On Spanning Trees with few Branch Vertices

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Emory University
Joint work with Ron Gould

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Spanning trees

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In the next few slides, spanning trees are more "desirable" the fewer branch vertices they have.

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In the next few slides, spanning trees are more "desirable" the fewer branch vertices they have.

- What conditions might lead to a desirable spanning tree?

One possible condition: independent sets

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And of course...it helps if the graph is claw-free.

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What are the best possible parameters?

Theorem (Matsuda, Ozeki, Yamashita 2012)

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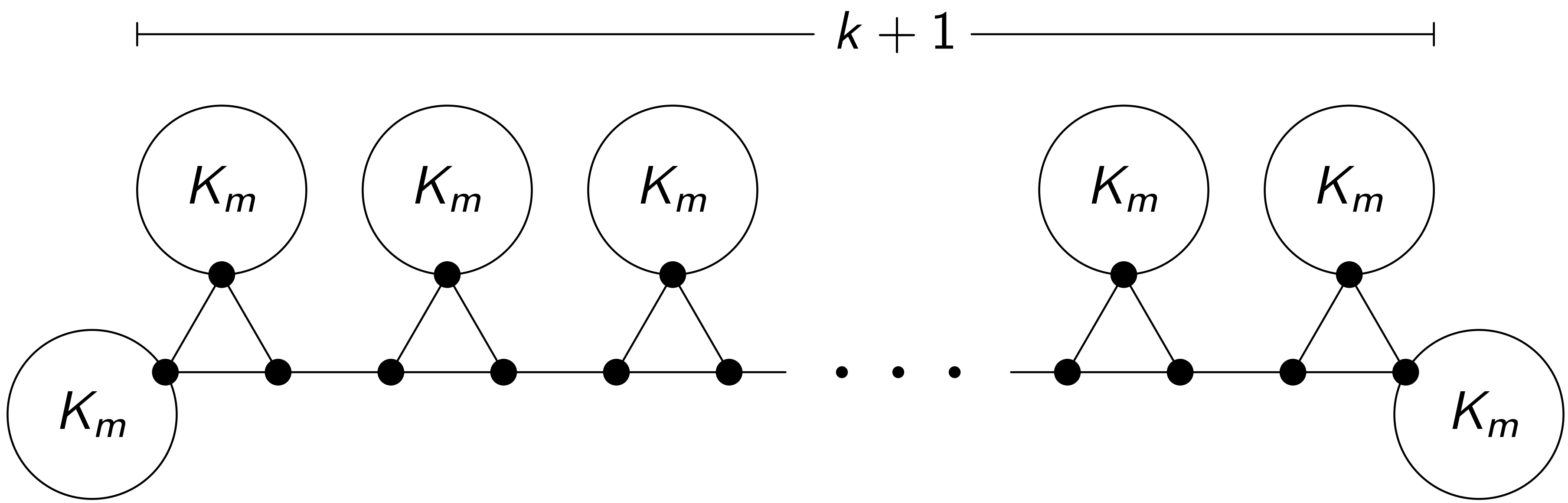
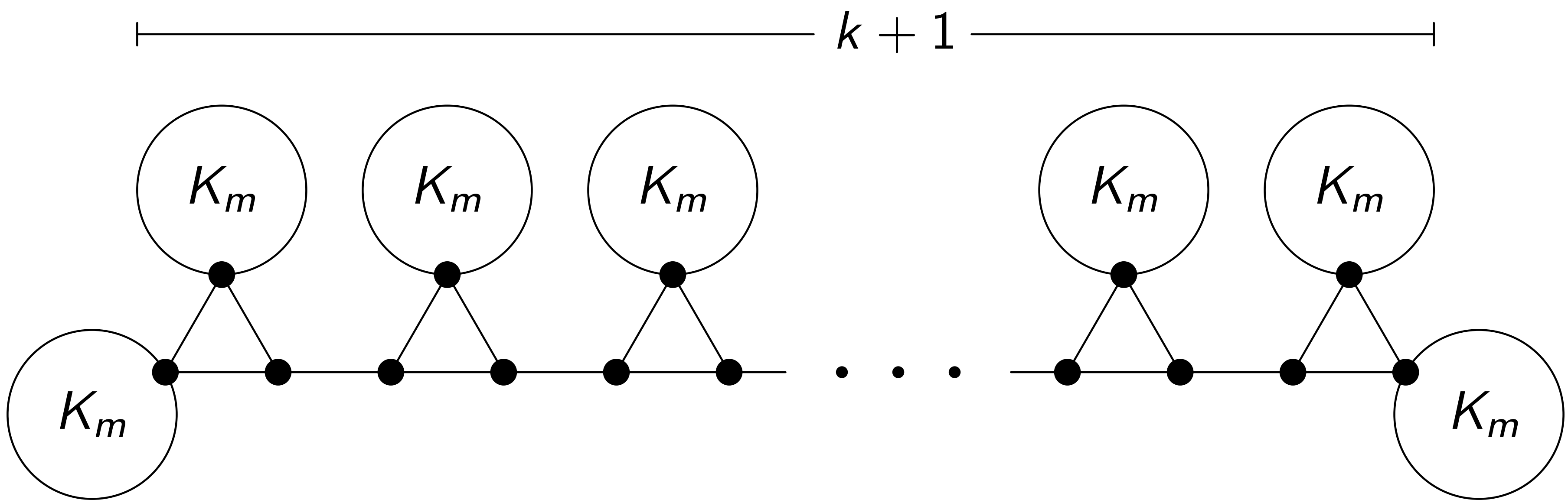
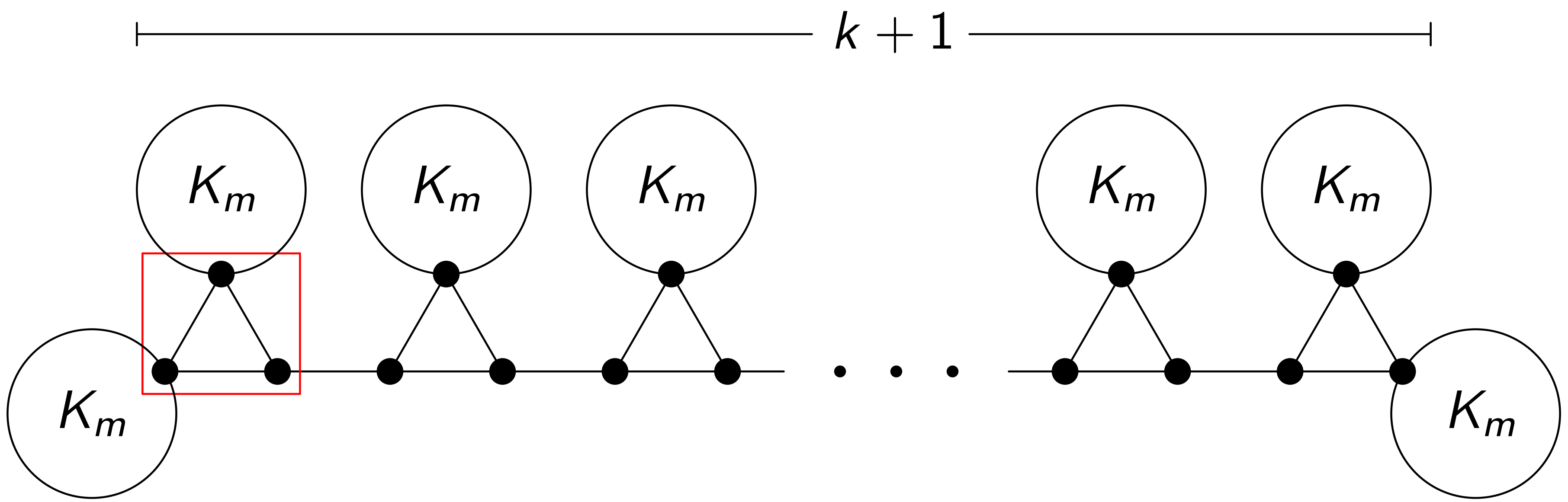


$k+1$
K_m
K_m
K_m
K_m
K_m
K_m
K_m



Connected and claw-free



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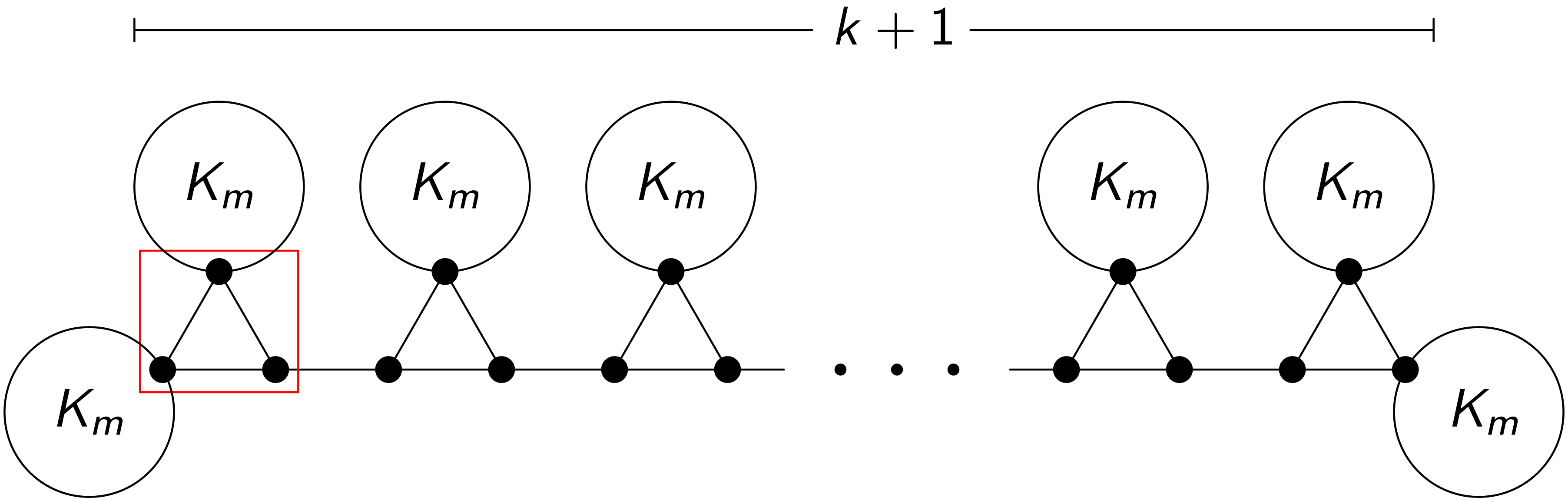

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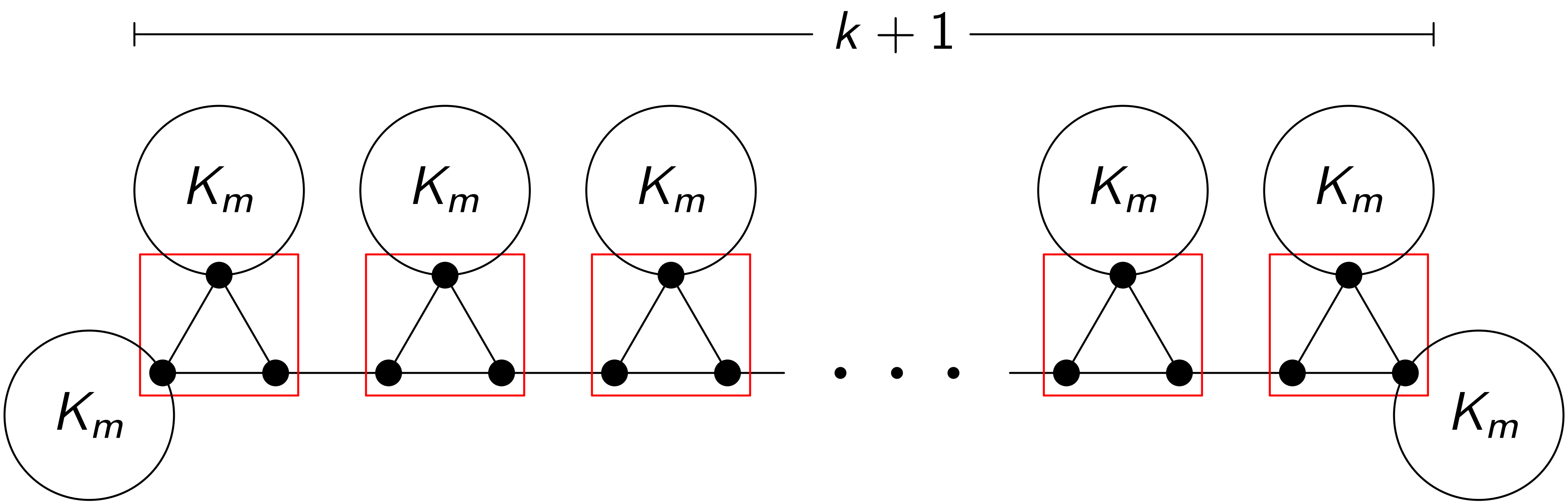
Any spanning tree must have a branch vertex in this triangle...



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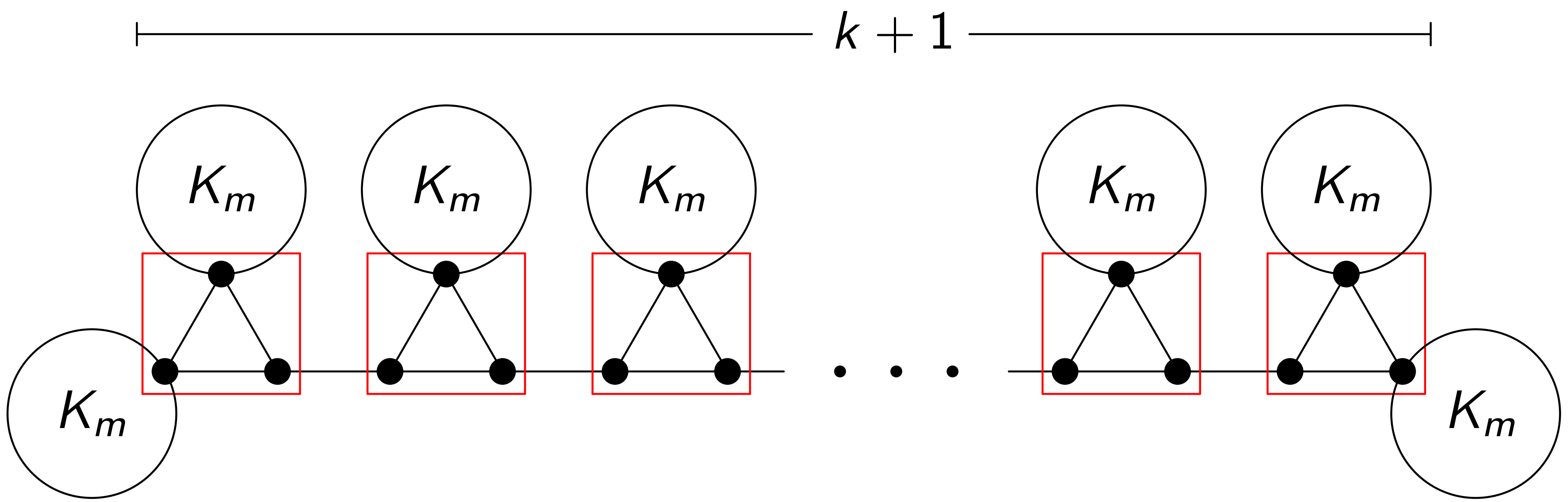
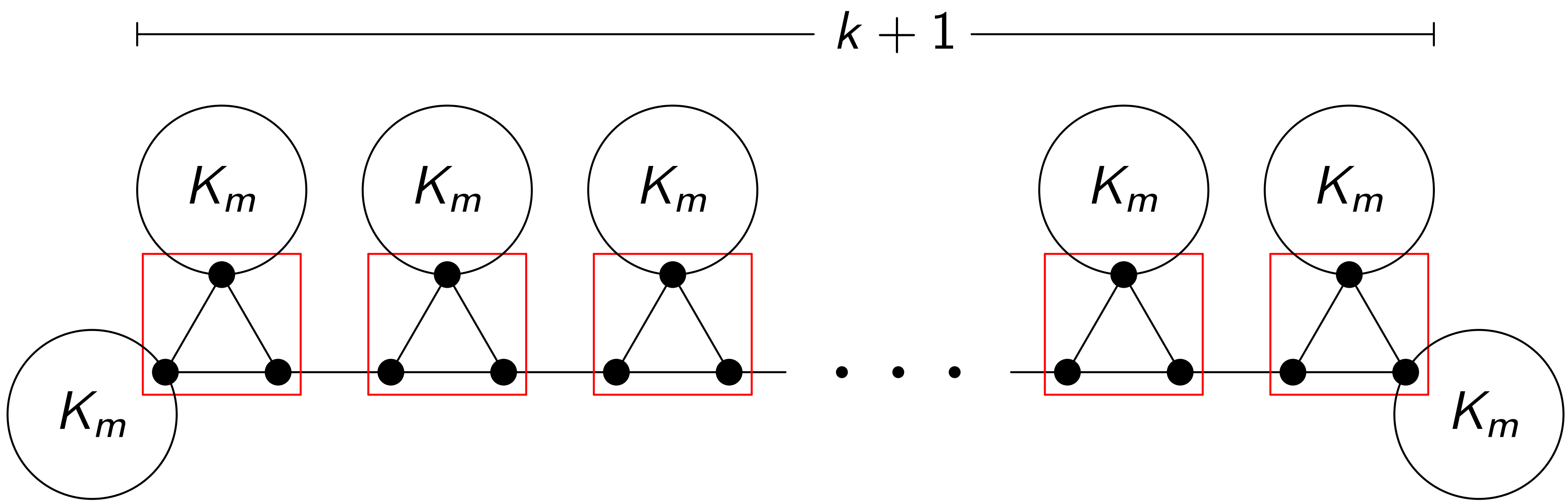


Connected and claw-free

Any spanning tree must have a branch vertex in this triangle...

...and each of these others...

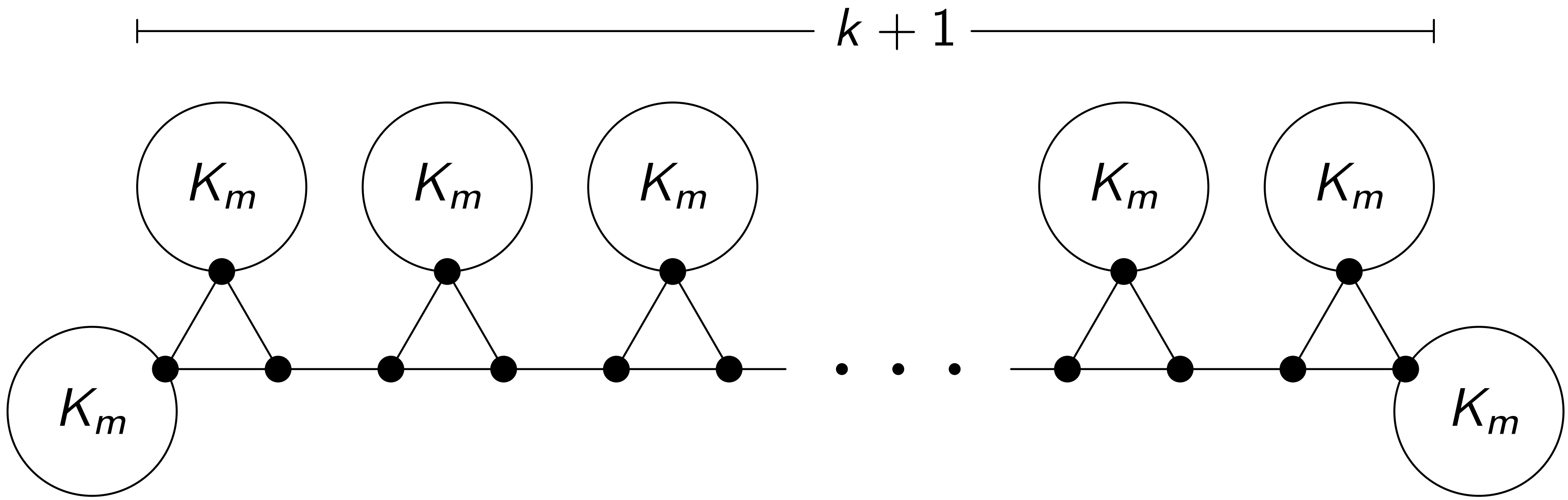


Connected and claw-free

Any spanning tree must have a branch vertex in this triangle...

...and each of these others...

...for a minimum of $k+1$ branch vertices.


$k+1$
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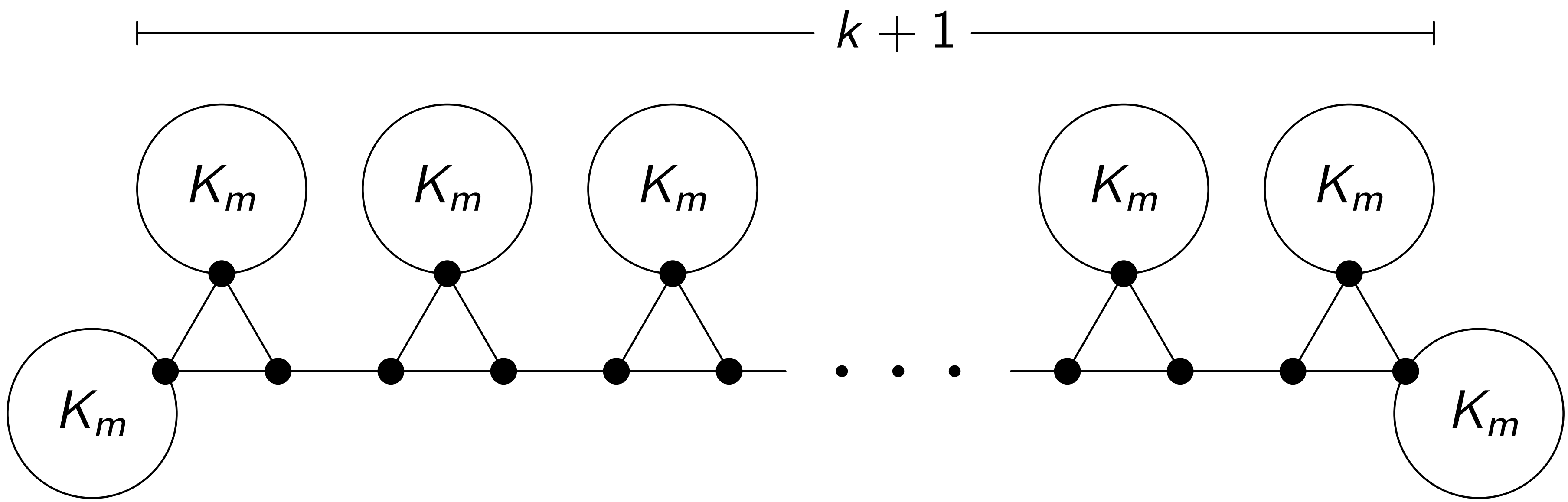
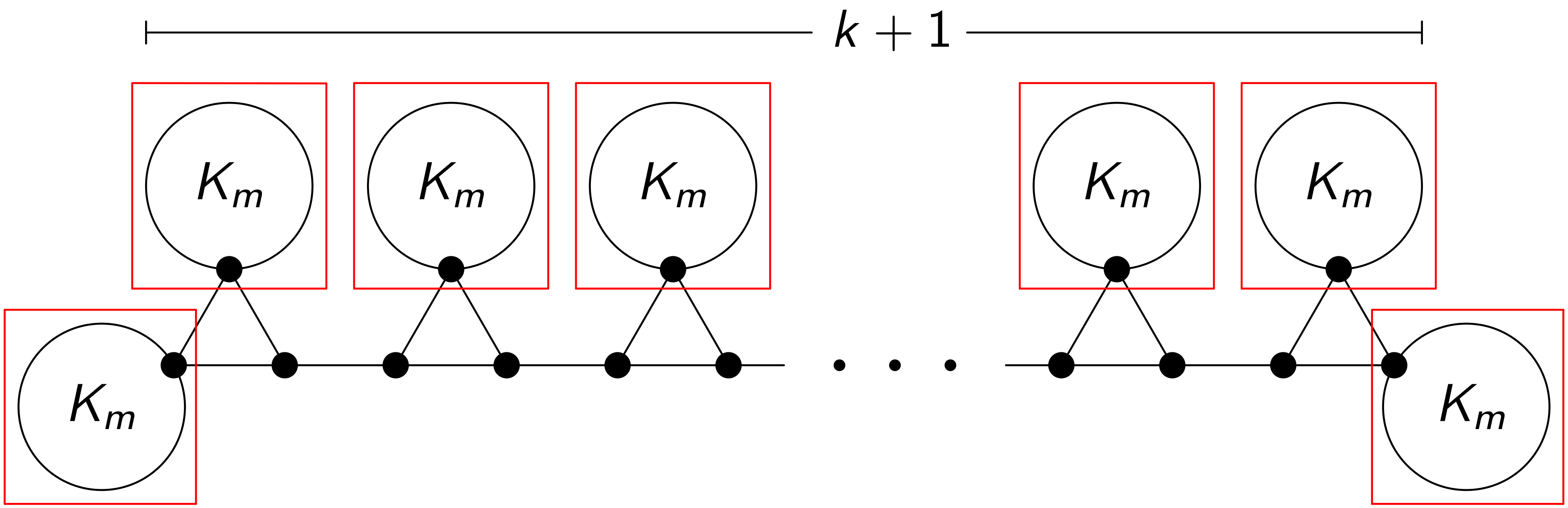


