently, this measurement is inaccurate for large-scale processing systems in which some subsets of system components cannot fail at the same time in real applications.
Definition (Harary, 1983)
The conditional connectivity of $G$ with respect to some property $P$ is the smallest cardinality of a set $S$ of vertices, if any, such that every component of the disconnected graph $G - S$ has property $P$.

- In 1989, Esfahanian proposed restricted connectivity
- In 1994, Latifi generalized it to the restricted $h$-connectivity
Restricted $h$-connectivity (Latifi et al. 1994)

For a given integer $h \geq 0$, a vertex (edge) subset $S$ of a connected graph $G$ is called an $h$-cut ($h$-edge-cut), if $G - S$ is disconnected and has the minimum degree $\delta(G - S) \geq h$. The $h$-super connectivity (edge connectivity) of $G$, denoted by $\kappa^{(h)}(G)(\lambda^{(h)}(G))$, is defined as the minimum cardinality over all . That is

$$\kappa^{(h)}(G) = \min\{|S| : S \text{ is an } h\text{-cut of } G\}.$$  

$$\lambda^{(h)}(G) = \min\{|S| : S \text{ is an } h\text{-edge-cut of } G\}.$$
Complexity (Oh, Choi, and Esfahanian, 1991)

The problem of finding the least cardinality $S$ such that $S$ is a conditional cut of $G$ is NP-complete.
Cayley graphs

\[Cay(\Gamma, S)\]

\(\Gamma = (Z, \circ)\) is a finite group, \(S\) is a nonempty subset of \(Z\) without identity. Cayley digraph \(Cay(\Gamma, S)\) is a digraph with vertices \(\Gamma\) and edges \(E(Cay(\Gamma, S)) = \{uv : v = u \circ s, u \in \Gamma, s \in S\}\). \(S^{-1} = \{s^{-1} : s \in S\} = S\), \(Cay(\Gamma, S)\) is an undirected graph.

\[Cay(Sym(n), T)\]

\(Sym(n)\) is the symmetric group on \(\{1, 2, \ldots, n\}\) and \(T\) is a set of transposition of \(Sym(n)\).

\(G(T)\) be the graph on \(n\) vertices \(\{1, 2, \ldots, n\}\) such that there is an edge \(ij\) in \(G(T)\) if and only if transposition \((ij) \in T\).

If \(G(T)\) is a star, \(Cay(Sym(n), T)\) be star graph; if \(G(T)\) is a path, \(Cay(Sym(n), T)\) be bubble-sort graph.
Star graphs $S_2, S_3, S_4$
Hierarchical Structure

Use $S_n^{j:i}$ to denote the subgraph of $S_n$ induced by all vertices with symbol $i$ in the $j$-th position, $I'_n$ be a set $\{2, \ldots, n\}$.

The first structural property (Akers and Krishnamurthy, 1989)

For a fixed dimension $j \in I'_n$, $S_n$ can be partitioned into $n$ subgraphs $S_n^{j:i}$, which is isomorphic to $S_{n-1}$ for each $i \in I_n$. Moreover, there are $(n-2)!$ independent edges between $S_n^{j:i_1}$ and $S_n^{j:i_2}$ for any $i_1, i_2 \in I_n$ with $i_1 \neq i_2$. 
Use $S_{n}^{j:i}$ to denote the subgraph of $S_{n}$ induced by all vertices with symbol $i$ in the $j$-th position, $I'_{n}$ be a set \{2, \ldots, n\}.

**The second structural property** (Shi et al., 2012)

For a fixed symbol $i \in I_{n}$, $S_{n}$ can be partitioned into $n$ subgraphs $S_{n}^{j:i}$, which is isomorphic to $S_{n-1}$ for each $j \in I'_{n}$ and $S_{n}^{1:i}$ is an independent vertex set of size $(n-1)!$. Moreover, there are a perfect matching between $S_{n}^{1:i}$ and $S_{n}^{j:i}$ for any $j \in I'_{n}$, and there are no edges between $S_{n}^{j_{1}:i}$ and $S_{n}^{j_{2}:i}$ for any $j_{1}, j_{2} \in I'_{n}$ with $j_{1} \neq j_{2}$. 
Two structures of $S_4$

Partition along dimension 4

Partition along symbol 1
Some results

Theorem (Rouskov et al., 1996)
If \( n \geq 3 \), then \( \kappa^{(1)}(S_n) = 2n - 4 \).

Theorem (Wan and Zhang, 2009)
If \( n \geq 4 \), then \( \kappa^{(2)}(S_n) = 6(n - 3) \).

Conjecture (Wan and Zhang, 2009)
If \( h \leq n - 2 \), then \( \kappa^{(h)}(S_n) = (h + 1)!(n - h - 1) \).

Theorem (Yang et al., 2010)
If \( n \geq 4 \), then \( \lambda^{(2)}(S_n) = 6(n - 3) \).
Strategy

For a subset $X \subseteq V(S_n)$ and $j \in I_n$, we use $U^X_j$ to denote the set of symbols in the $j$-th position of vertices in $X$, formally, $U^X_j = \{p_j : p_1 \ldots p_j \ldots p_n \in X\}$.

**Lemma (Li and Xu, 2014)**

Let $H$ be a subgraph of $S_n$ with vertex-set $X$. For a fixed $h \in I_{n-1}$, if $\delta(H) \geq h$, then there exists some $j \in I'_n$ such that $|U^X_j| \geq h + 1$.

