Quasi-surfaces: chromatic numb. and Euler formula

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A standard Quasi-surface T_k
$k \geq 2$

generalizes both

the 2-sphere S_0 and

the k-book $\mathbb{B}_k$ $k \geq 0$

$\langle$ the equator $\leftrightarrow$ the spine $\rangle$

Let c be a circle in $\mathbb{R}^3$,

called an **event horizon**.

<the equator: a spine>

For $i = 1, 2, \cdots, k$,

let M_i be a closed 2-disk

with $\partial M_i = c$,

called the *i-th membrane*.

< the i-th page>

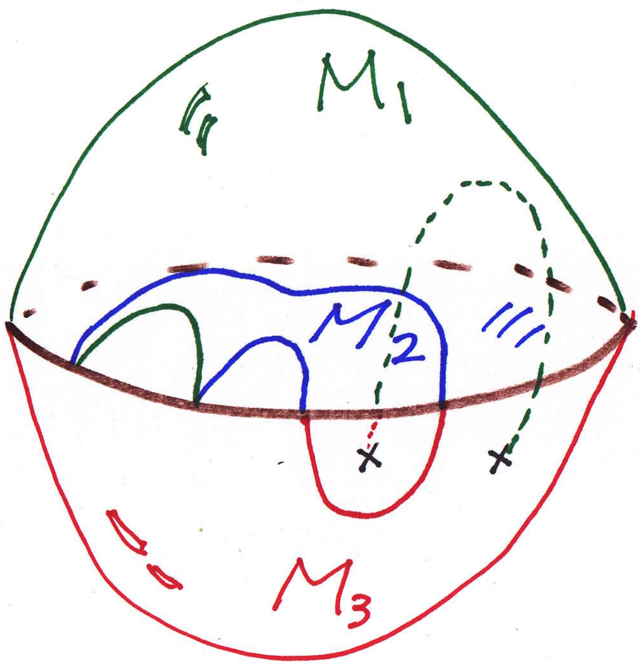
For $k \geq 2$, a standard Quasi-surface, denoted T_k, is the disjoint union of M_i with identifying ∂M_i

$$\bigcup_{i=1}^{k} M_i \quad \text{with } \partial M_i = C$$

C: event hor.

T_k

$k=3$



Jordan Curve Th:

Every simple closed curve bounds exactly two regions in $\mathbb{R}^2$.

Fails in T_k if $k \geq 3$.

Overview

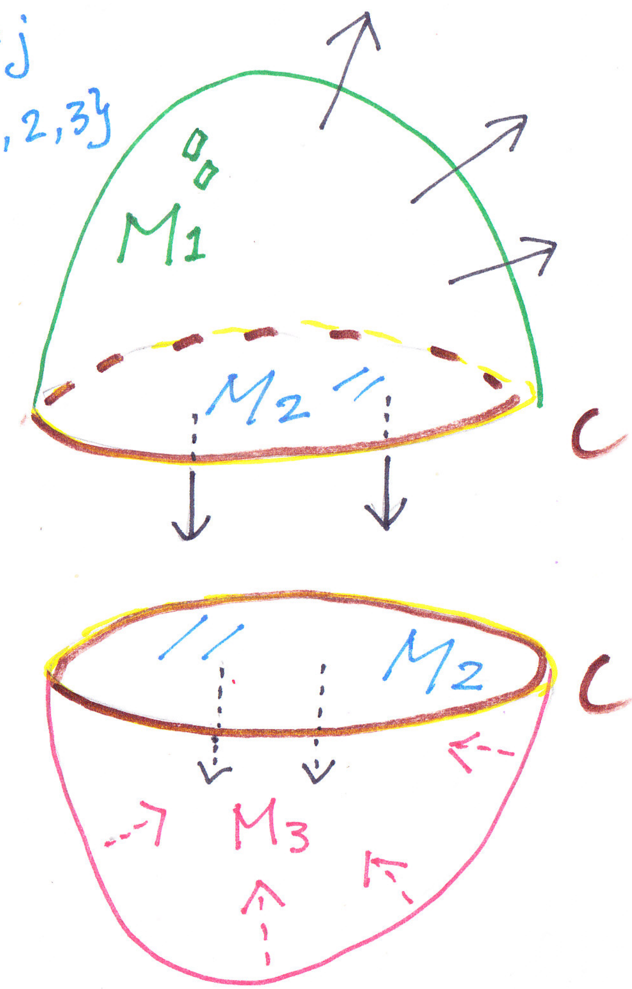
	χ chromatic numb.	$\mathcal{E}$ Euler formula	Orientable ?
S_0	4CT	$\mathcal{E}(S_0)=2$	Orientable
$\mathcal{B}_k$	book thickness if $bt(G) \leq k$ $\chi(G) \leq 2k+2$ [1979]	undefined	Orientable
T_k	Vertex density on $\mathbb{C}$ Matters	well-defined for T_k ?	Non-Orientable if $k \geq 3$.