$$|V(G)| \quad =$$



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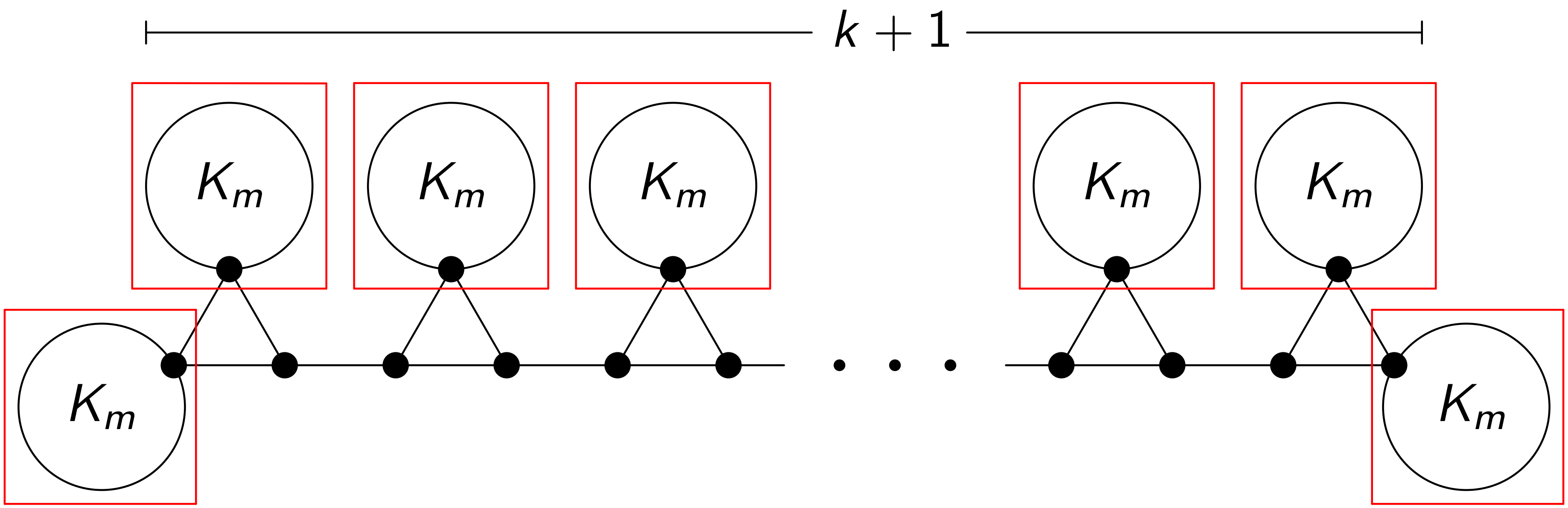
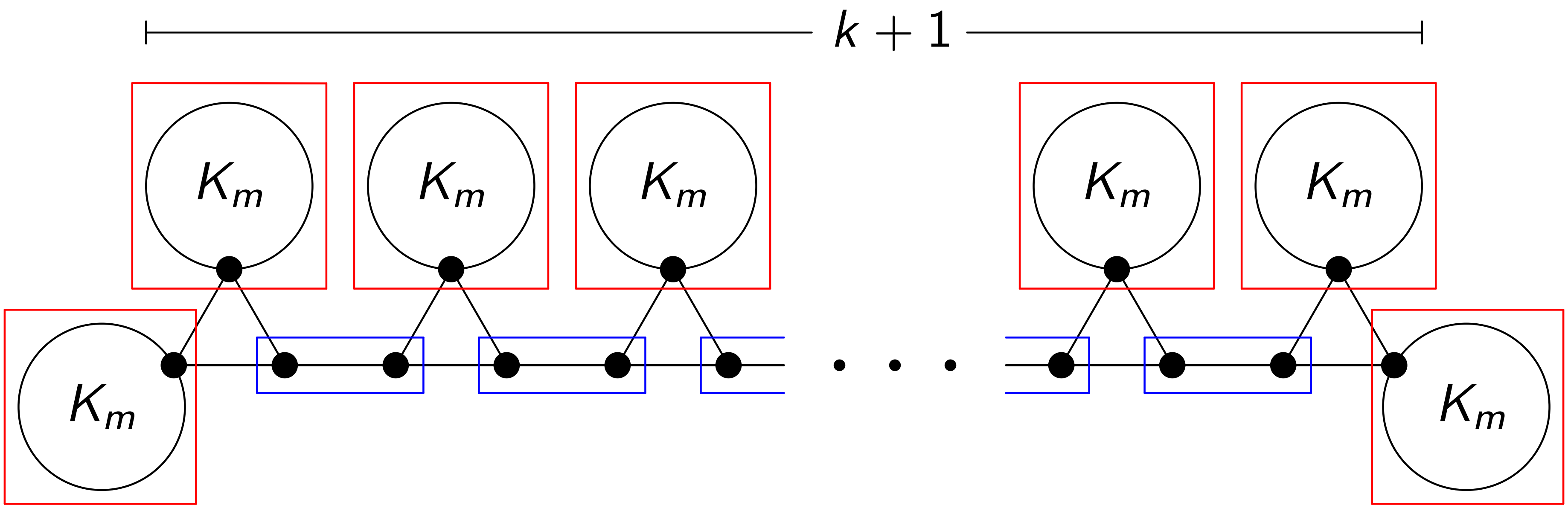


$$|V(G)| \quad = \quad m(k+3)$$



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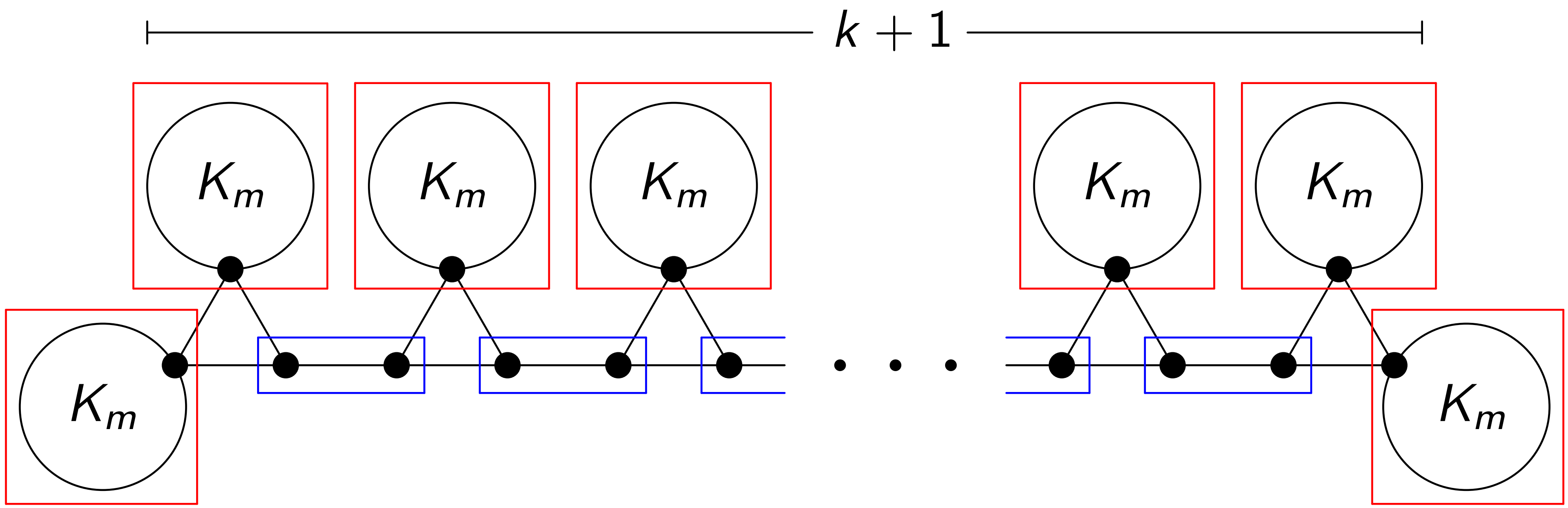
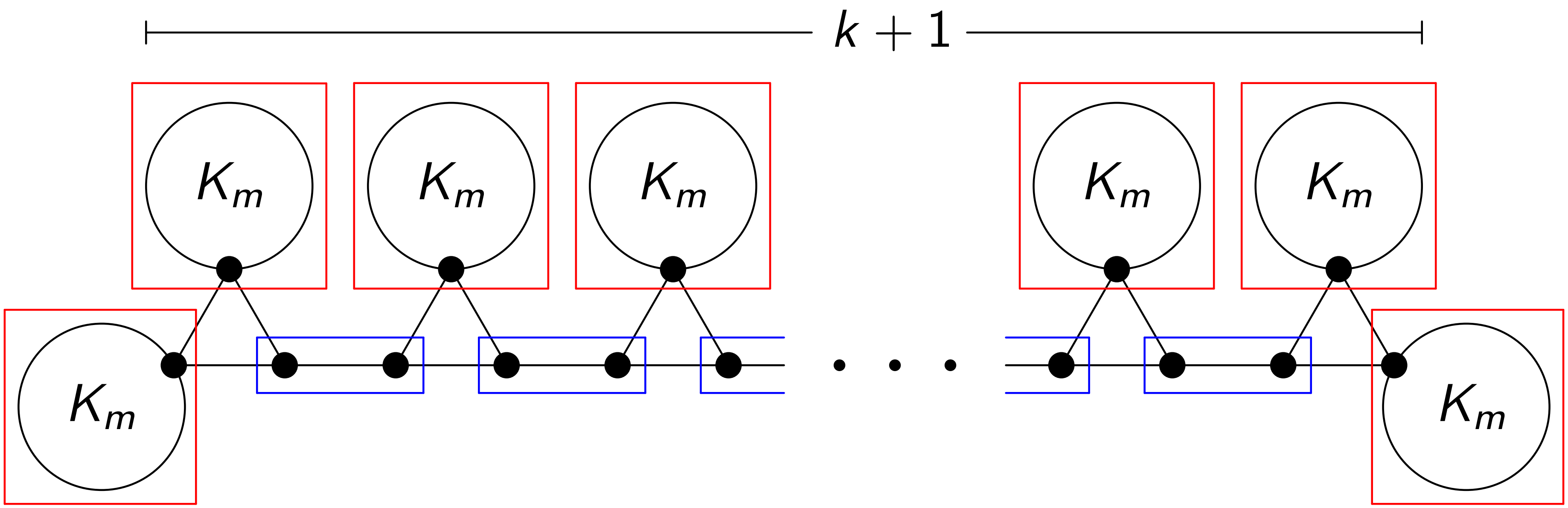
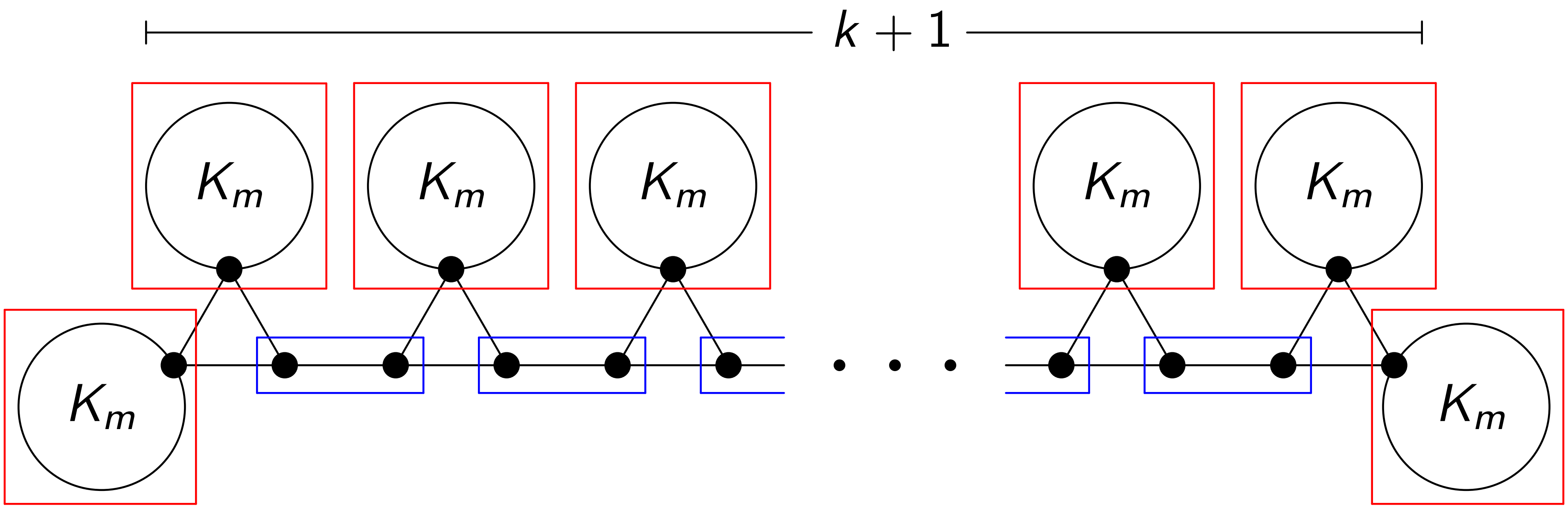


$$|V(G)| \quad = \quad m(k+3) + 2k$$



$$|V(G)| \;=\; m(k+3) + 2k = mk + 3m + 2k$$



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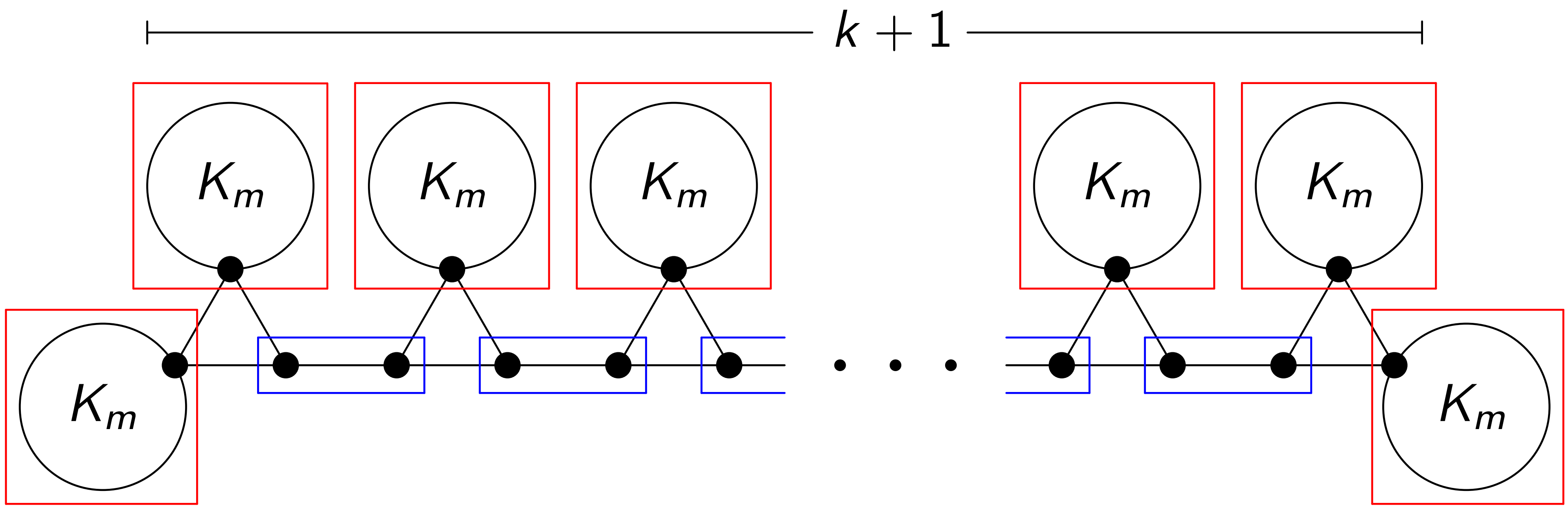

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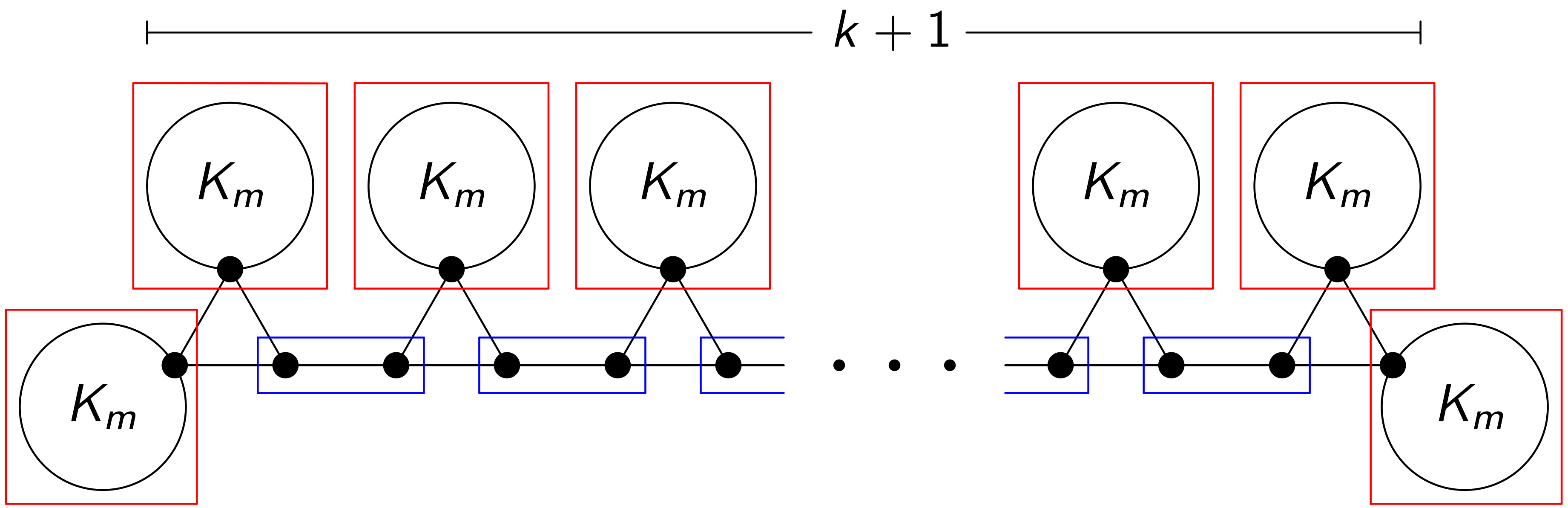

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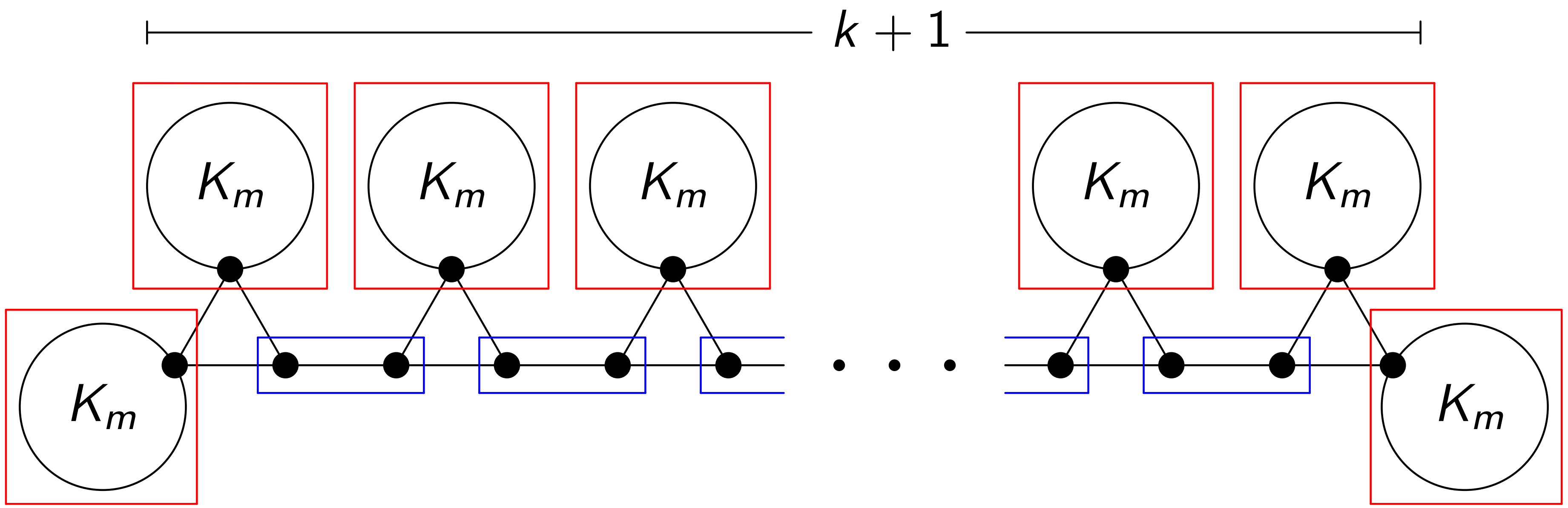
X



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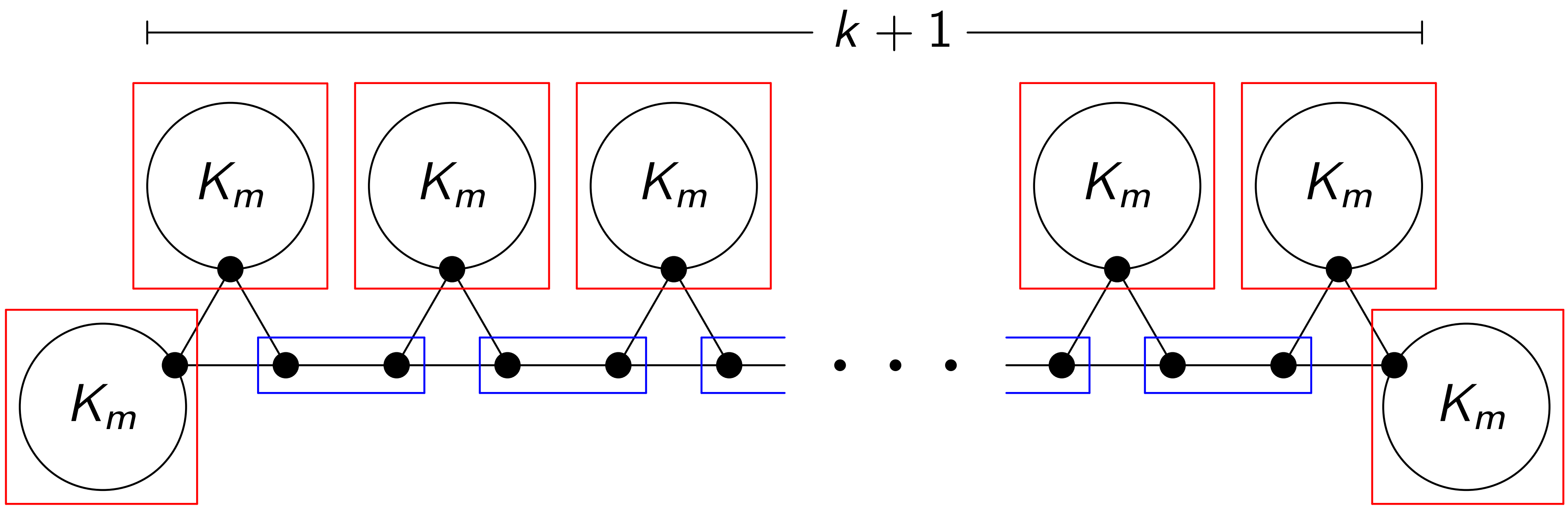