**Theorem (Li and Xu, 2014)**

If $0 \leq h \leq n - 2$, then $\kappa^{(h)}(S_n) \geq (h + 1)!(n - h - 1)$, $\lambda^{(h)}(S_n) \geq (h + 1)!(n - h - 1)$. 
Some results

**Theorem (Li and Xu, 2014)**

If $0 \leq h \leq n - 2$, then $\kappa^{(h)}(S_n) = \lambda^{(h)}(S_n) = (h + 1)!(n - h - 1)$.

**Conjecture (Wan and Zhang, 2009)**

If $h \leq n - 2$, then $\kappa^{(h)}(S_n) = (h + 1)!(n - h - 1)$.

The Conjecture is proved to be correct, can we say anything more?
Introduction  Main Results  Further Problems

(n,k)-star, A generalization of $S_n$

**Definition (Akers and Krishnamurthy, 1989)**

The $n$-dimensional star graph $S_n$ has vertex-set $P(n)$ and has an edge between any two vertices if and only if one can be obtained from the other by **swapping** the 1-th digit and the $i$-th digit for $i \in I_n'$, that is, two vertices $x = p_1p_2 \ldots p_i \ldots p_n$ and $y$ are adjacent if and only if $y = p_ip_2 \ldots p_{i-1}p_1p_{i+1} \ldots p_n$ for some $i \in I_n'$.

**Definition (Chiang et al., 1995)**

An $(n,k)$-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$, a vertex $p = p_1p_2 \ldots p_i \ldots p_k$ being linked a vertex $q$ if and only if $q$ is

(a) $p_ip_2 \ldots p_{i-1}p_1p_{i+1} \ldots p_k$, where $i \in I_k'$ (**swap** $p_1$ with $p_i$), or

(b) $p_1'p_2p_3 \ldots p_k$, where $p_1' \in I_n \setminus \{p_i : i \in I_k\}$ (**replace** $p_1$ by $p_1'$).
Definition, $t$-Split

A $t$-split graph $G^t$ of $G$ is a graph obtained from $G$ by replacing each vertex $x$ by a set $V_x$ of $t$ independent vertices, and replacing each edge $e = xy$ by a perfect matching $E_e$ between $V_x$ and $V_y$.

Lemma

Let $G$ be a connected graph and $G^t$ be a $t$-split graph of $G$. Then $\kappa^{(h)}(G^t) \leq t \kappa^{(h)}(G)$ and $\lambda^{(h)}(G^t) \leq t \lambda^{(h)}(G)$. 

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(4, 2)-star graph $S_{4,2}$ and its 2-split graph
Relationship between $S_n$ and $S_{n,k}$

**Theorem (Li and Xu, 2017+)**

For any $k$ with $2 \leq k \leq n - 1$, there is an $(n - k)!$-split graph of $S_{n,k}$ that is isomorphic to a star graph $S_n$.

**Theorem (Li and Xu, 2017+)**

For $2 \leq k \leq n - 1$ and $n - k \leq h \leq n - 2$, 

$$
\kappa^{(h)}(S_{n,k}) = \lambda^{(h)}(S_{n,k}) = \frac{(h + 1)!(n - h - 1)}{(n - k)!}.
$$
Bubble-sort graphs

Definition (Akers and Krishnamurthy, 1989)
The $n$-dimensional bubble-sort graph $B_n$ has $n!$ vertices labeled by distinct permutations on $\{1, 2, ..., n\}$, and has an edge between any two vertices if and only if one can be obtained from the other by swapping the $i$-th digit and the $(i + 1)$-th digit where $1 \leq i \leq n - 1$. 
Bubble graphs $B_2, B_3, B_4$
Some Results

Theorem (Akers and Krishnamurthy, 1989)

1. $B_n$ has regular degree $n - 1$;
2. $\kappa(B_n) = \lambda(B_n) = n - 1$;

Structure (Akers and Krishnamurthy, 1989)

For a fixed $t \in \{1, n\}$, $B_n$ can be partitioned into $n$ subgraphs $B_n^{t:j}$ isomorphic to $B_{n-1}$ for each $j \in I_n$, moreover, there are $(n - 2)!$ independent edges between $B_n^{t:j_1}$ and $B_n^{t:j_2}$ for any $j_1, j_2 \in I_n$ with $j_1 \neq j_2$.

Theorem (Yang et al., 2010)

If $n \geq 3$, then $\kappa^1(B_n) = 2n - 4$; if $n \geq 4$, then $\kappa^2(B_n) = 4n - 12$.

How about $\lambda^h(B_n), \kappa^h(B_n)$ for more bigger $h$?
Construction of the upper bound;

The structure of $B_n$;

How an $h$-cut (edge cut) of large networks distributed in the small $B_{n-1}$;

Prove the lower bound using the $(h - 1)$-supper connectivity of $B_{n-1}$. 
Fault tolerance in $B_n$

**Theorem (Li and Xu, 2017+)**

$$\kappa^h(B_n) = \lambda^h(B_n) = 2^h(n - 1 - h) \text{ for any } h \text{ with } 2h \leq n.$$ 

**Corollary**

If $n \geq 3$, then $\kappa^1(B_n) = 2n - 4$; if $n \geq 4$, then $\kappa^2(B_n) = 4n - 12$. 

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Problem

How about the \( h \)-super connectivity (edge connectivity) of \( B_n \) for \( h \geq 2k + 1 \)?

\( h \)-Atom

For a given integer \( h (\geq 0) \), a vertex\( (\\text{edge}) \) subset \( S \) of a connected graph \( G \) is called an \( S \) be an \( h \)-cut(\( h \)-edge-cut), of \( G \), the minimum connected component of \( G - S \) is an \( h \)-atom(\( h \)-edge atom) of \( G \).

Observation and Problems

The \( h \)-atom of \( S_n, Q_n \) is regular, How about \( h \)-atom for general Cayley graphs or regular graphs.
Thanks For Your Attention !