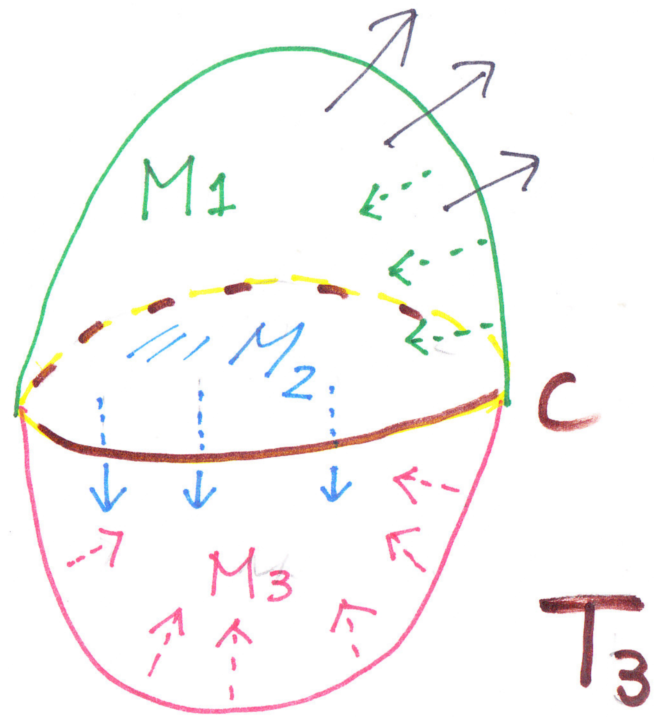
Proof) Why T_k is non-orientable? if $k \geq 3$

$M_i \cup M_j \cong S_0$: orientable

$i \neq j$

$i, j \in \{1, 2, 3\}$





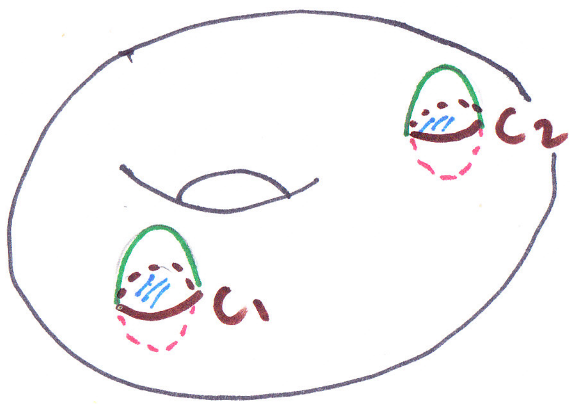
non orientable

More space exploration

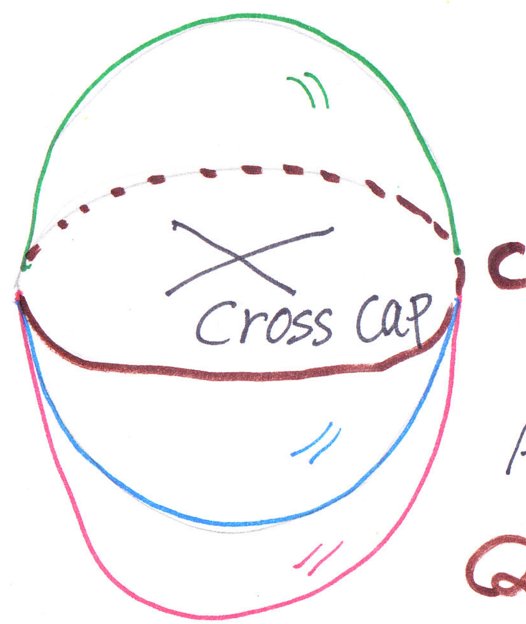
generalize quasi-surfaces

The event horizon(s) c_i can be located anywhere.

ex.



Two standard **Quasi-surfaces** with S_1 (torus)



A standard **Quasi-surface** with N_1 (proj. plane)

Now do graph theory

we want to embed a graph G

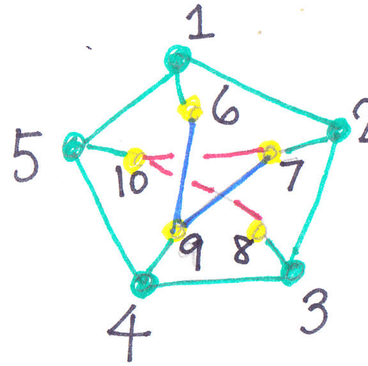
into T_k.