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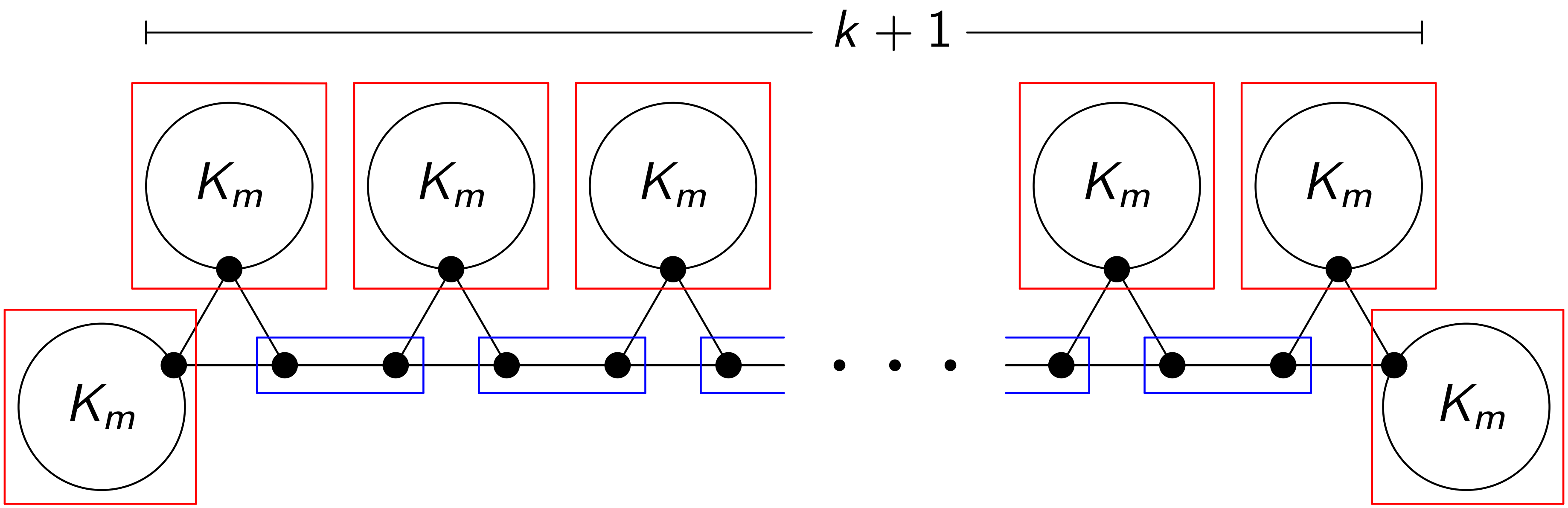

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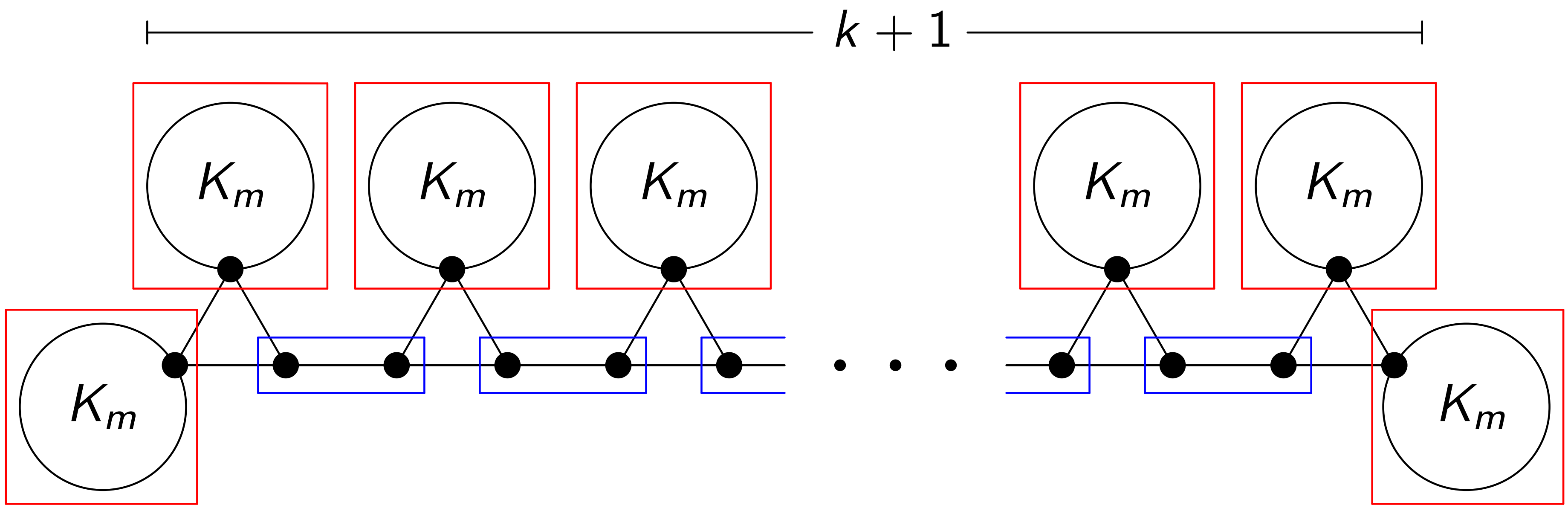

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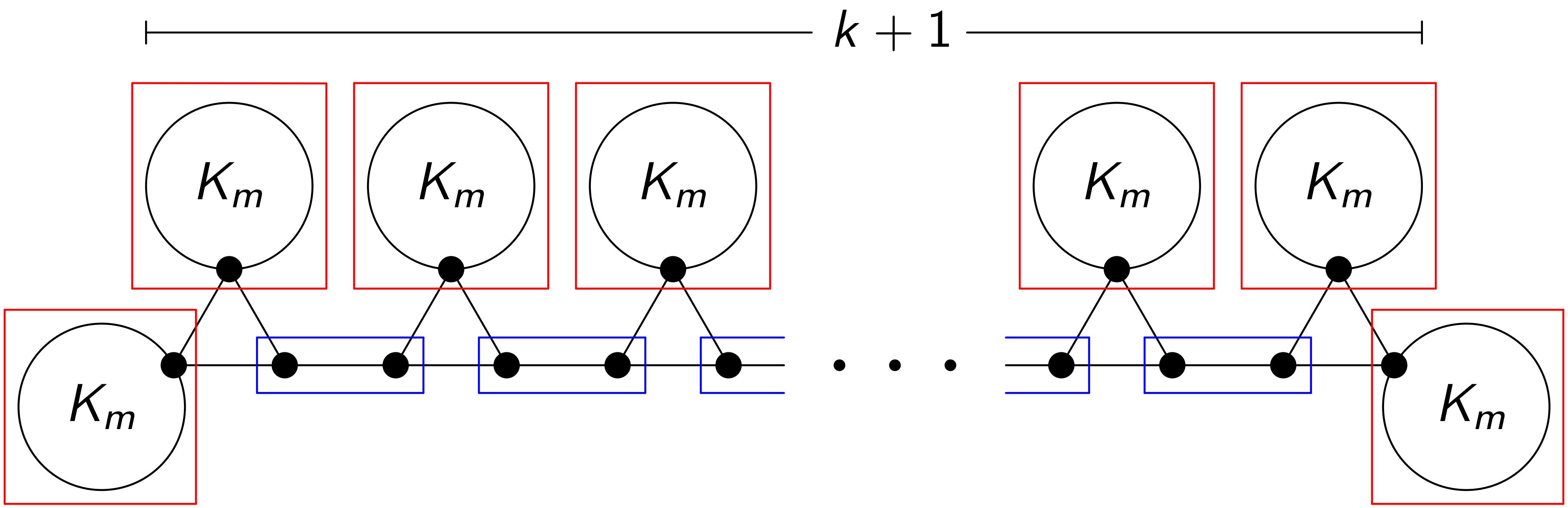

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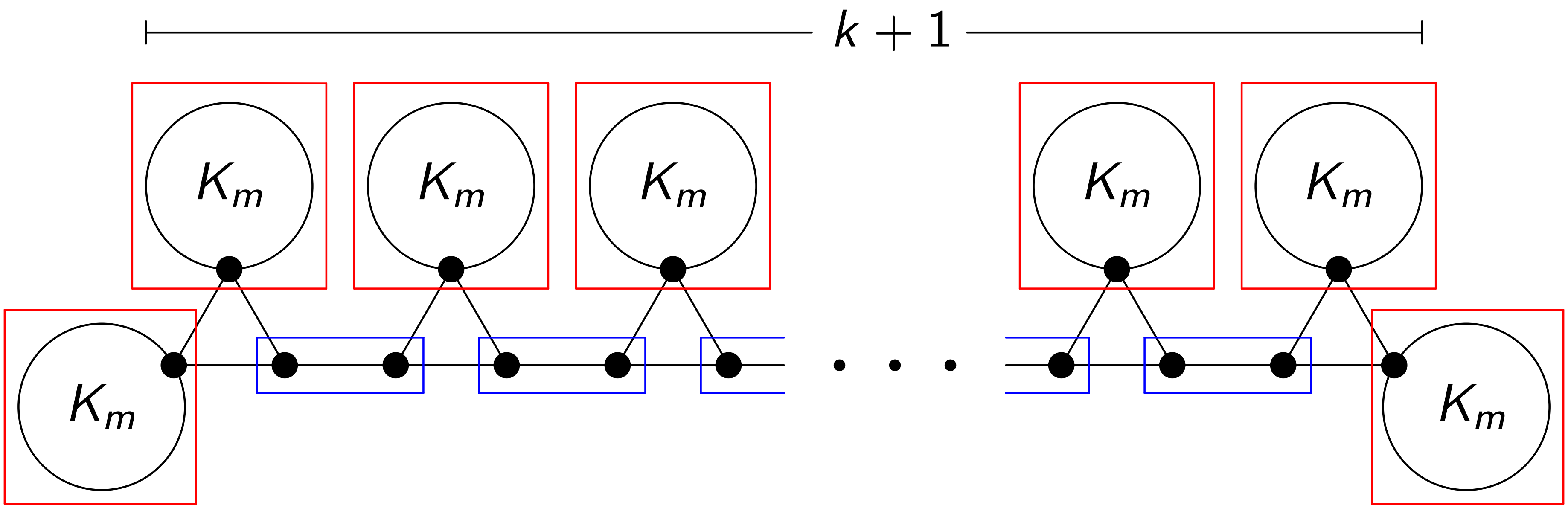

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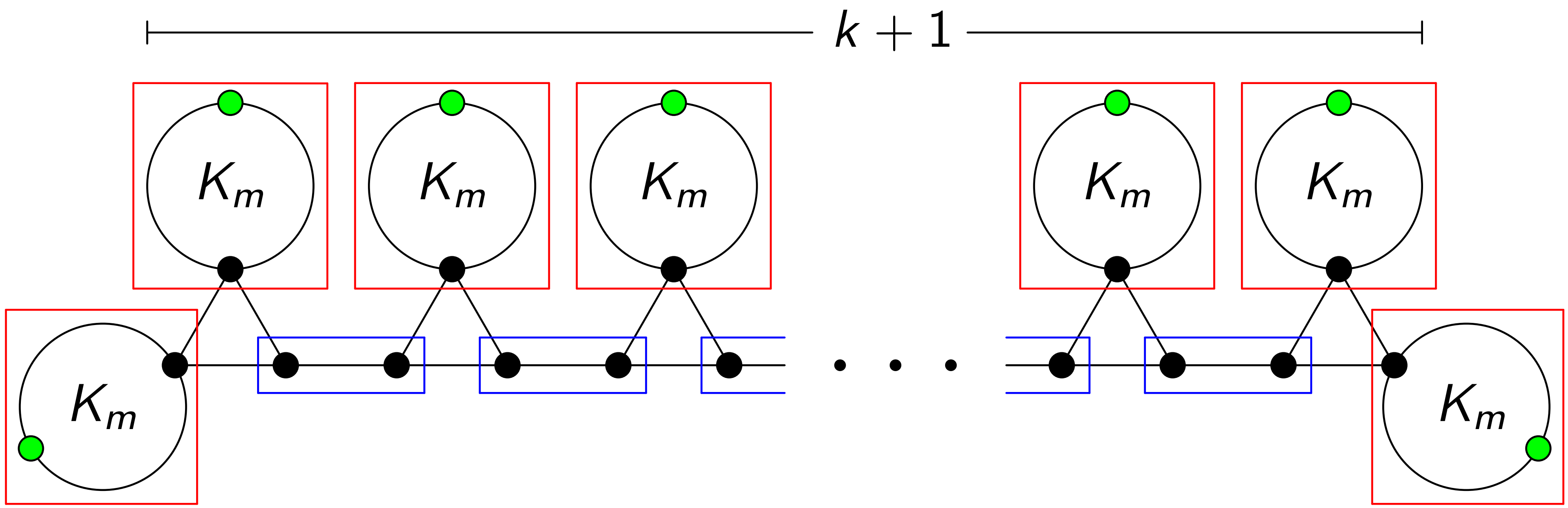

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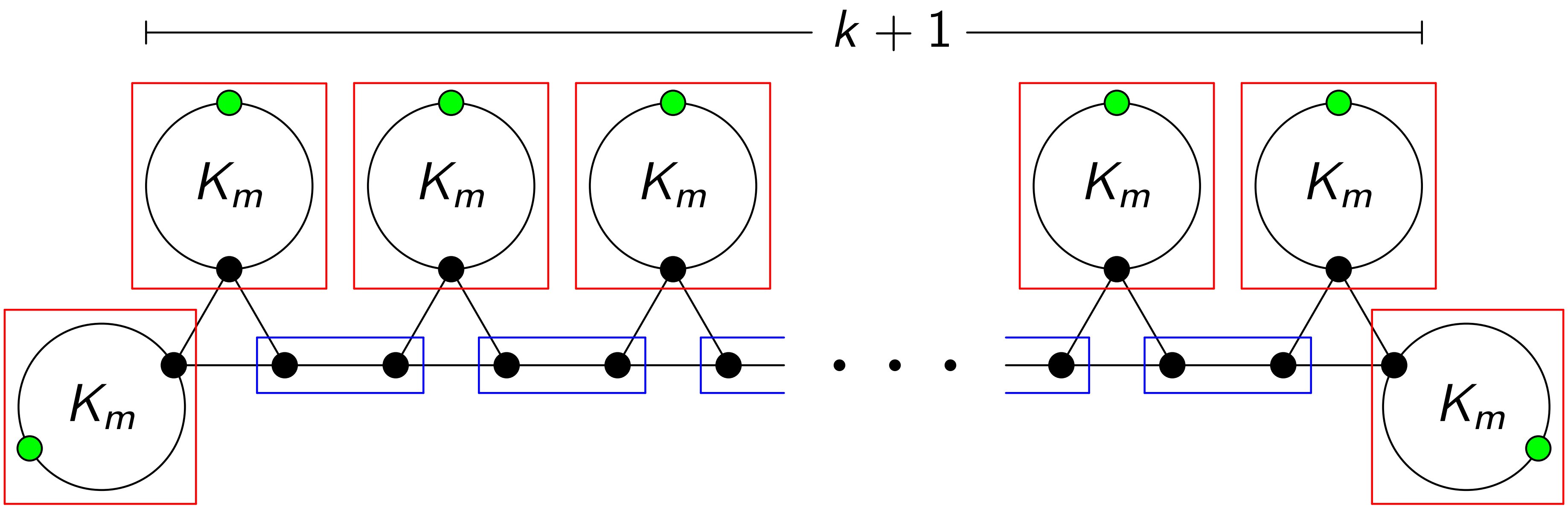

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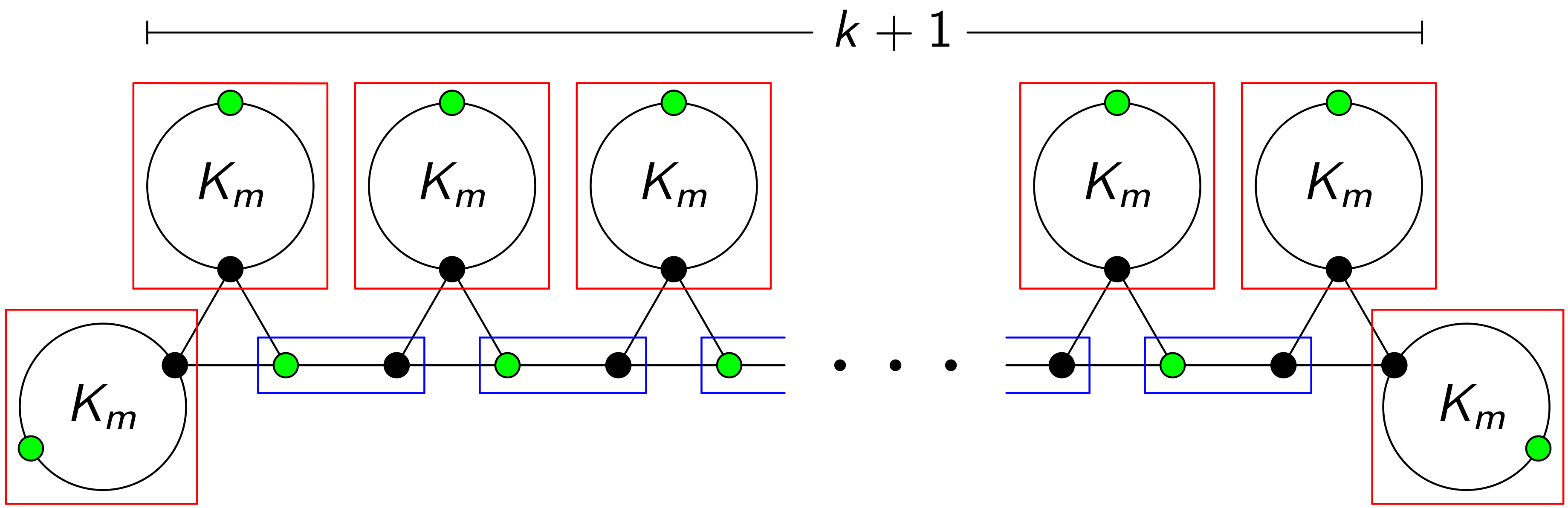

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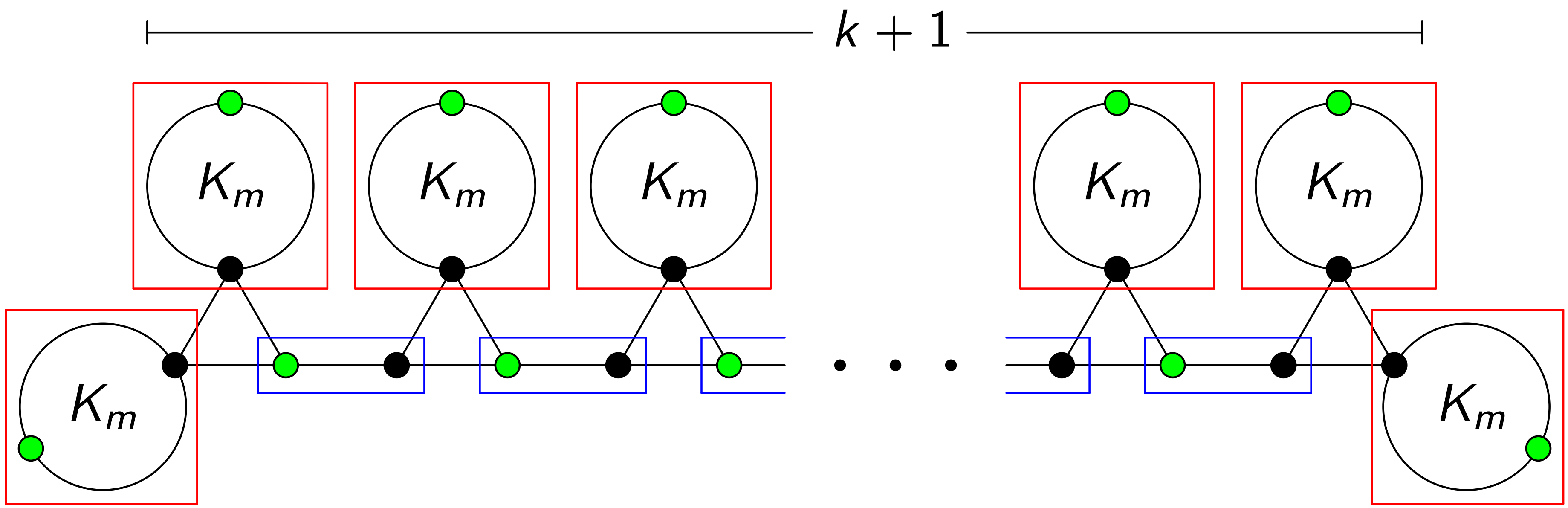

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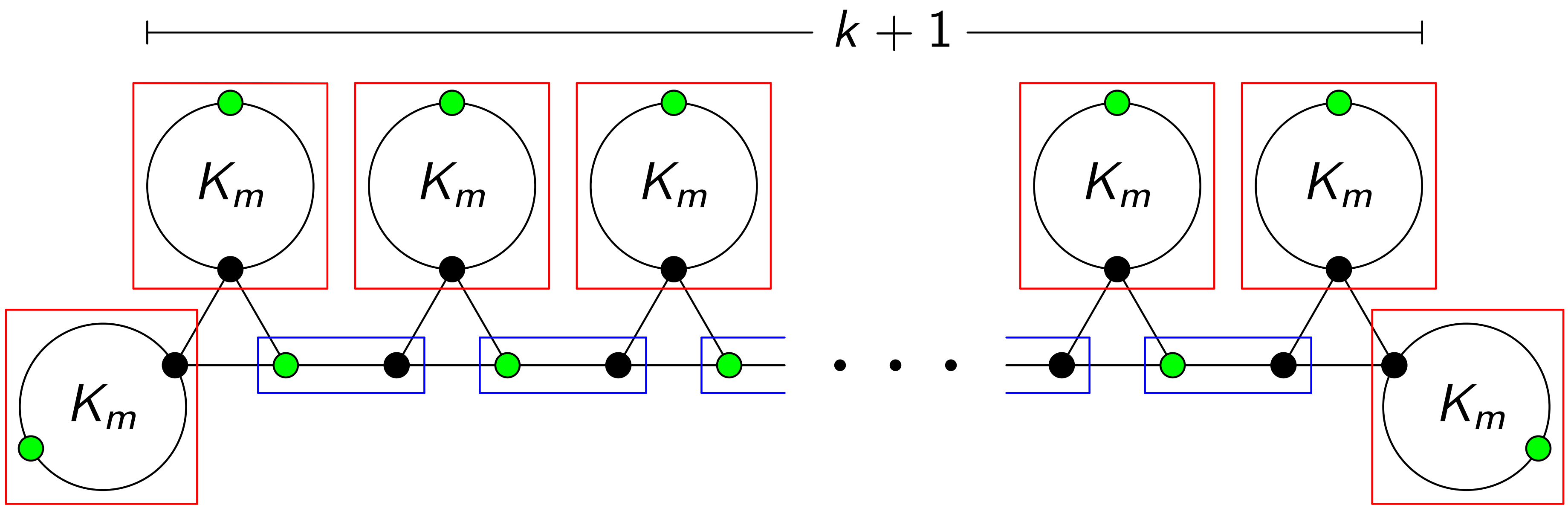

