Templates for Minor-Closed Classes of Binary Matroids

Kevin Grace* and Stefan van Zwam

Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana

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Matroids

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Example: Consider the following matrix over $\mathbb{GF}(2)$:
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\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
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0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\] Some sets of columns are dependent, and some are independent.
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Some sets of columns are dependent, and some are independent.
This matroid also can be represented by a graph.

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The edges are the elements of the matroid.
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<tr>
<th>1</th>
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Minors and Duality

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- Duals of graphic matroids are called *cographic* matroids.
Robertson and Seymour

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- Graph Minors Project

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\[
\begin{array}{c|c|c|c}
X & \text{columns from } \Lambda & 0 & Y_0 \quad Y_1 \quad C \\
\hline
\text{incidence} & \text{unit and} & A_1 \\
\text{matrix of} & \text{zero} & \text{rows} \\
\text{a graph} & \text{columns} & \text{from } \Delta
\end{array}
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Respecting a Template

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\text{incidence matrix of a graph} & \text{unit and zero columns} & \text{rows from } \Delta \\
\end{array} \]

(i) \( C, X, Y_0 \) and \( Y_1 \) are disjoint finite sets.
Respecting a Template

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(ii) $A_1 \in (\text{GF}(2))^{X \times (C \cup Y_0 \cup Y_1)}$. 
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- incidence matrix of a graph
- unit and zero columns
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(iii) $\Lambda$ is a subgroup of the additive group of $(\text{GF}(2))^X$. 
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Conforming to a Template

A matrix $A$ *conforms* to a template $\Phi$ if it is formed from a matrix $A'$ that respects $\Phi$ by adding a column of $Y_1$ to each column of $Z$.
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A matroid $M$ conforms to $\Phi$ if there is a matrix $A$ that conforms to $\Phi$ such that $M$ is isomorphic to the vector matroid of $M(A)/C\setminus Y_1$. 

$M(\Phi)$ is the set of matroids conforming to $\Phi$. 
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$\mathcal{M}(\Phi)$ is the set of matroids conforming to $\Phi$. 
Theorem (Geelen, Gerards, and Whittle 2015)

Let $\mathcal{M}$ be a proper minor-closed class of binary matroids. Then there exist $k, l \in \mathbb{Z}_+$ and frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that

$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$, $\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$, and if $\mathcal{M}$ is a simple vertically $k$-connected member of $\mathcal{M}$ with at least $l$ elements, then either $\mathcal{M}$ is a member of at least one of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$, or $\mathcal{M}^*$ is a member of at least one of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$. 
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Minors and Weak Conforming

- A *template minor* of a template $\Phi$ is a template $\Phi'$ obtained from $\Phi$ by repeatedly performing one of several operations.

- Every matroid in $M(\Phi')$ is a minor of a matroid in $M(\Phi)$.

- If $\Phi'$ is a template minor of $\Phi$, then every matroid conforming to $\Phi'$ weakly conforms to $\Phi$.

- We write $\Phi' \preceq \Phi$ if every matroid weakly conforming to $\Phi'$ also weakly conforms to $\Phi$.

- The relation $\preceq$ is a preorder on the set of frame templates.
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There exist $k, l \in \mathbb{Z}^+$ such that no simple, vertically $k$-connected matroid with at least $l$ elements either conforms or coconforms to $\Phi$. 
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To use templates to study a minor-closed class $\mathcal{M}$:

1. Find a matroid $\mathcal{N}$ not in $\mathcal{M}$.
2. Find all templates such that $\mathcal{N}$ is not a minor of any matroid conforming to that template.
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Seymour (1981) showed:

> Cographic matroids are 1-flowing.
> The class of 1-flowing matroids is minor-closed.
> All 1-flowing matroids are binary.
> $AG(3, 2)$ is not 1-flowing.

Conjecture (Seymour’s 1-flowing Conjecture, 1981)

The set of excluded minors for the class of 1-flowing matroids consists of $U_{2,4}$, $AG(3, 2)$, $T_{11}$, and $T^*_11$. 
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*The set of excluded minors for the class of 1-flowing matroids consists of $U_{2,4}$, $AG(3, 2)$, $T_{11}$, and $T_{11}^*$.***
It can be shown that to each of $\Phi_{Y_0}$, $\Phi_{Y_1}$, $\Phi_C$, $\Phi_X$, and $\Phi_{CX}$ conforms a matroid with an $AG(3, 2)$-minor.
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**Theorem (G. and Van Zwam, 2017)**

*There exist $k, l \in \mathbb{Z}_+^+$ such that every simple, vertically $k$-connected, 1-flowing matroid with at least $l$ elements is either graphic or cographic.*
All Minors Are Not Created Equal

If we consider highly connected matroids of sufficient size in a minor-closed class, we often can reduce the number of excluded minors.

Example: A 3-connected graph with at least 11 edges is planar if and only if it contains no $K_3$, $K_3$-minor.

$\text{EX}(M_1, M_2, \ldots)$: the class of binary matroids with no minor in the set $\{M_1, M_2, \ldots\}$. 
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Theorem (G. and Van Zwam, submitted)
There exist $k, l \in \mathbb{Z}^+$ such that a vertically $k$-connected matroid with at least $l$ elements is in $\text{EX}(\text{PG}(3,2) \setminus \mathcal{L}, \mathcal{M}^\ast(\mathcal{K}_6), \mathcal{L}^{11})$ if and only if it is an even-cycle matroid.

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Theorem (G. and Van Zwam, submitted)
There exist $k, l \in \mathbb{Z}^+$ such that a cyclically $k$-connected matroid with at least $l$ elements is in $\text{EX}((\mathcal{M}(\mathcal{K}_6)), \mathcal{H}^\ast_{12})$ if and only if it is an even-cut matroid.
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There exist $k, l \in \mathbb{Z}_+$ such that a vertically $k$-connected matroid with at least $l$ elements is in $\mathcal{E}(PG(3, 2) \setminus e, M^*(K_6), L_{11})$ if and only if it is an even-cycle matroid.
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Thank you!