We consider

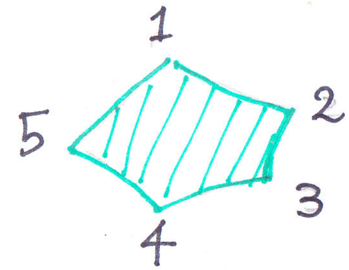
2-cell emb $G \sqsubset T_k$.

ex. a non-2-cell emb

$G \subset T_3$



Why 2-cell? ① traditional ② another reason to use 2-cell emb.

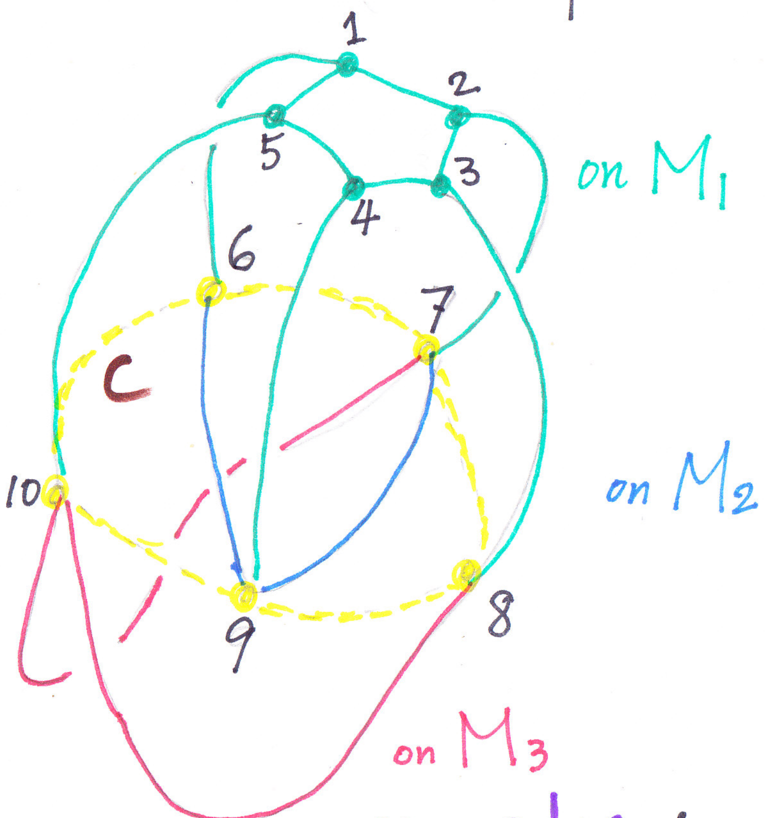


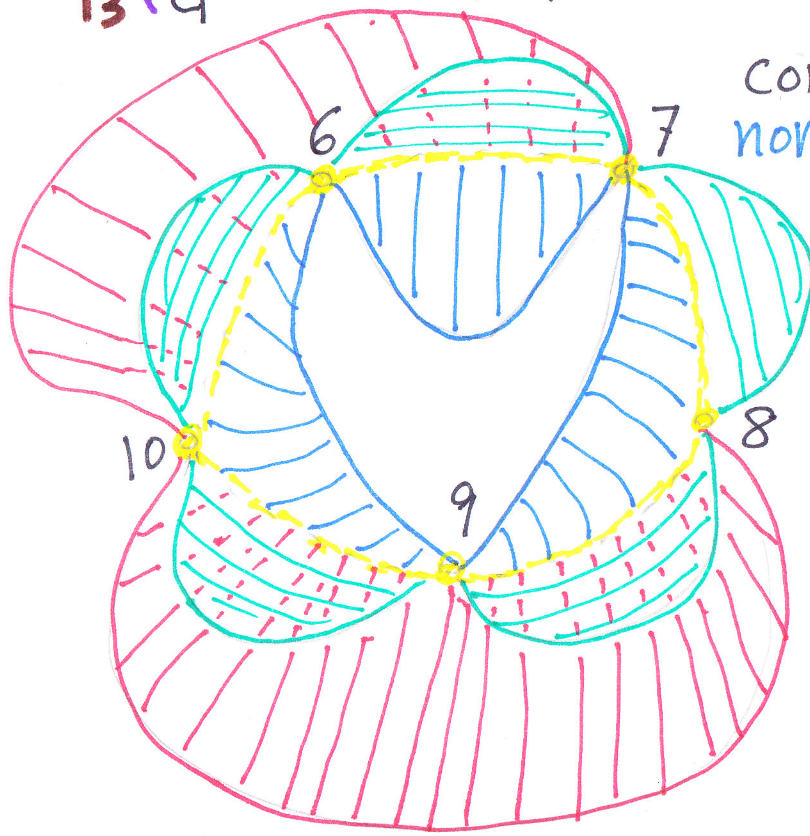
a 2-cell

Two faces in $T_3 \setminus G$

+

a connected non-2-cell





no edge in C

Consequences of

$\exists$ a 2-cell emb $\varphi : G \hookrightarrow T_k$

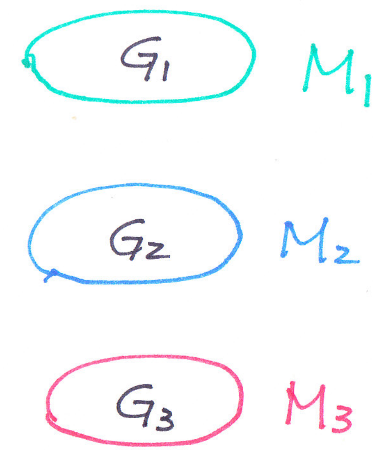
$\Rightarrow$ 1. G is connected

2. $\varphi^{-1}(c)$ is a cycle in G

where c is the ev. hor. of T_k $(k \geq 3)$

★ no vacancy in c

$\exists ?\ \mathcal{E}(T_k)$ yes!



Th. $\forall G \sqsubset T_k \quad \mathcal{E}(G) = k.$

Pf) $T_k = \bigcup_{i=1}^{k} M_i$ with $\partial M_i = c$

$G_i := G \cap M_i$; then $\mathcal{E}(G_i) = \mathcal{E}(S_0) \underbrace{-1}_{\text{a face}} = 1$

$$\mathcal{E}(G) = \sum_{i=1}^{k} \mathcal{E}(G_i) + \underbrace{(k-1)n_c - (k-1)m_c}_{0}$$

$$= \boxed{k}$$

$n_c := |V(G) \cap c|$

$m_c := |E(G) \cap c|$

$n_c = m_c$ b/c $G \cap c$ is a cycle

$\square$

Next, consider $\chi(G)$ if $G \sqsubset T_k$.

To prove a chromatic number theorem, finding an upper bound of min deg $\delta(G)$ is a key.

Usually, Euler formula $\overset{※}{\rightarrow}$ edge max. formula

$\rightarrow \delta(G) \rightarrow \chi(G)$