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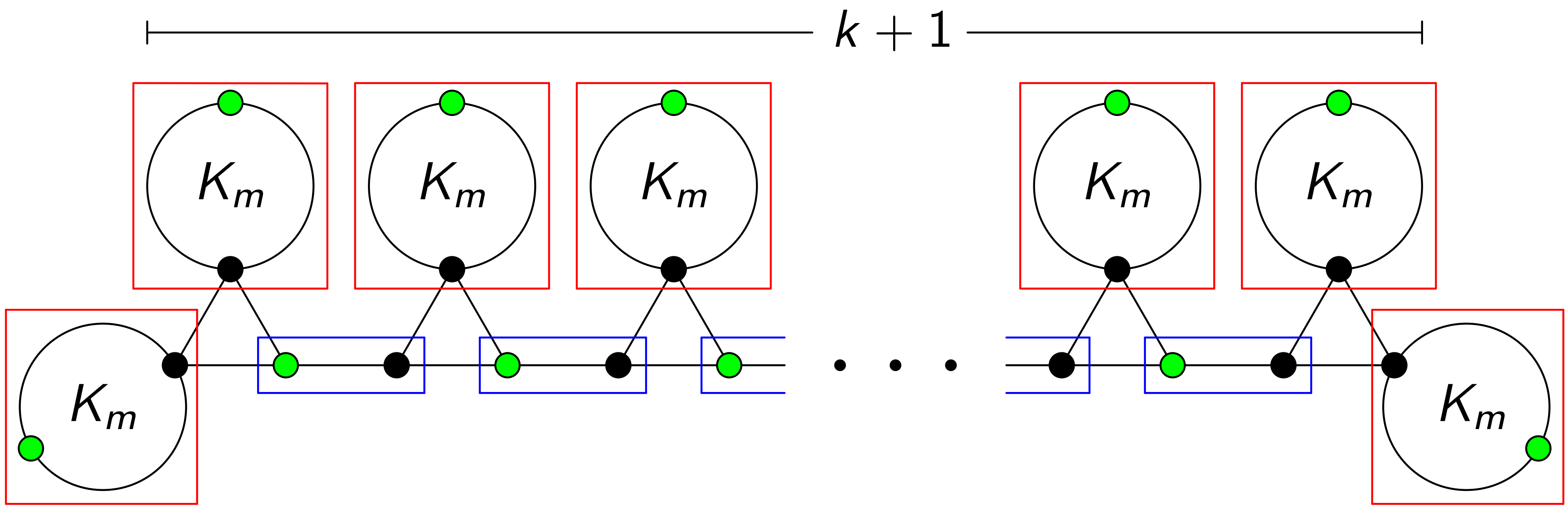

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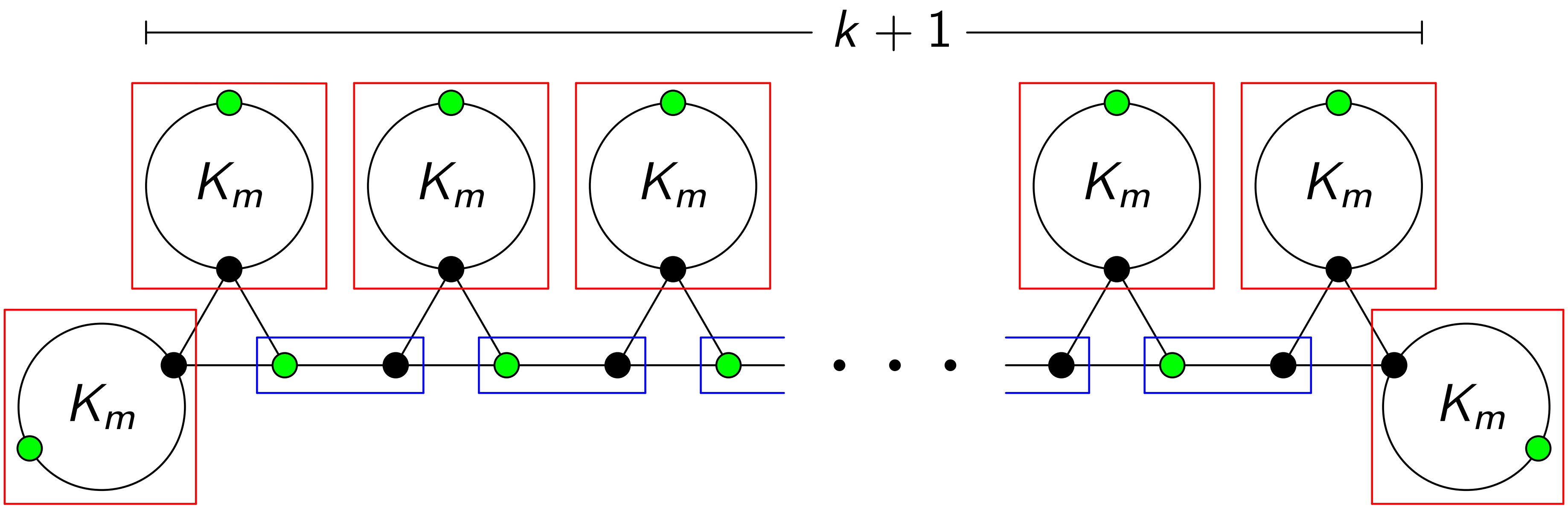

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Let k be a non-negative integer and let G be a connected claw-free graph. Then G contains *either* a spanning tree with at most k branch vertices, *or* an independent set of $2k+3$ vertices. *This is best possible.*

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Theorem (Kano, et. al. 2012)

Let k be a non-negative integer and let G be a connected claw-free graph. Then G contains either a spanning tree with at most $k + 2$ leaves, or an independent set of $k + 3$ vertices whose degrees add up to at most $|V(G)| - k - 3$.

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Corollary

Let G be a connected claw-free graph. Then G contains either a spanning tree with at most 2 leaves, or an independent set of 3 vertices with at most $|V(G)|-3$ outgoing edges.

Conjecture (Matsuda, Ozeki, Yamashita 2012)

Let k be a non-negative integer and let G be a connected claw-free graph. Then G contains *either* a spanning tree with at most k branch vertices, *or* an independent set of $2k+3$ vertices with at most $|V(G)|-3$ outgoing edges. *This is best possible.*

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- By the theorem of Kano et. al. above (with $k = 4$), G has a spanning tree with at most 6 leaves.

Among spanning trees with at most 6 leaves, choose a tree T such that:

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(T1) T has as few branch vertices as possible.

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How many different structures could T possibly have?

First case: T has only 5 leaves (the fewest possible):

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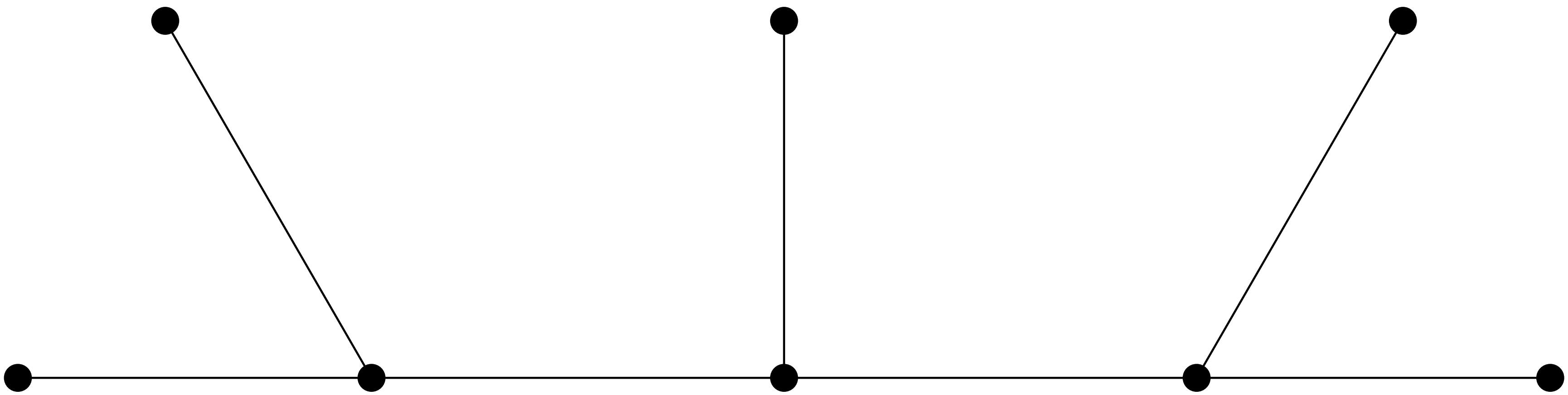

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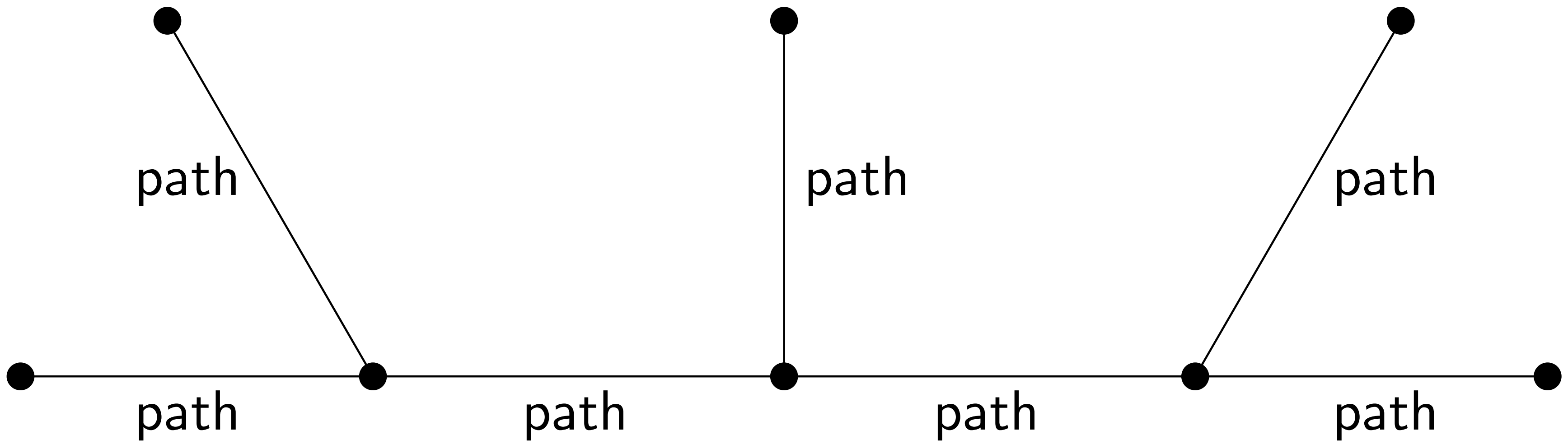

Second and third cases: T has 6 leaves, but only 3 branch vertices.

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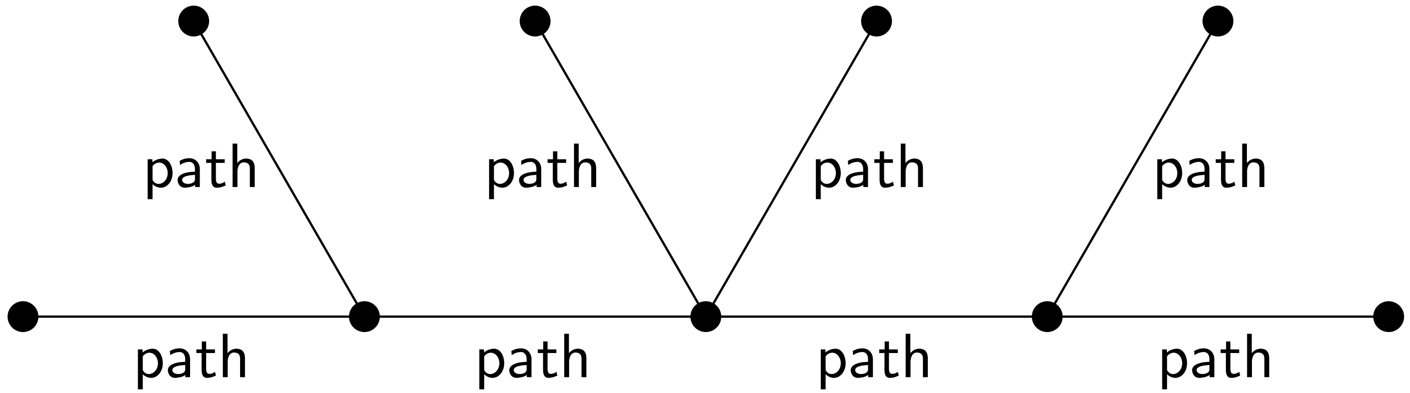

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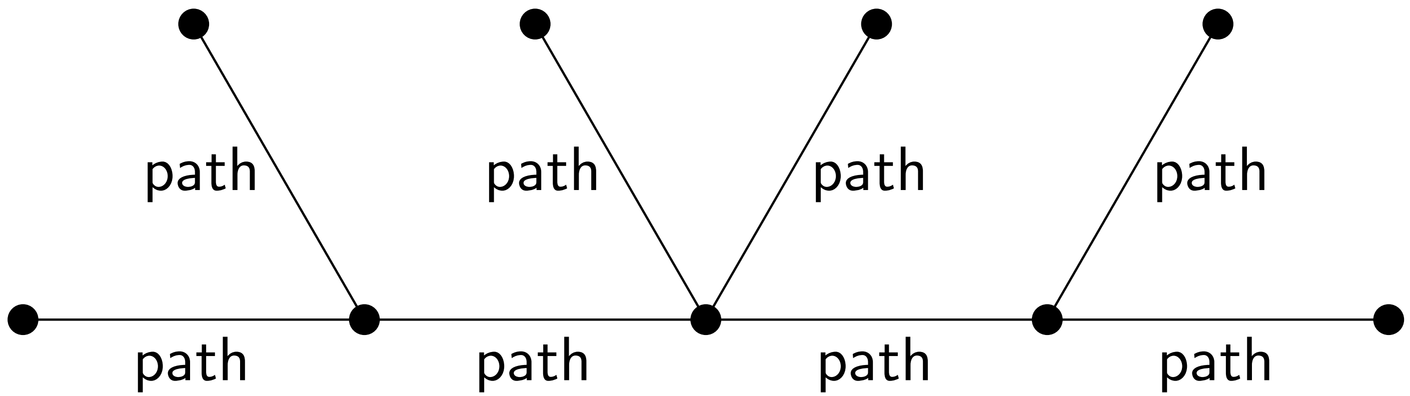

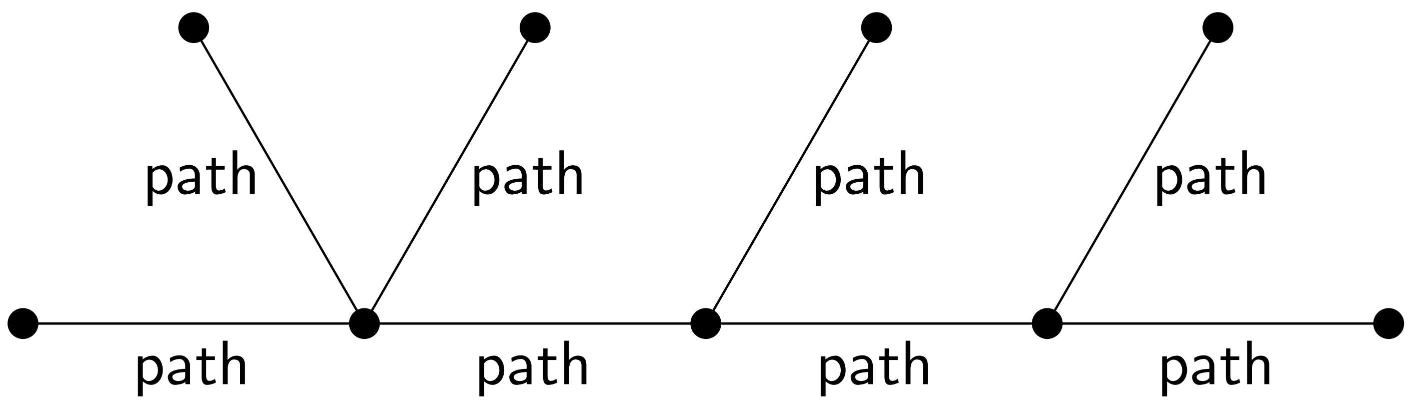

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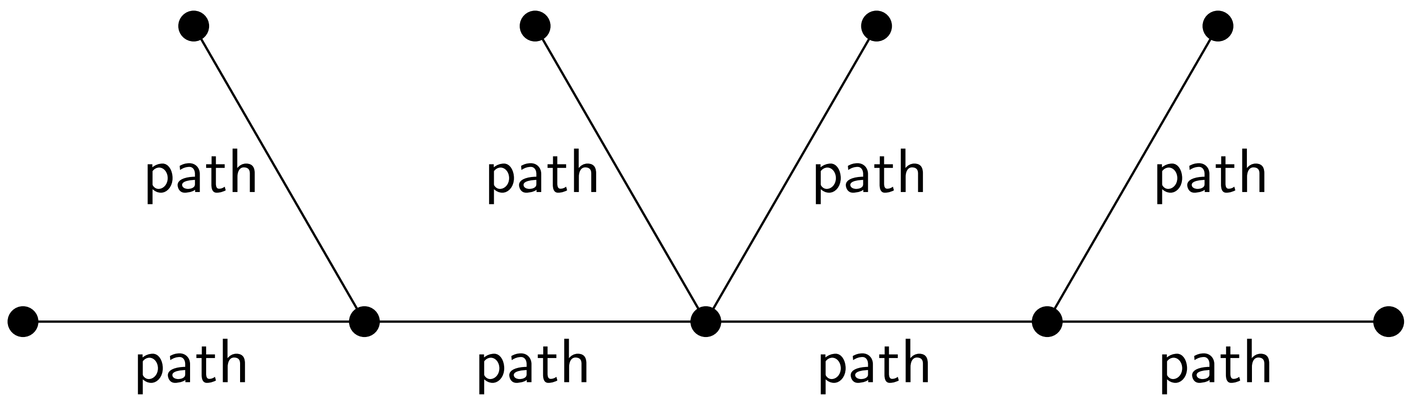

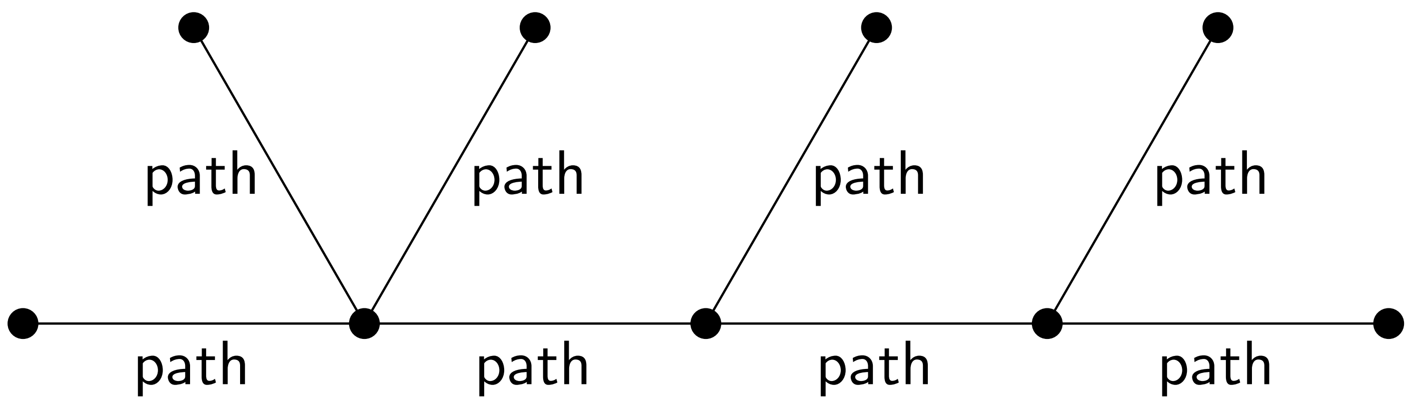

(T3) If choosing between trees of these two types, we always choose one of the first type.

Fourth and fifth cases: T has 4 branch vertices (and therefore 6 leaves)

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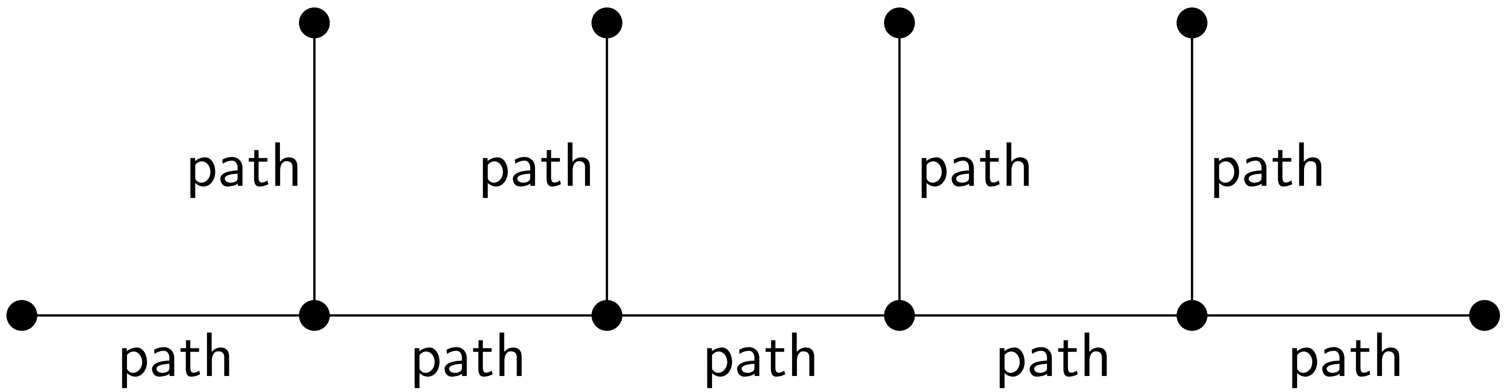

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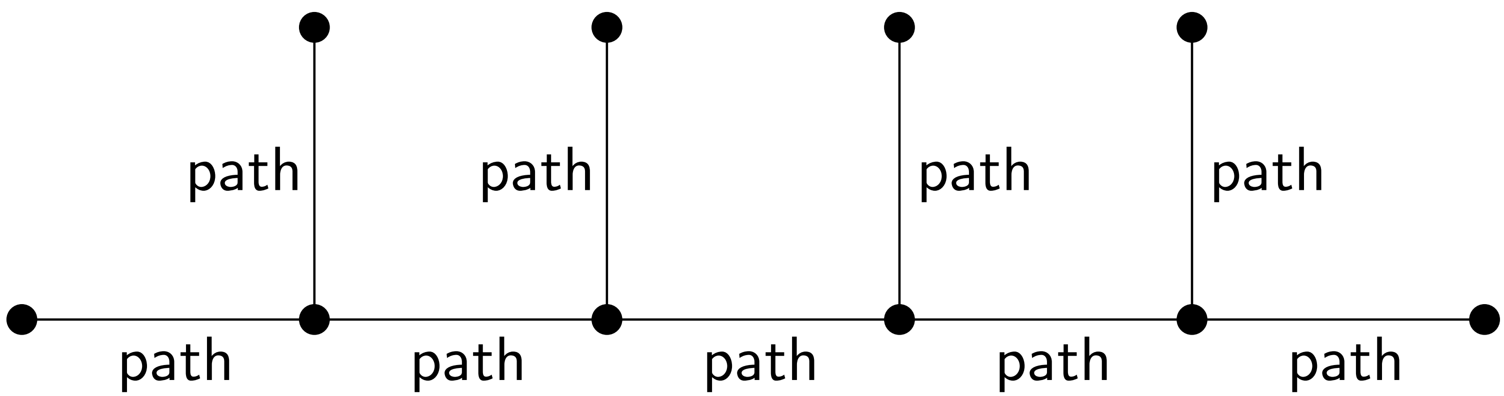

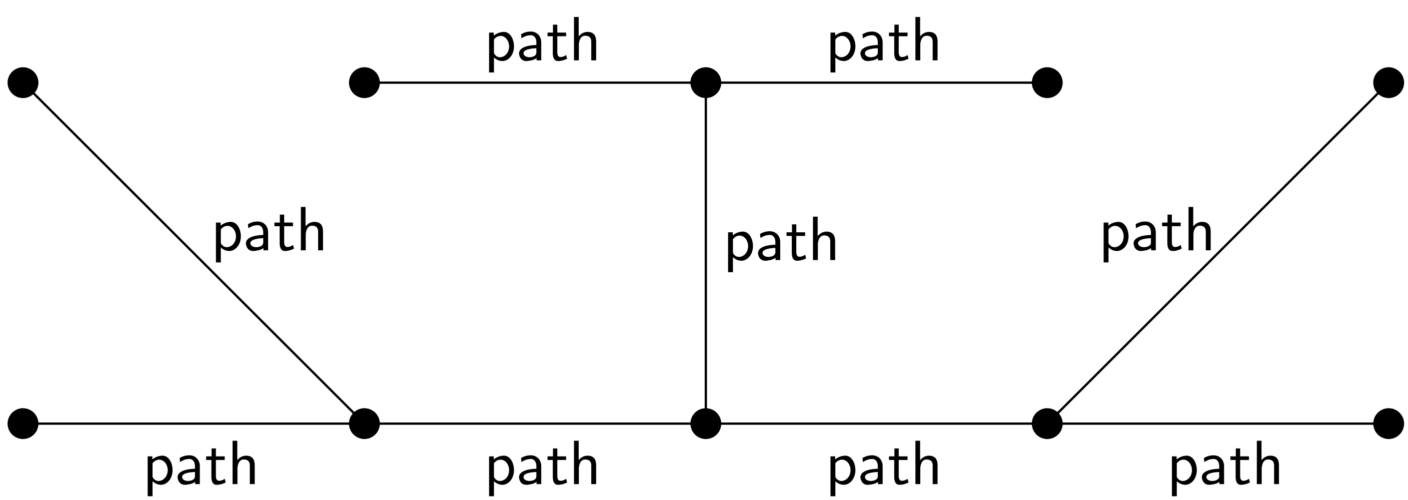

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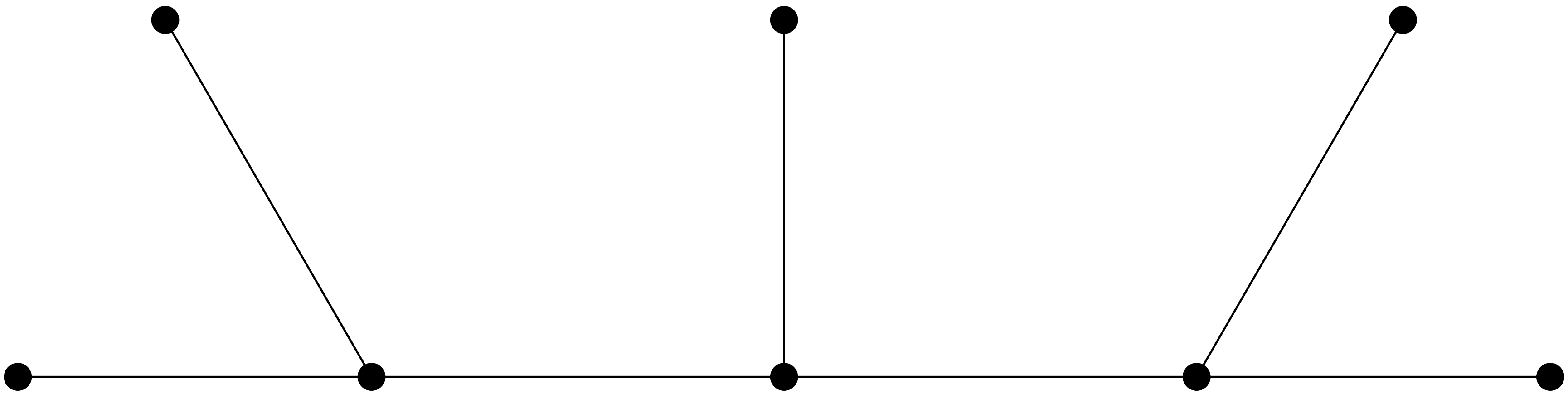
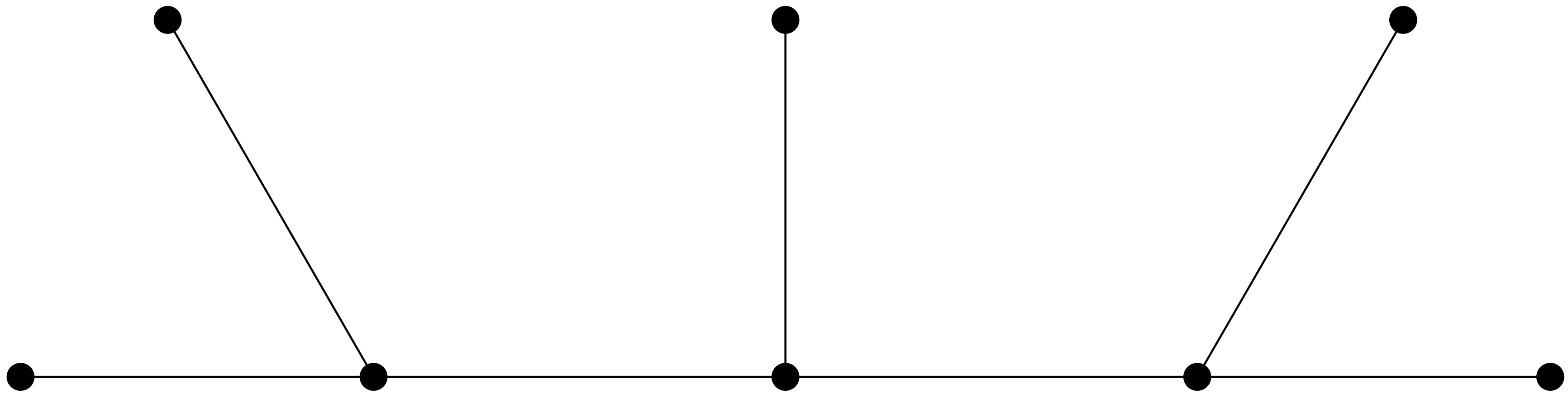


First case:

- Choose independent set X



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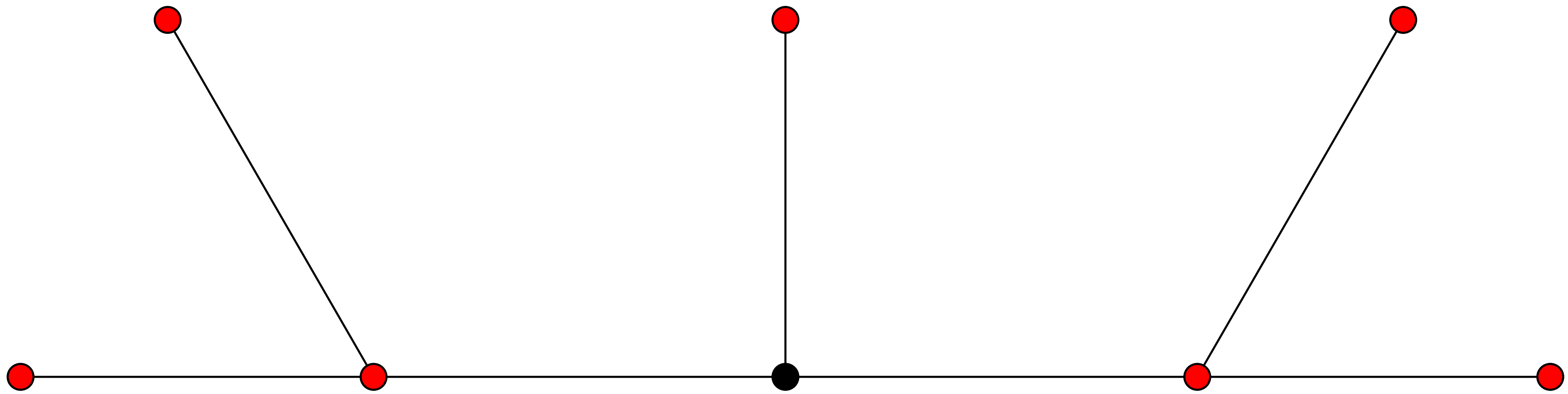

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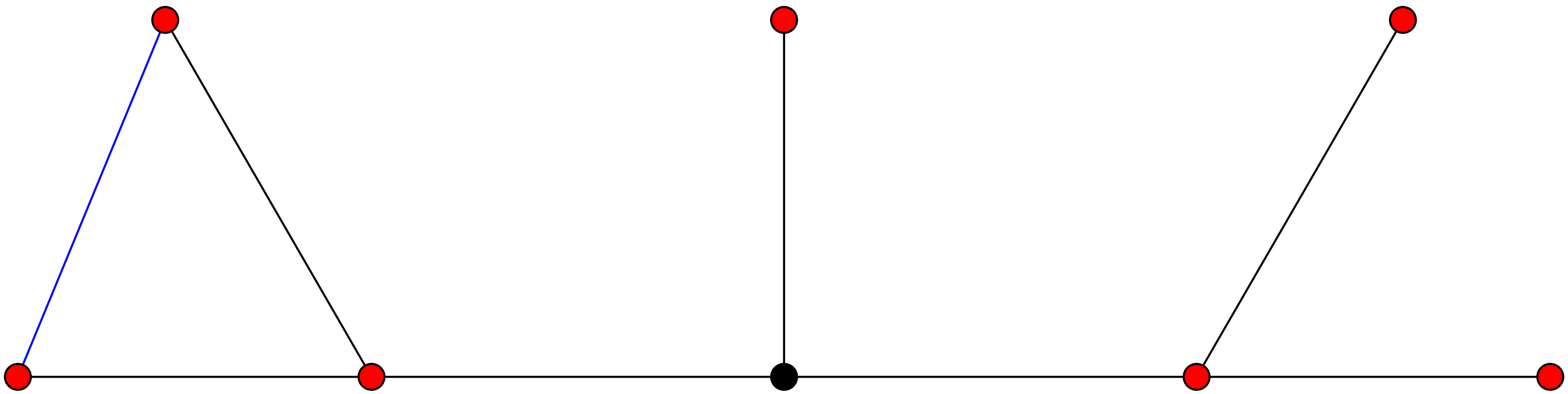

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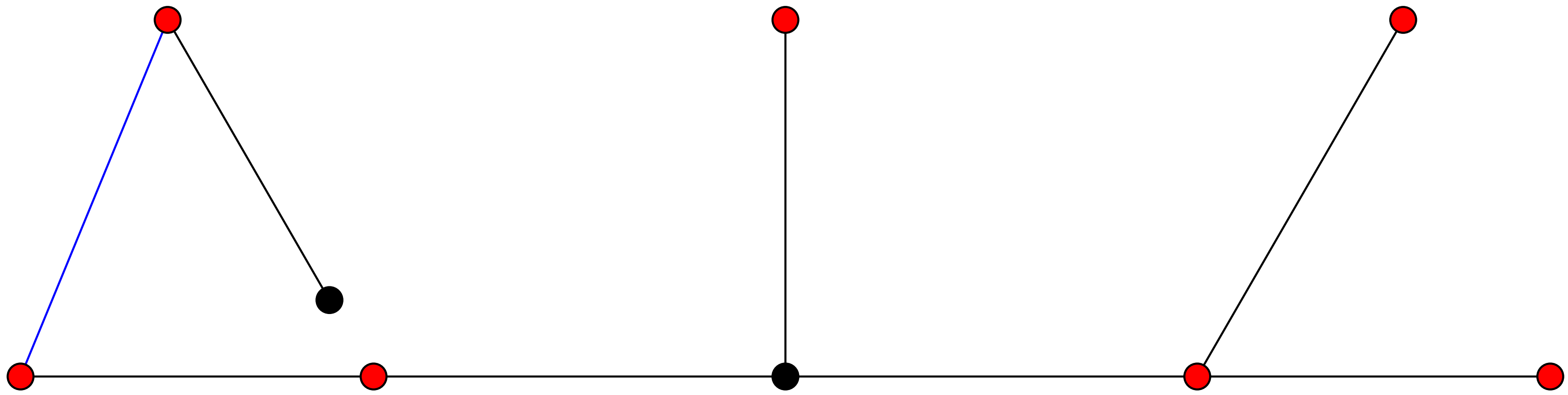

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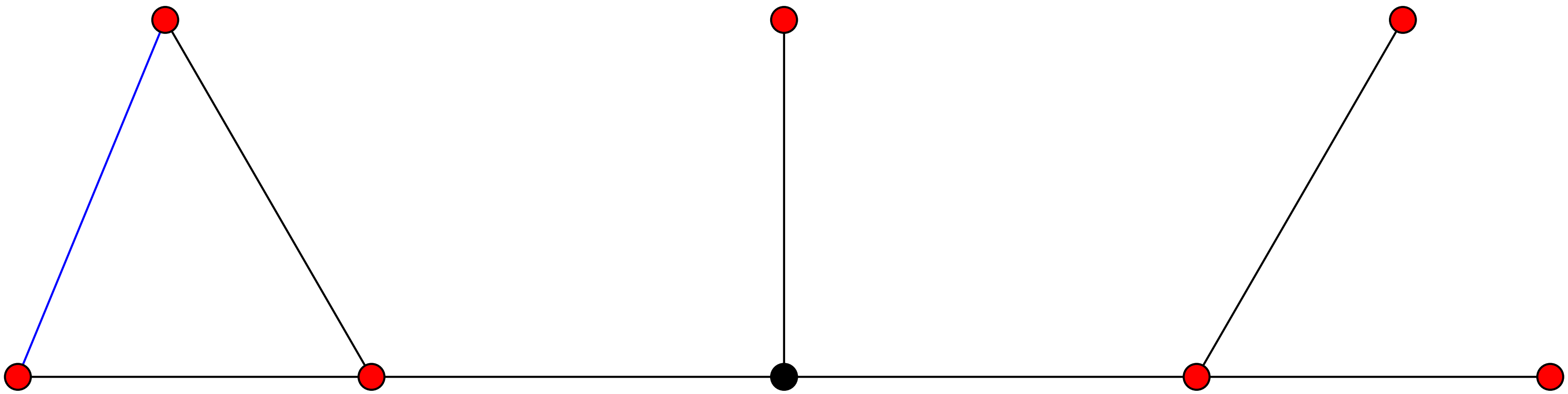

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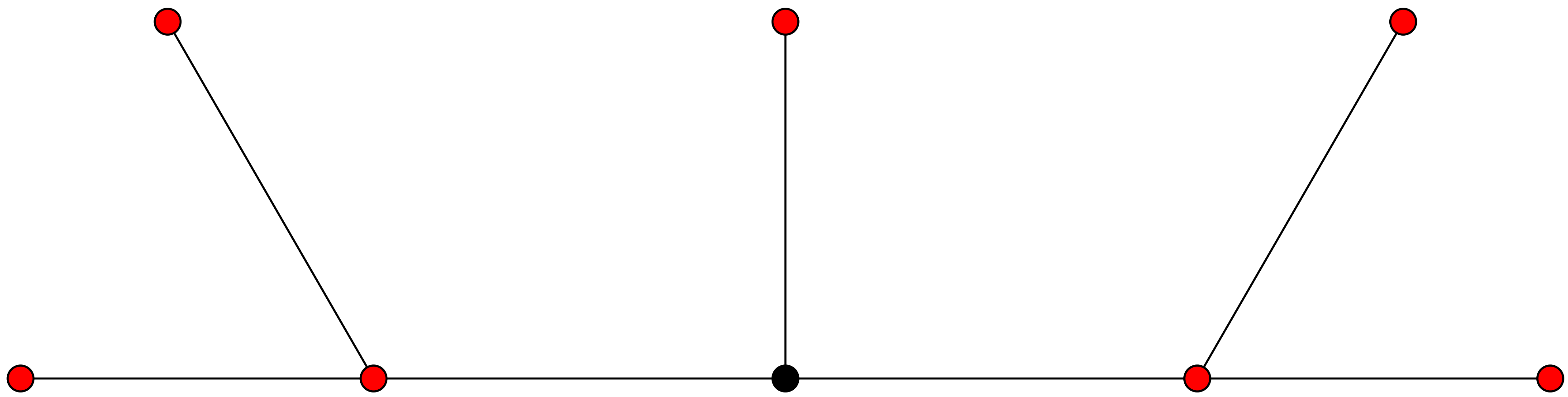

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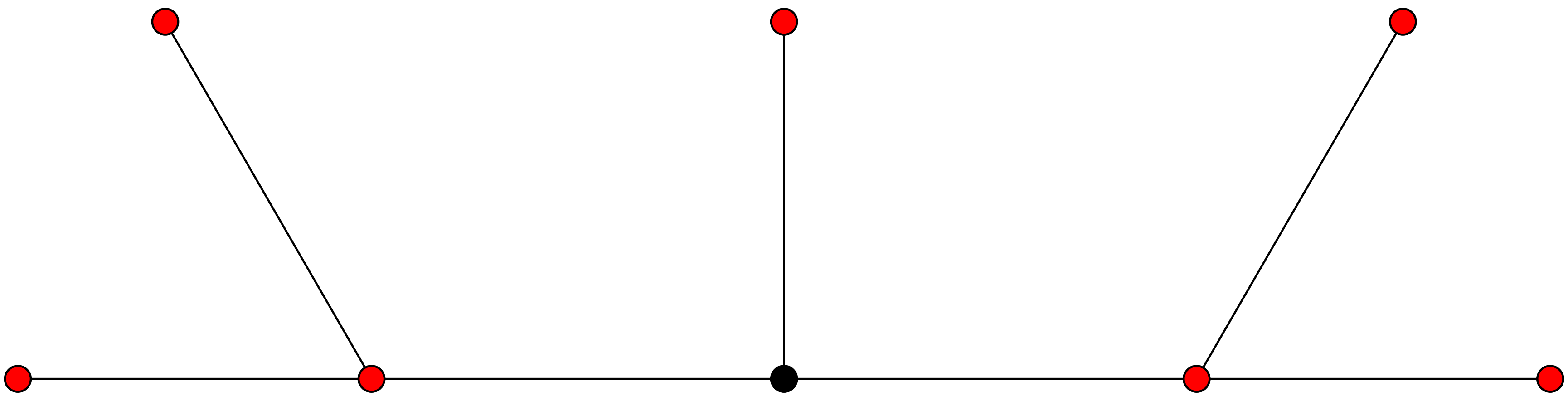
- Partition the tree



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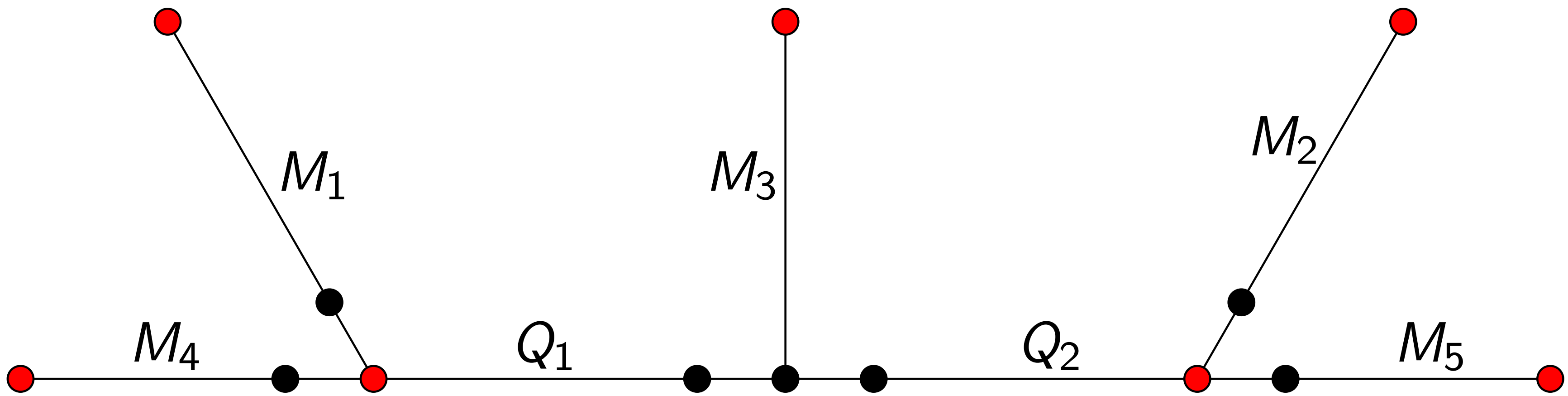

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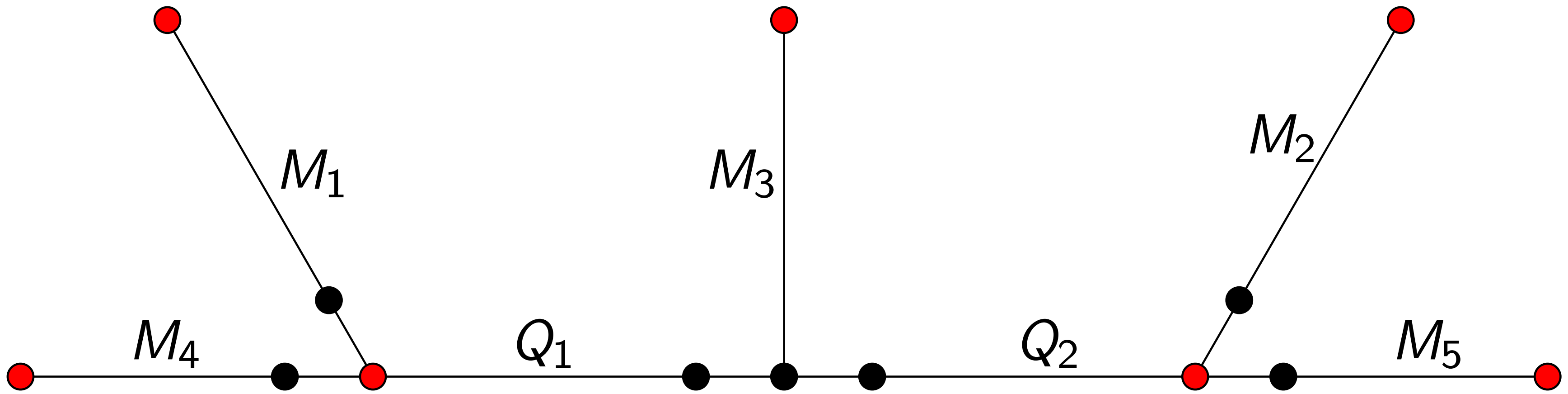

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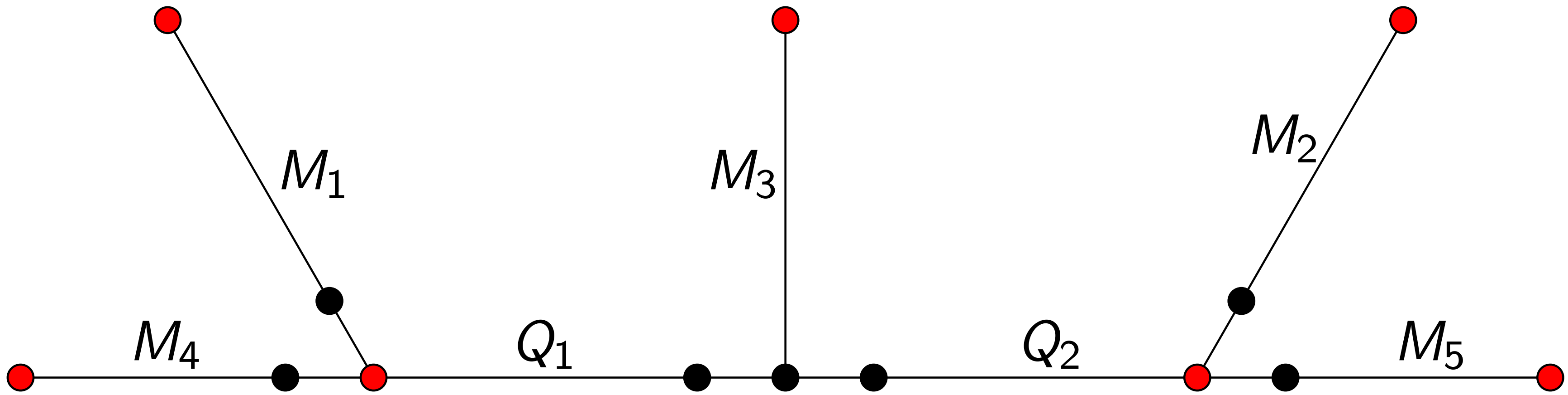