※ uses Jordan Curve Th.

Not in T_k, $k \geq 3$

Euler formula $\not\rightarrow \chi(G)$

$M_i \cong$ 2-disk

deduces edge max. formula in T_k

(no Euler formula used)

Lemma 1 $G \sqsubset T_k$

$$\alpha_1 \leq 3(\alpha_0 - k) + (k-2)n_c$$

where $n_c = |V(G) \cap c|$

$$\alpha_1 = |E(G)|$$

$$\alpha_0 = |V(G)|$$

$V(G) \subseteq C$ in T_k $\Leftrightarrow$ G has a k-page book emb.

or $bt(G) \leq k$

$\alpha_1 \leq 3(\alpha_0 - k) + (k-2)n_c$ with $n_c = \alpha_0$

$\alpha_1 \leq (k-2+3)\alpha_0 - 3k$

$$\delta \leq 2(k+1) - \frac{6k}{\alpha_0}$$

$$\delta \leq 2k+1$$

$bt(G) \leq k$

$\Rightarrow \chi(G) \leq 2k+2$ --- confirm

Bernhart & Kainen [1979]

book thickness

<u>Conj</u>. If $2 \leq bt(G)$,

then $\chi(G) \leq 2k$.

note: If $bt(G) < 2$, then

G is outerplanar.

$\chi(\text{outerplanar}) \leq 3$

sharp

ex. $G = K_3$

$bt(K_3) = 1$

$\chi(K_3) = 3$

Lemma 2

If $2n_c \le \alpha_0 = |V(G)|$, then $\delta(G) \le 3+k$;
$\chi(G) \le 4+k$.

Lemma 3

If $2kn_c \le \alpha_0$ then $\delta \le 6$;
$\chi(G) \le 7$.

Conj (G. Turner)

If $G \sqsubset T_k$ and

$\forall i \in \{1, 2, \cdots, k\}$ $\mathrm{int}(M_i) \cap V(G) \neq \phi$,

then $\chi(G) \leq 6$.

(idea behind: Apply 4CT to each membrane M_i)

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Thank you!