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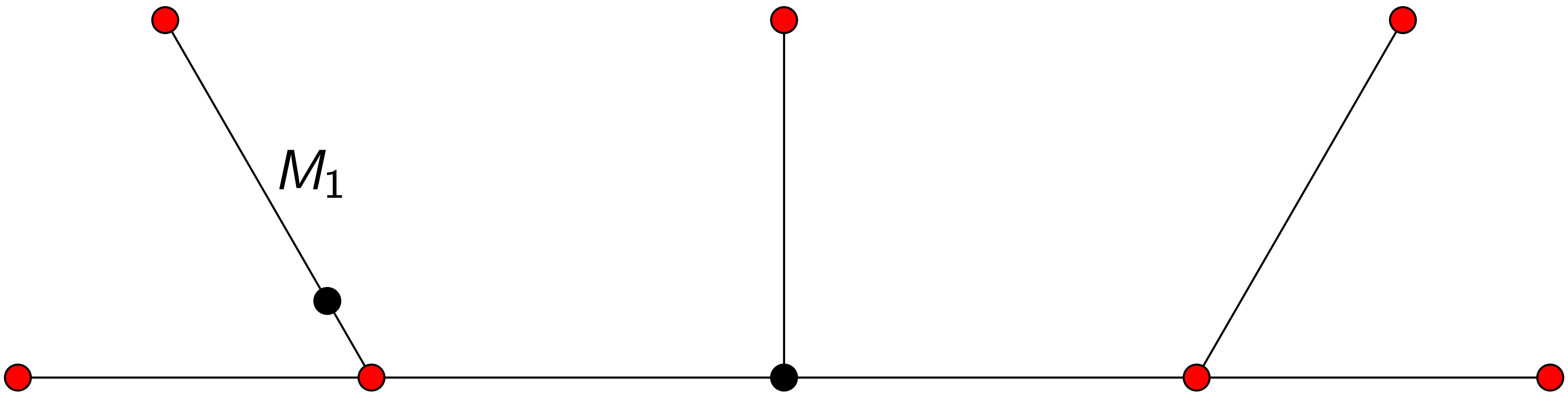
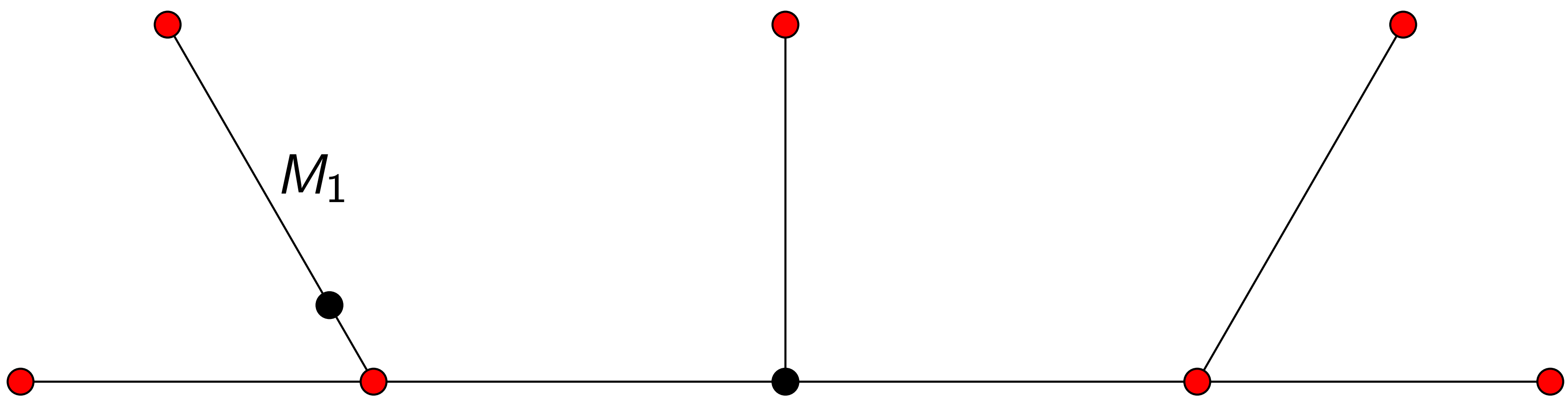


First case:

- Consider one part
- Show certain neighbor sets must be disjoint



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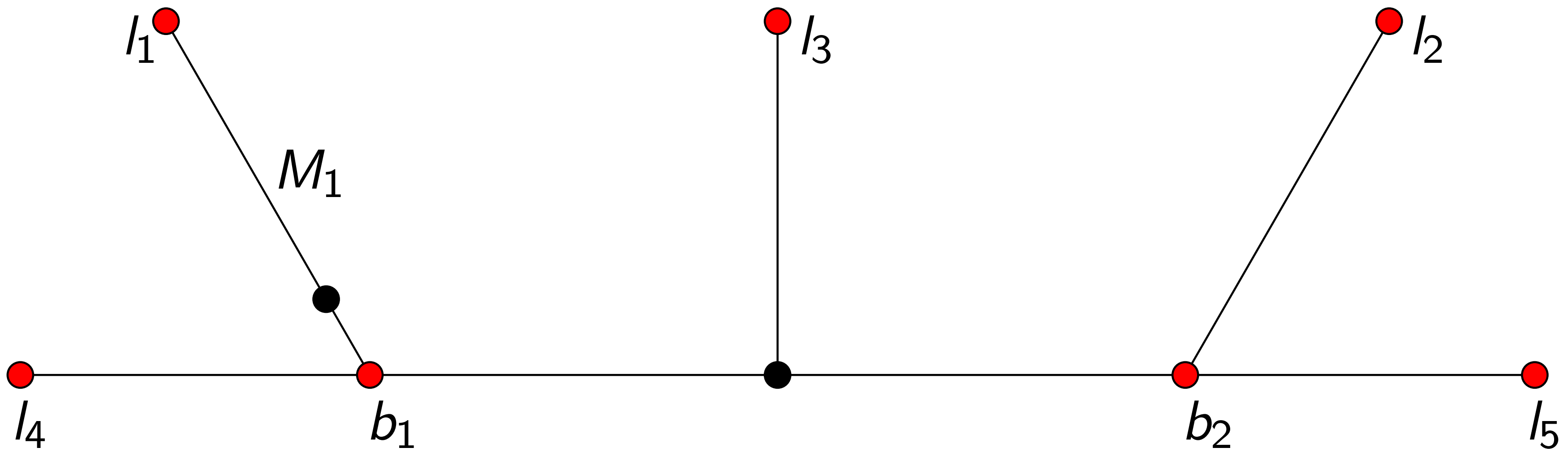

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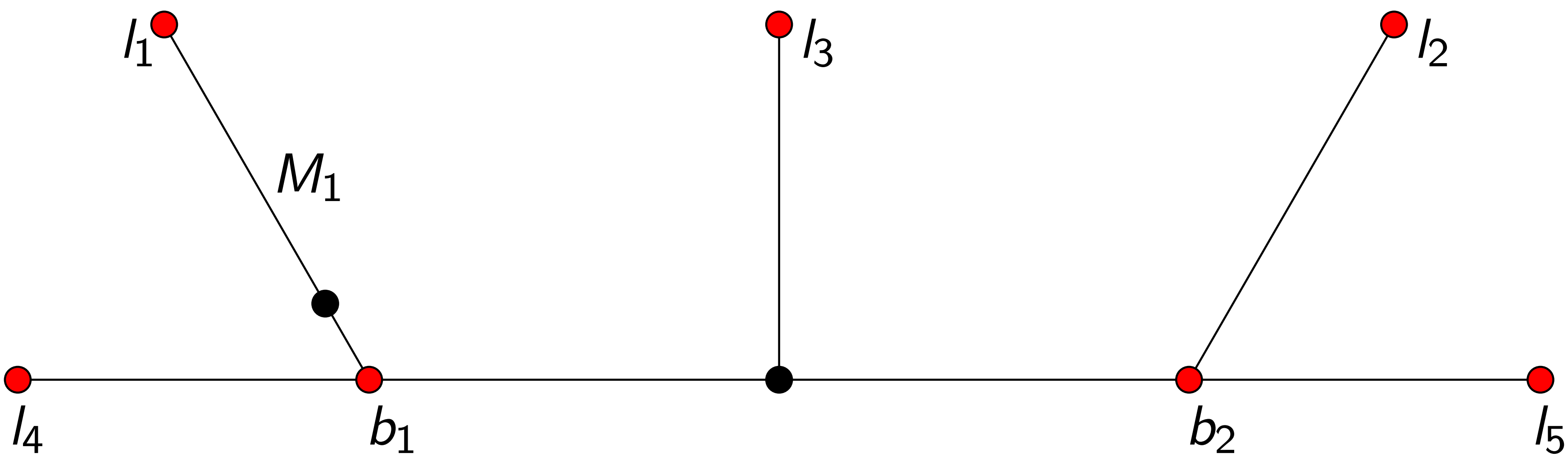

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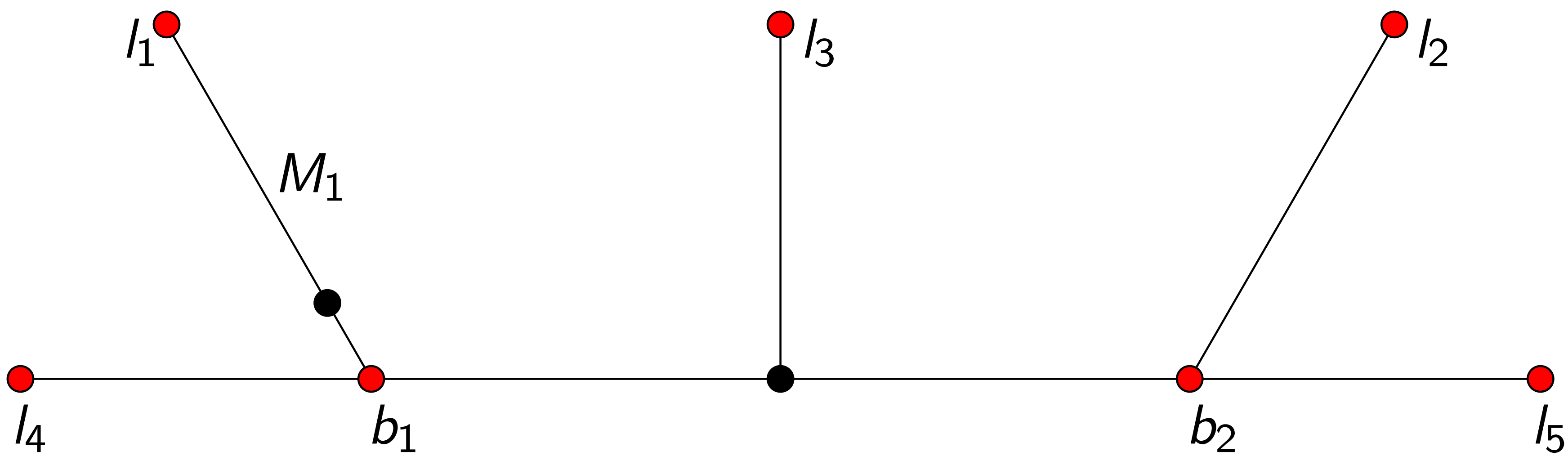

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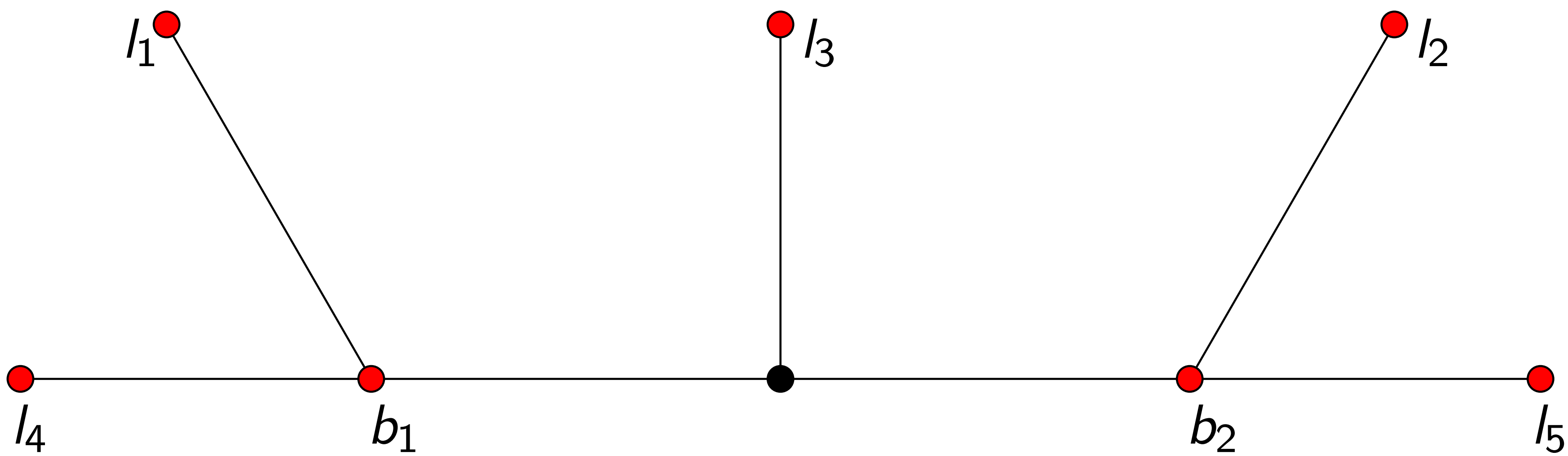

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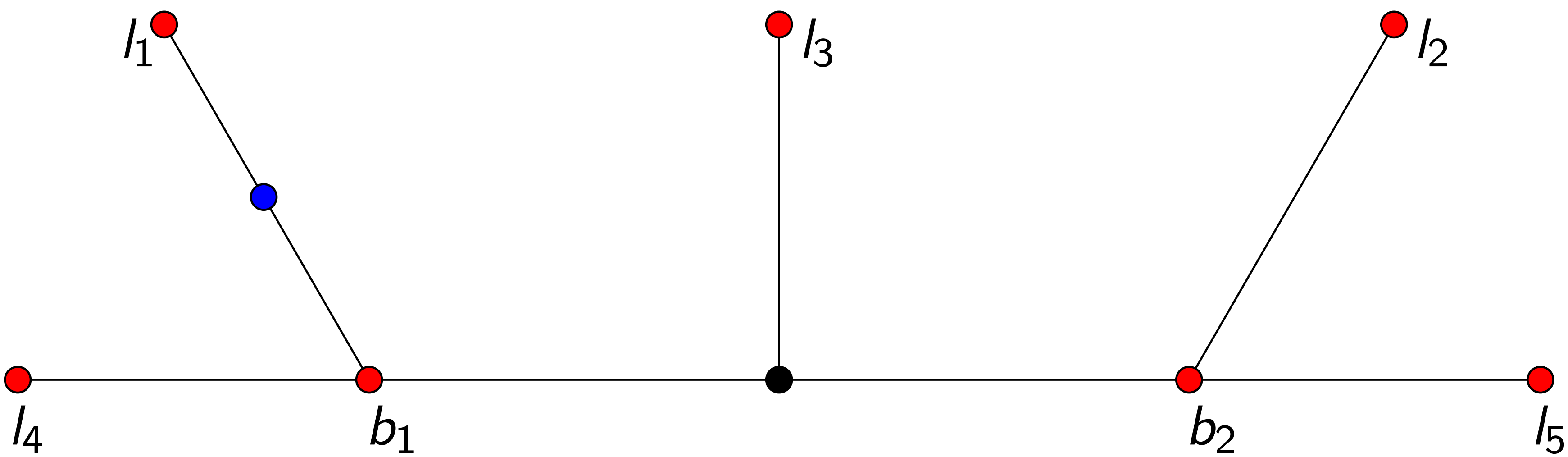

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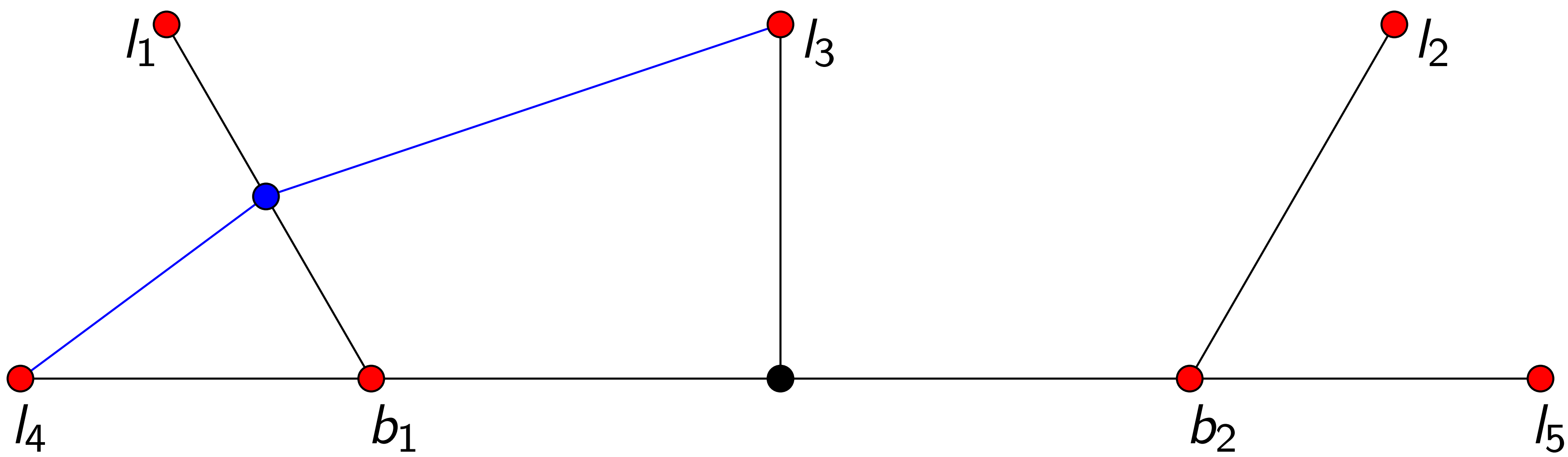

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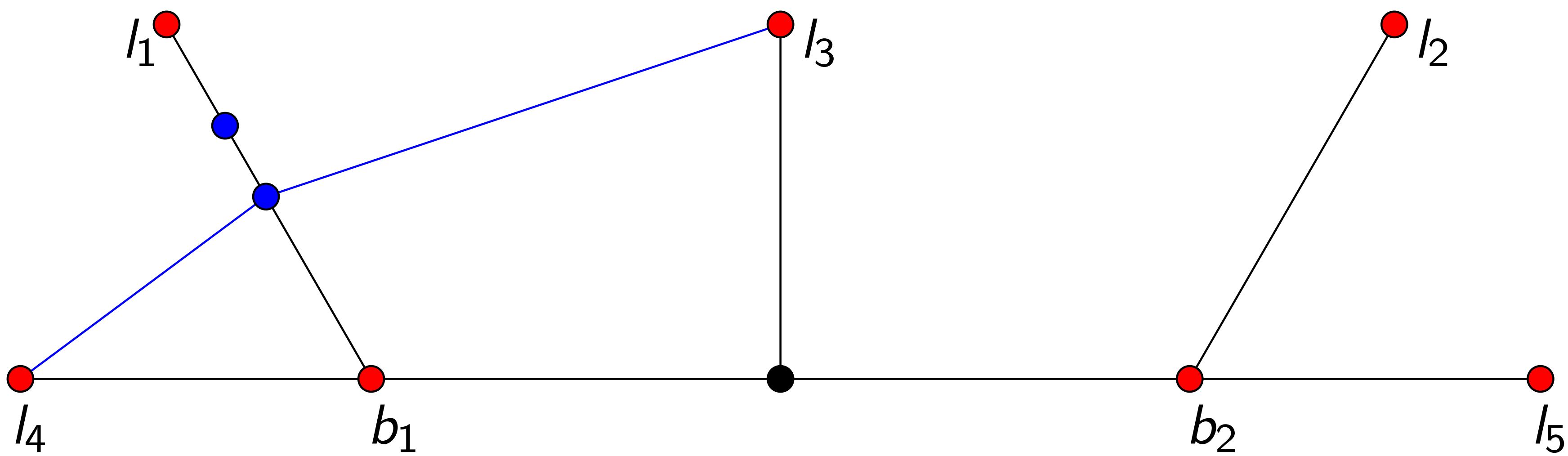

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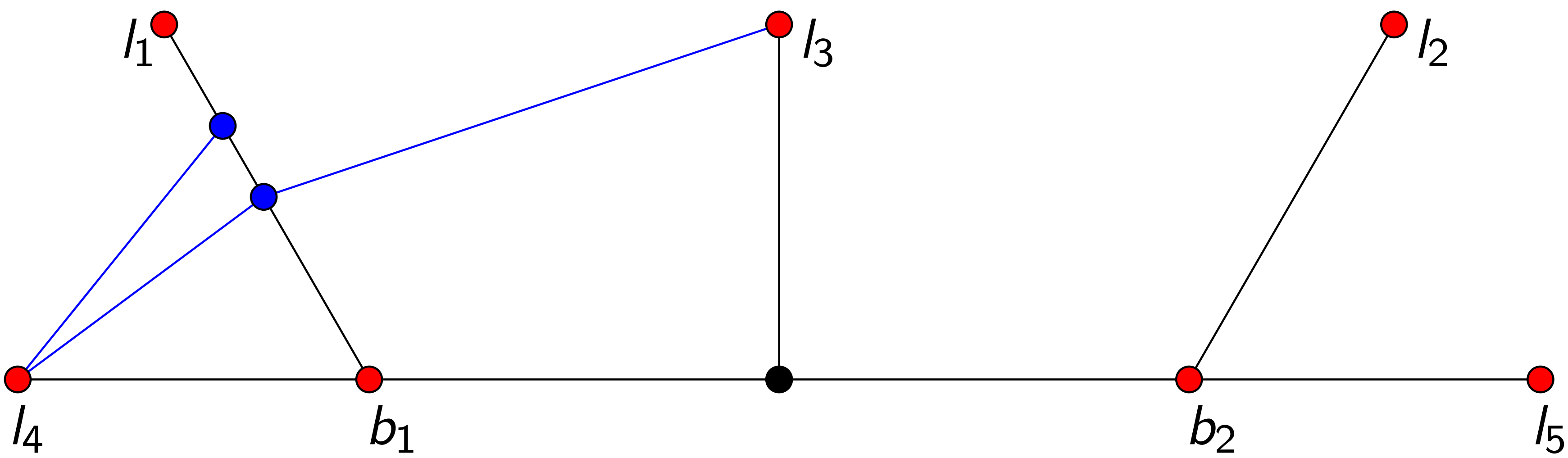

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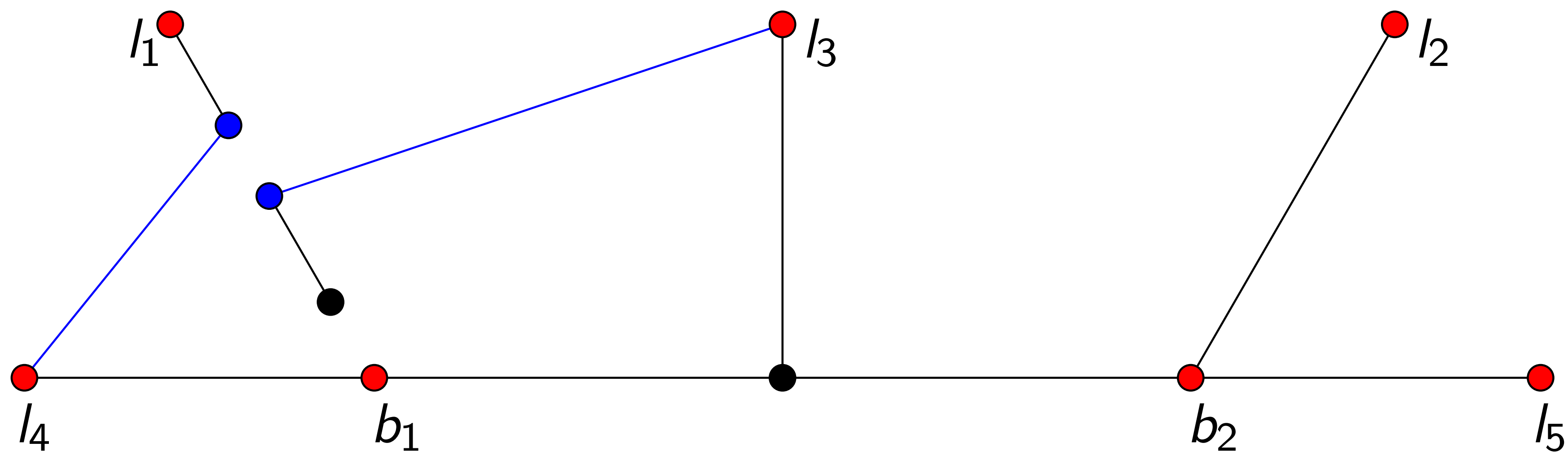

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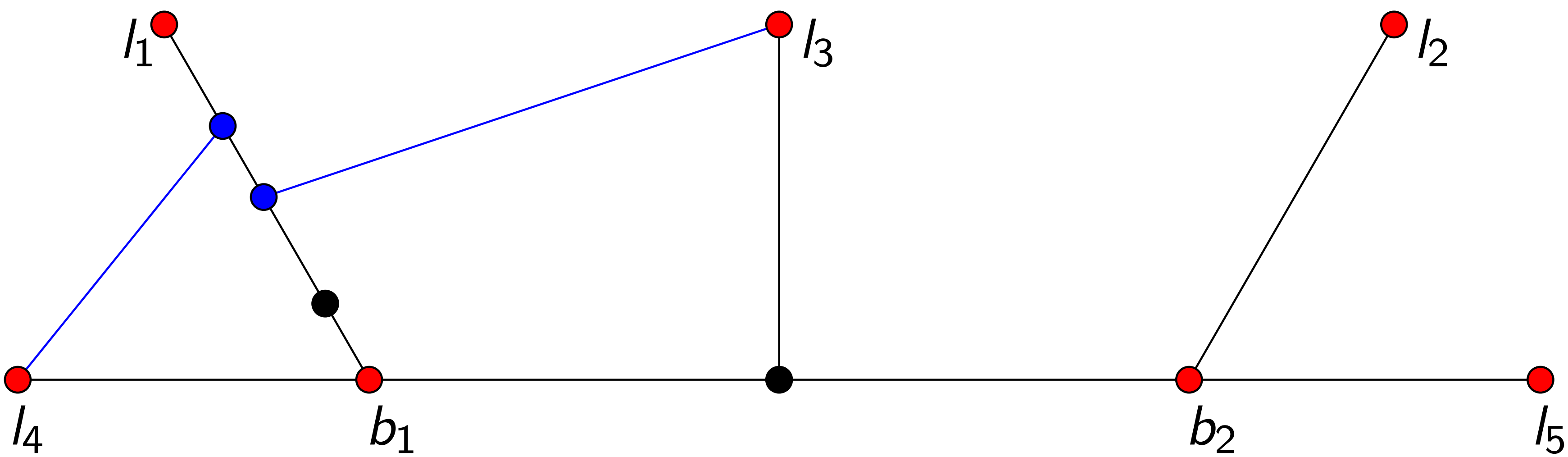

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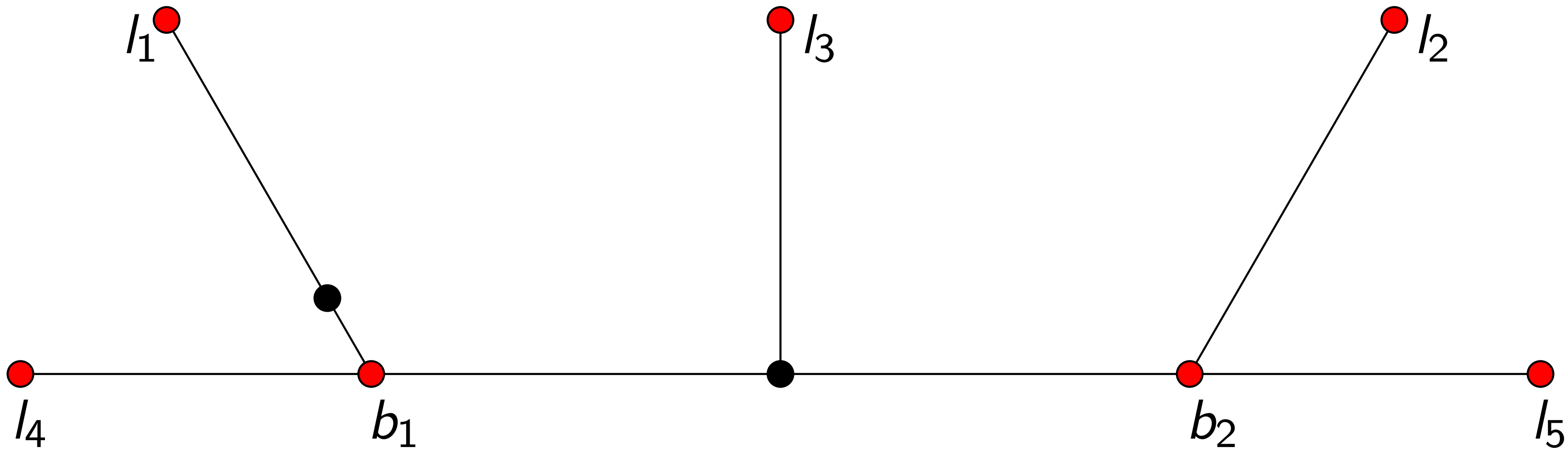

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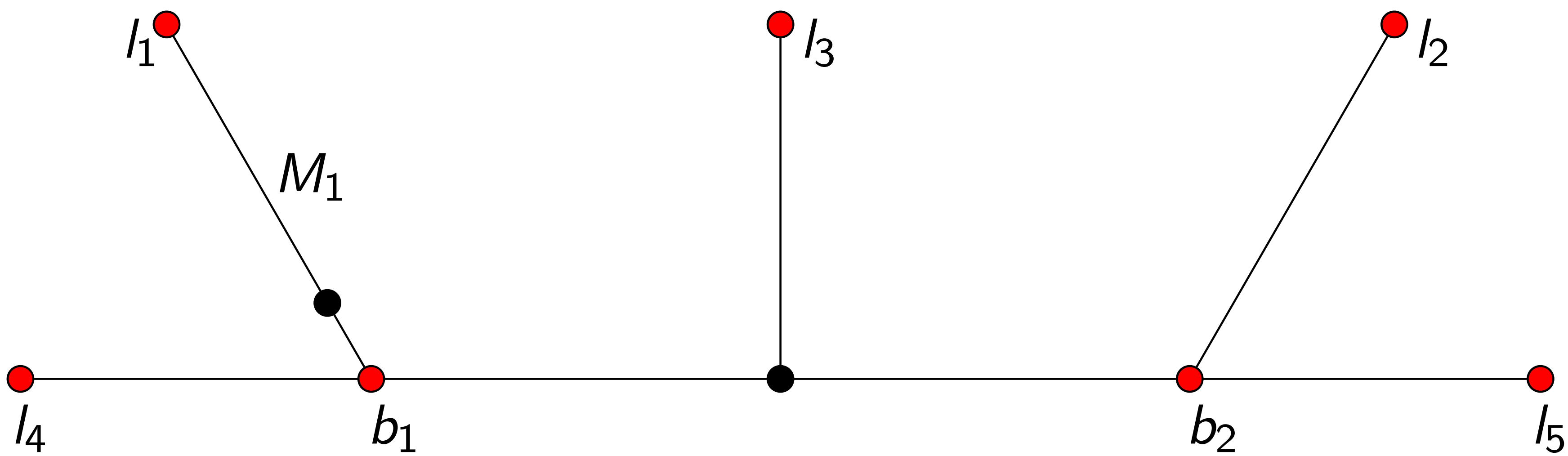

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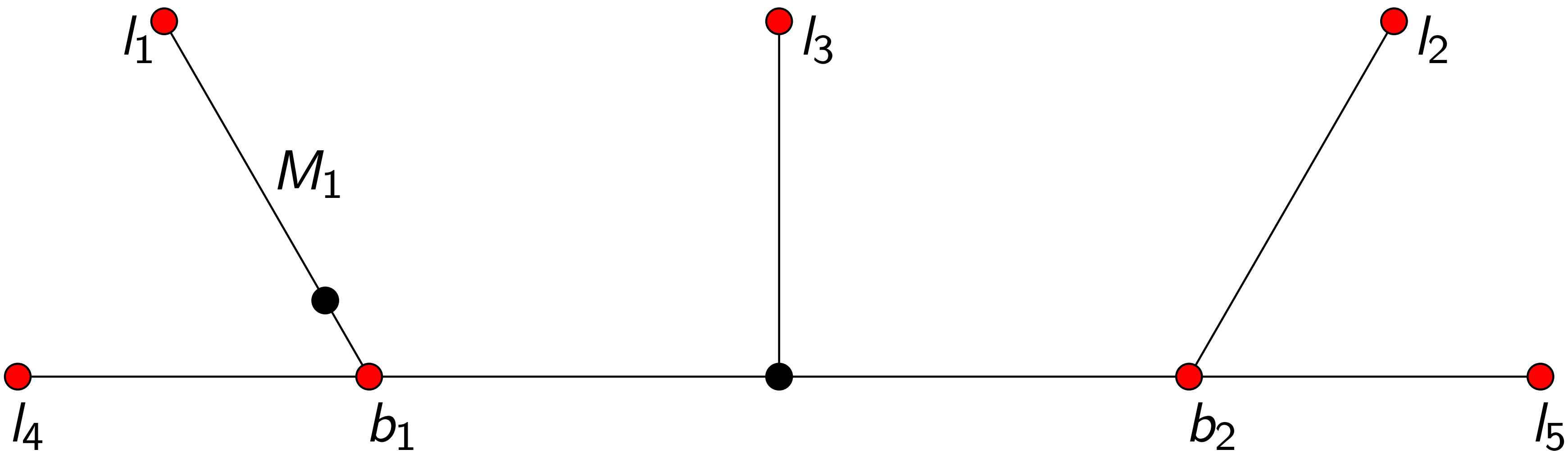

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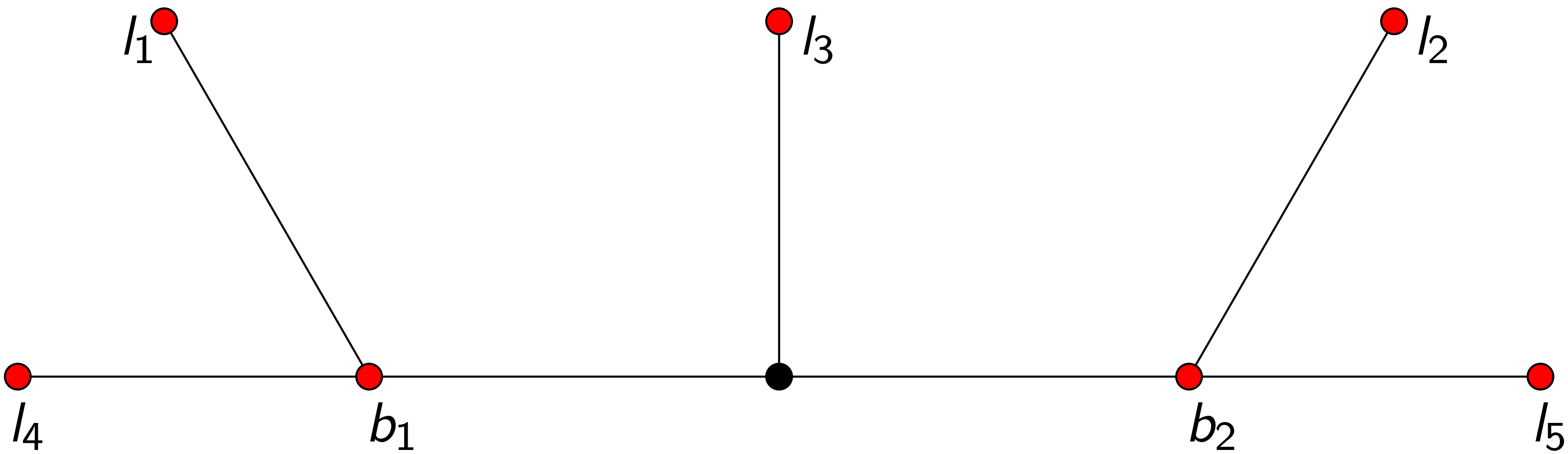

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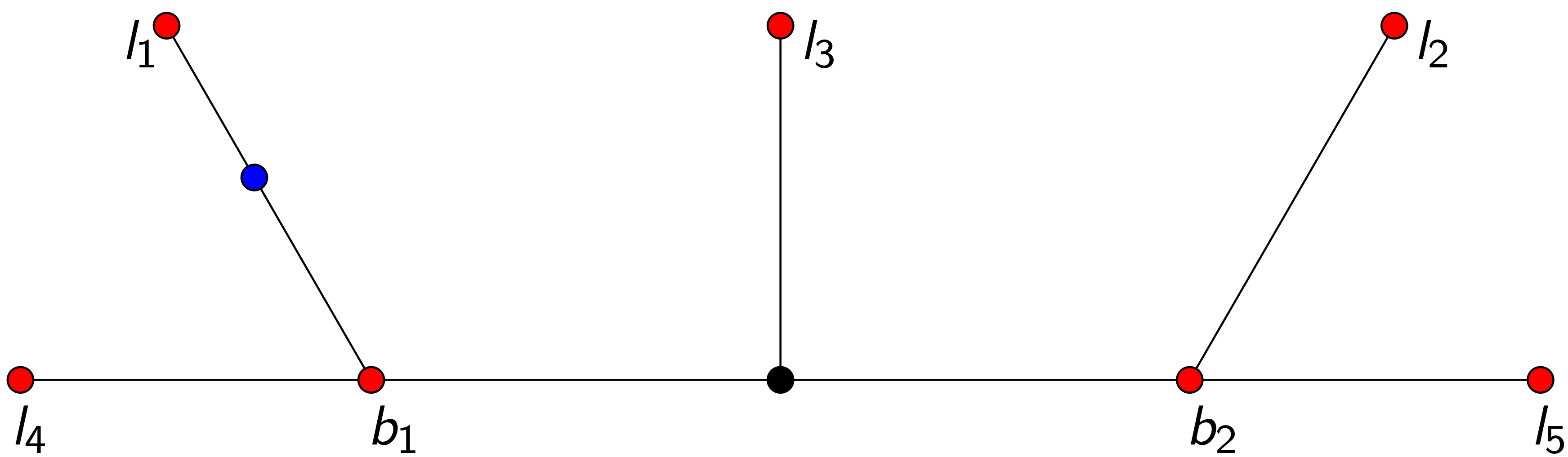

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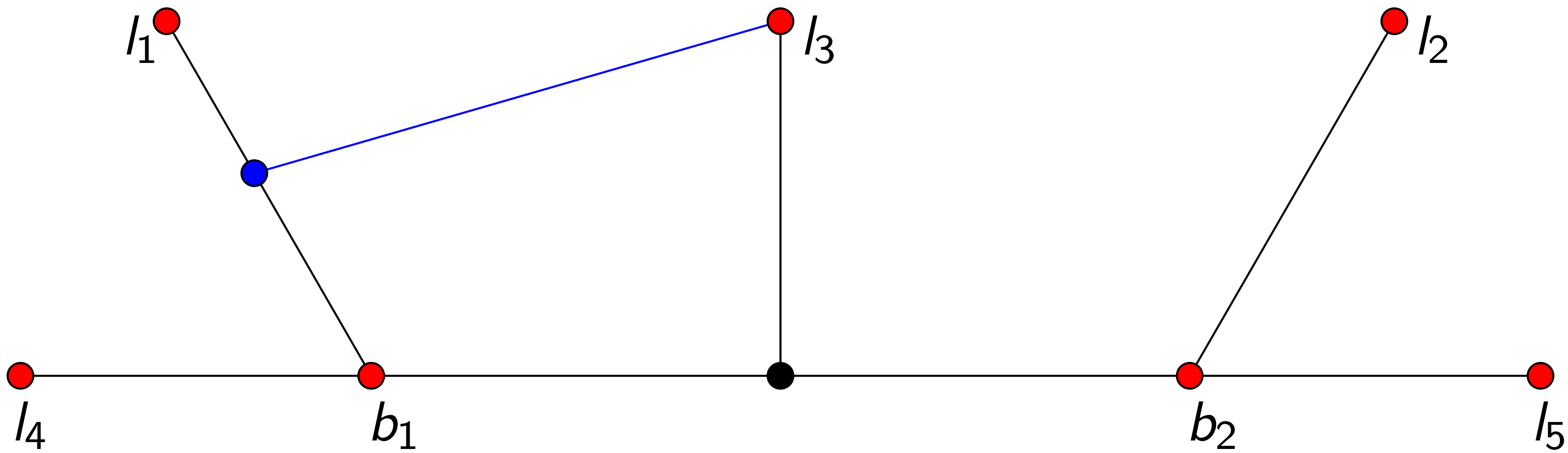

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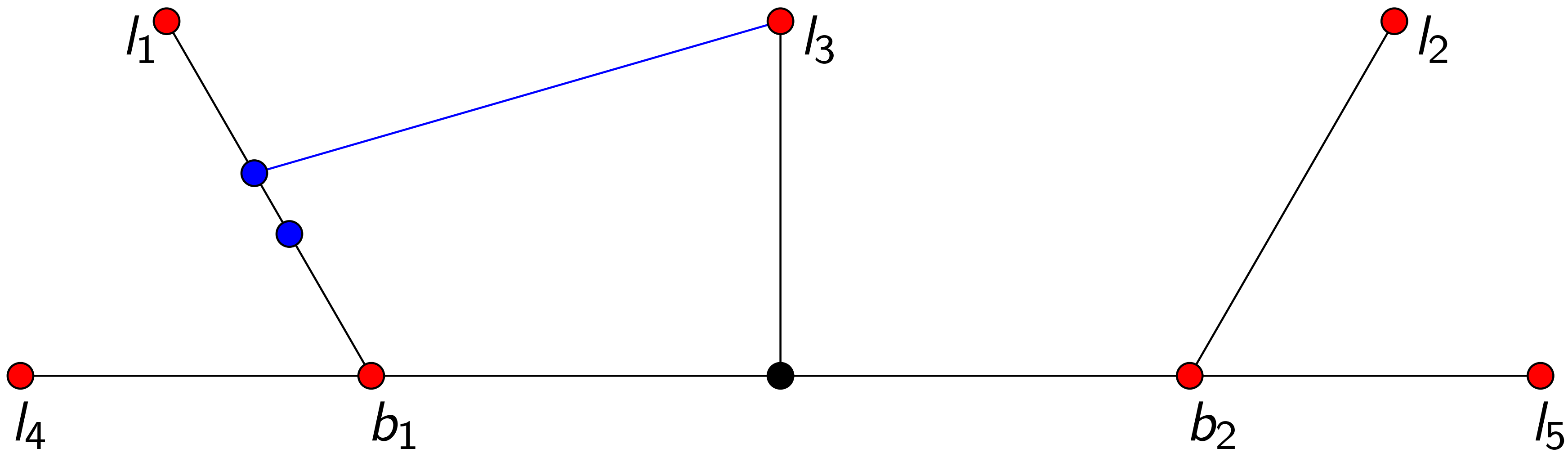

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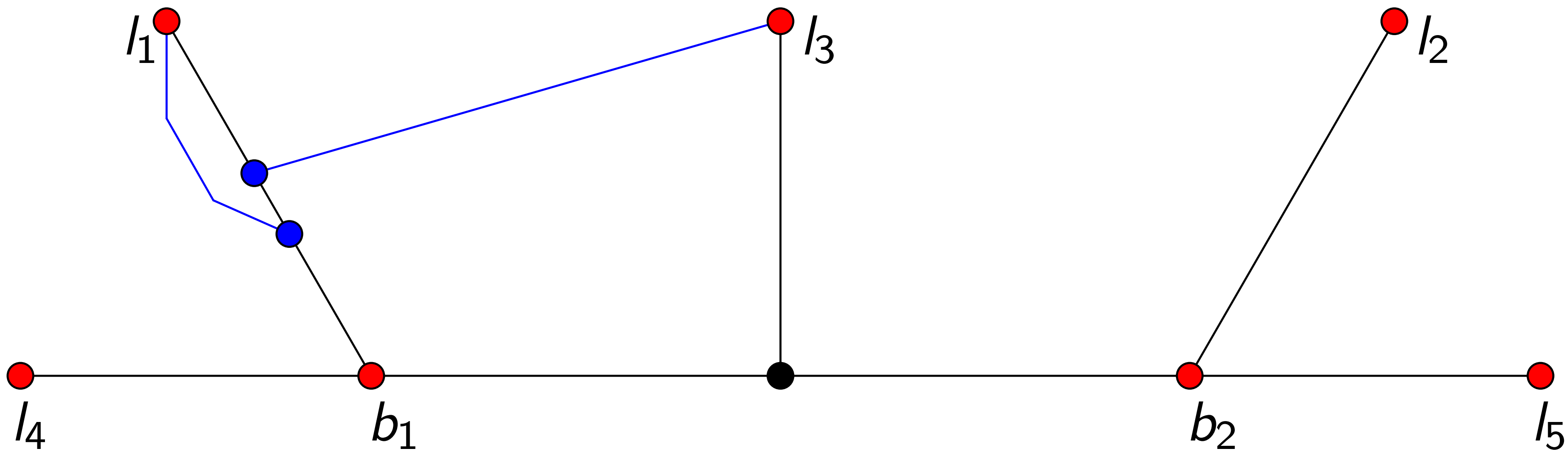

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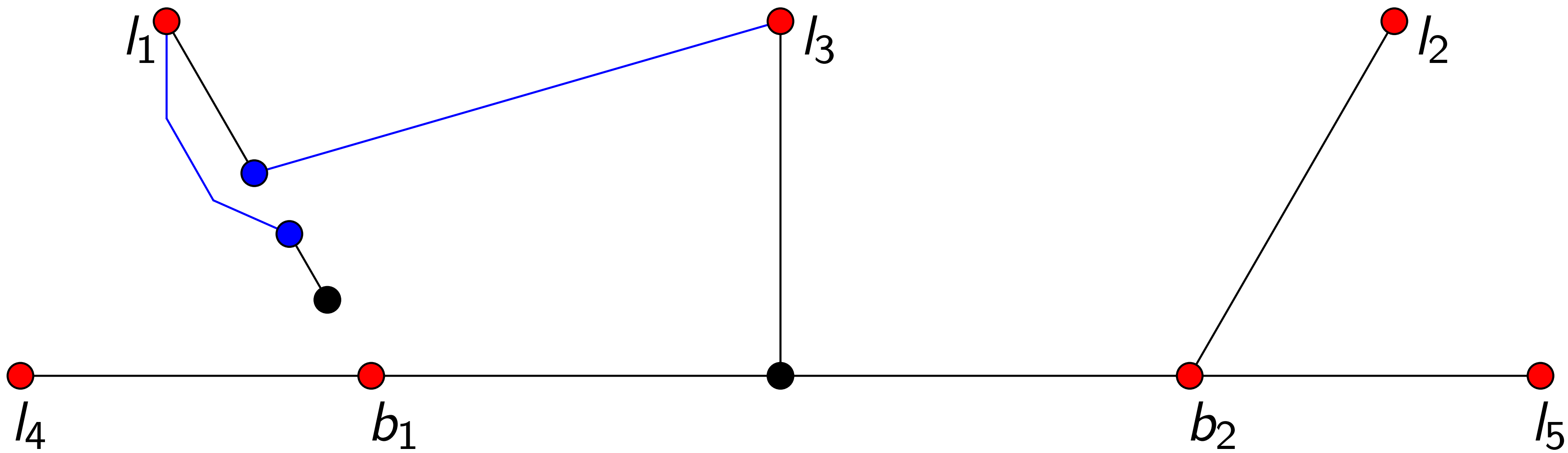

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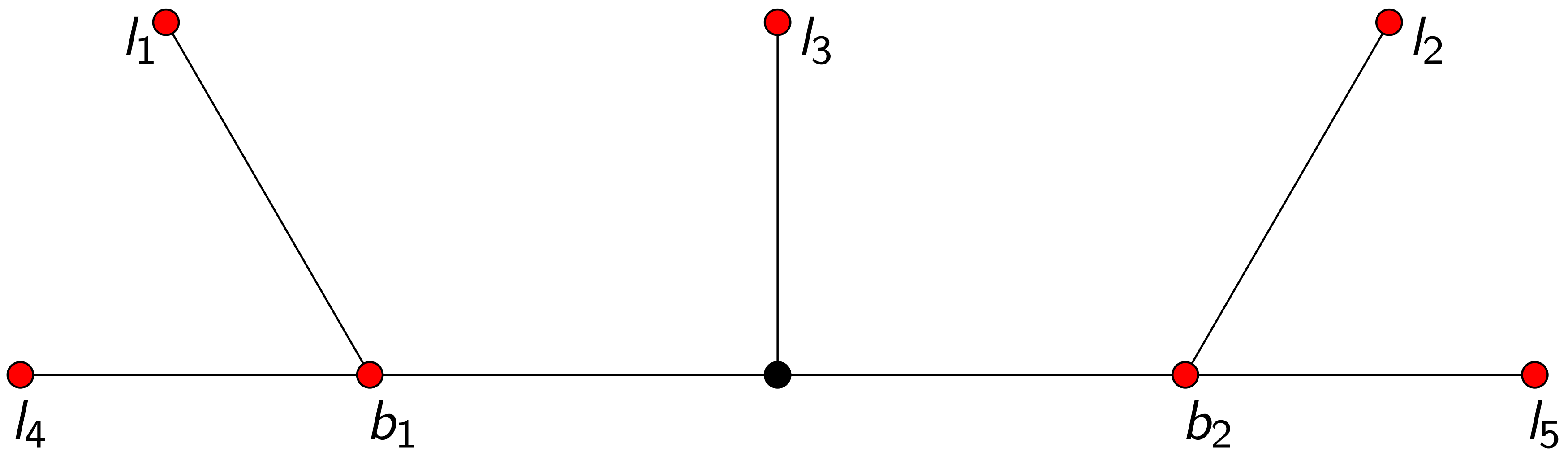
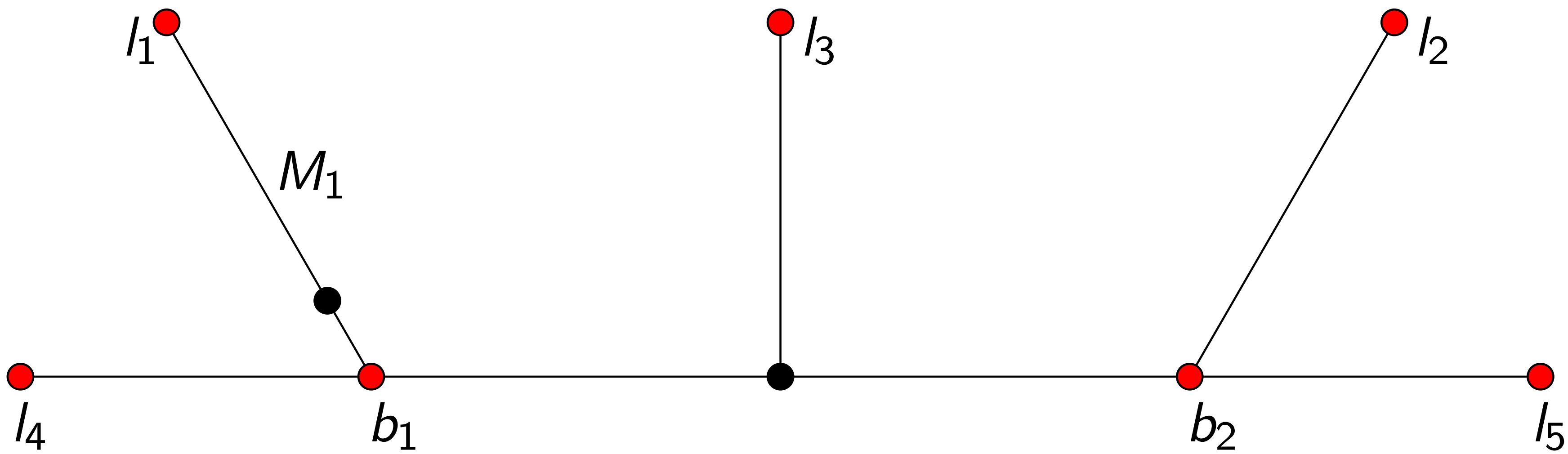

l_1
l_3
l_2
l_4
b_1
b_2
l_5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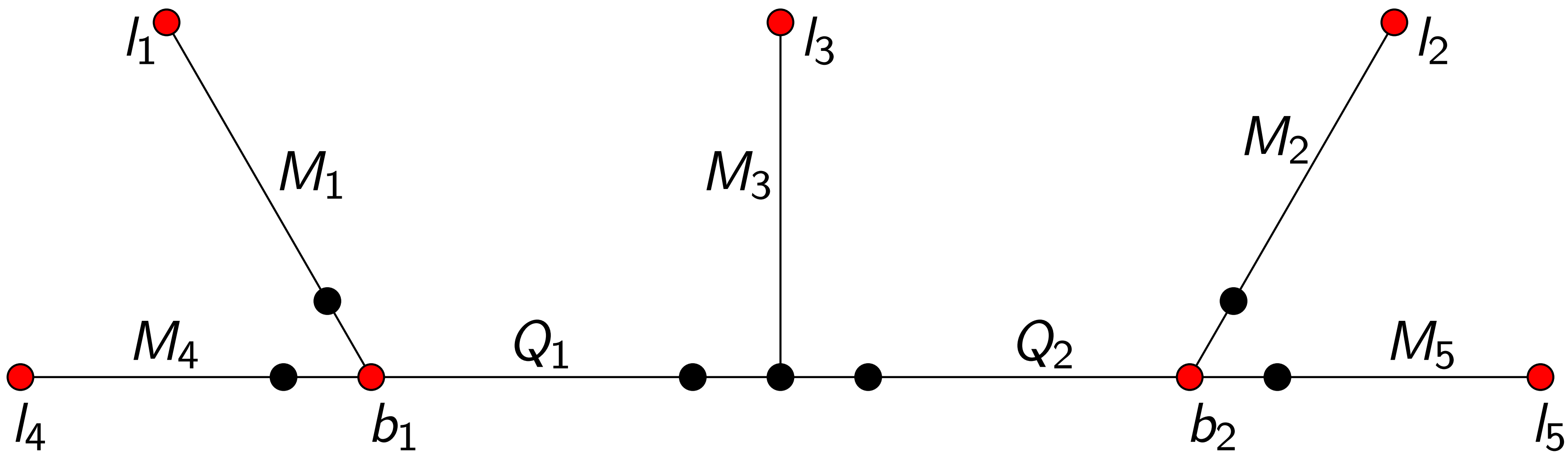
l_1
l_3
l_2
M_1
M_3
M_2
M_4
Q_1
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l_4
b_1
b_2
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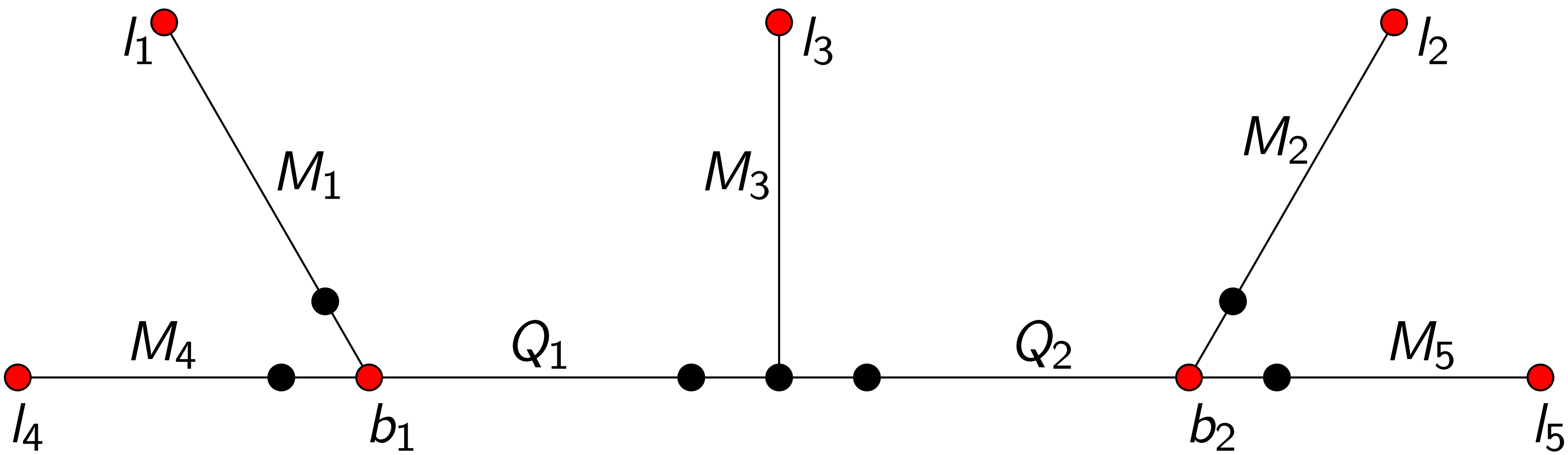

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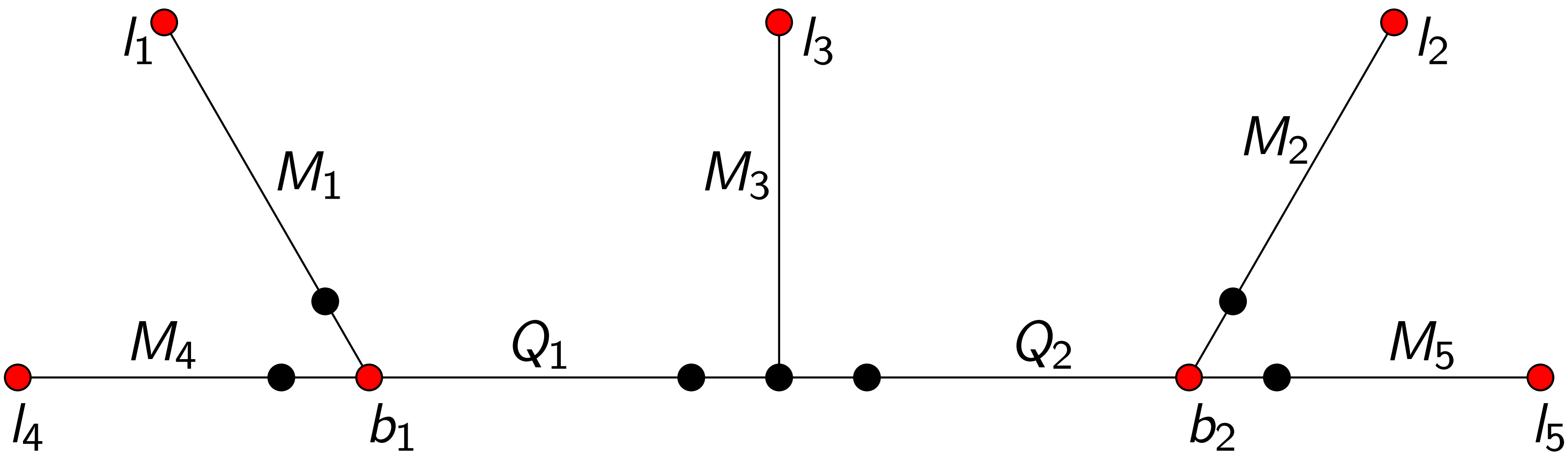

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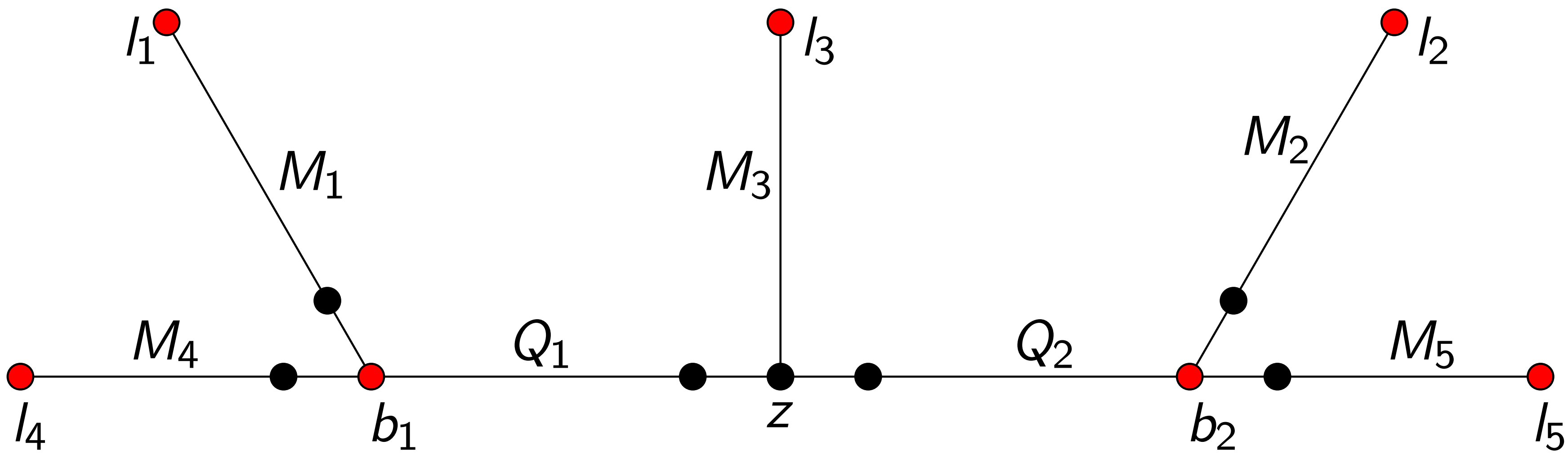

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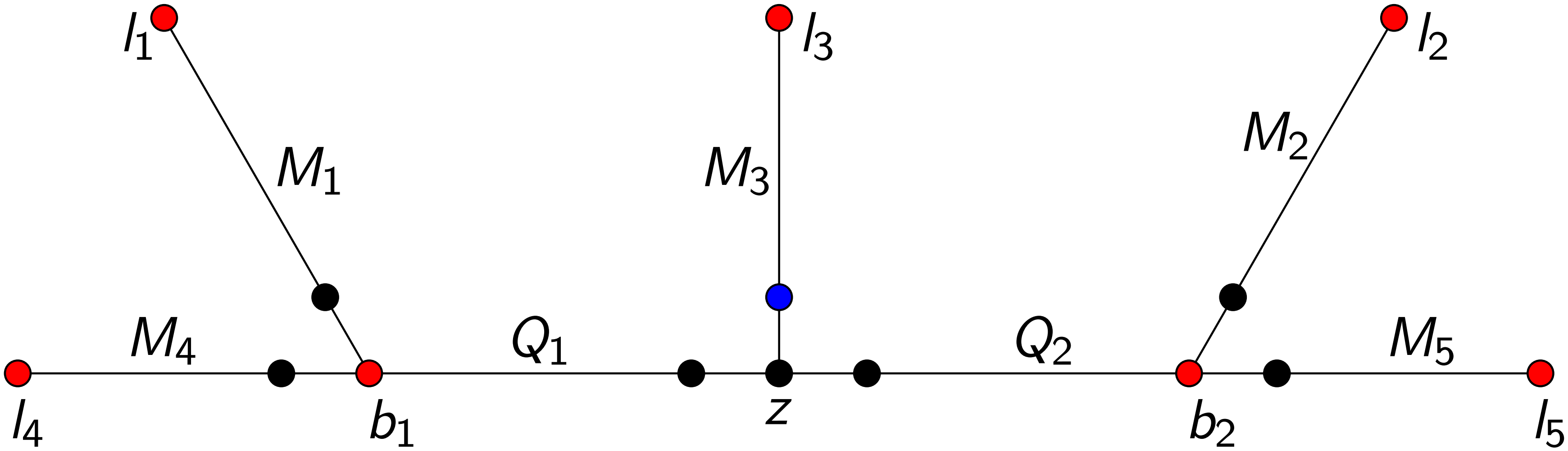

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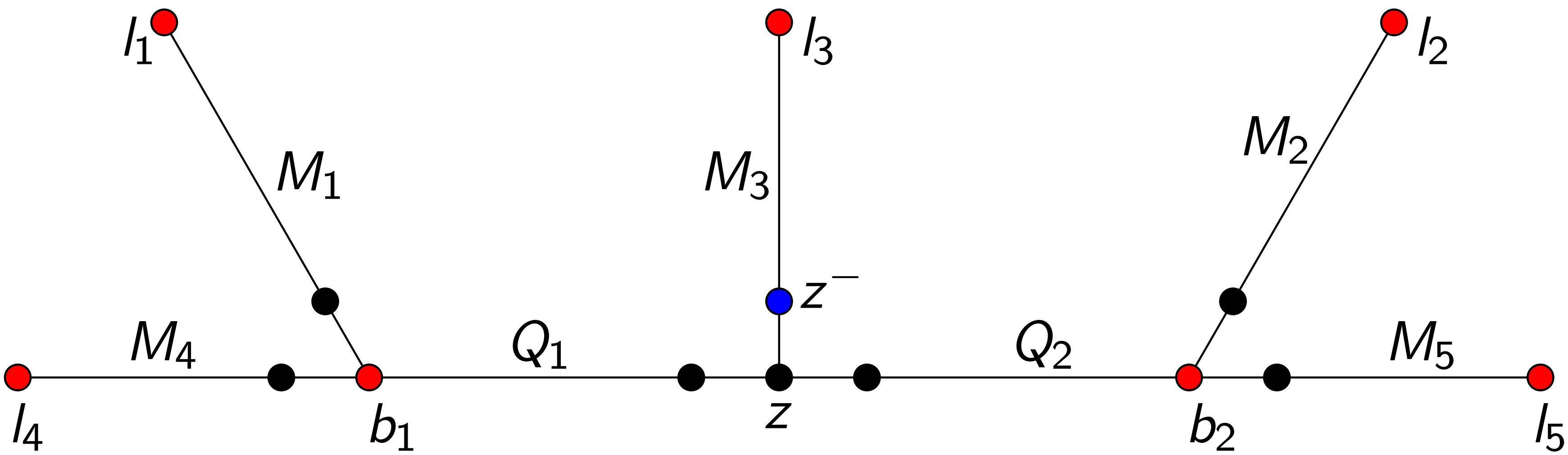

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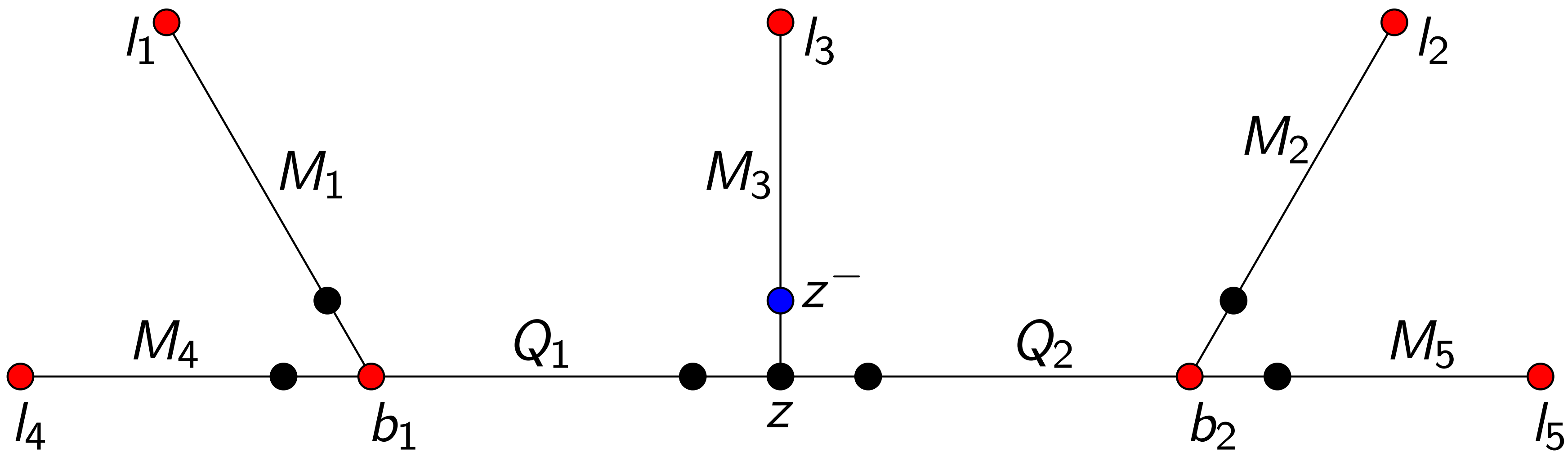

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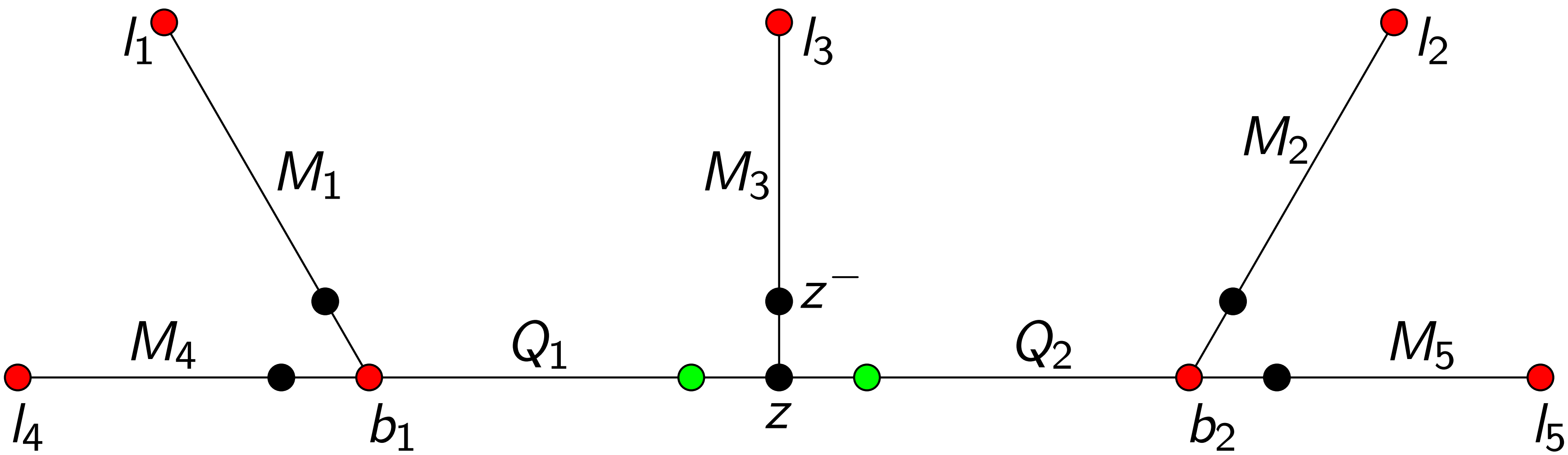

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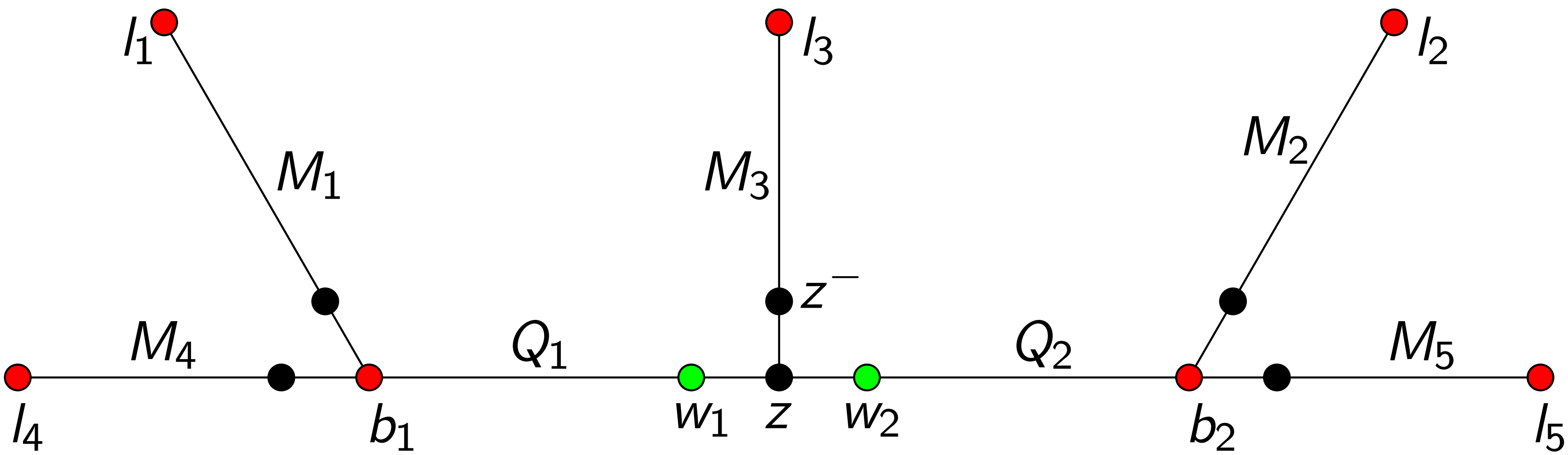

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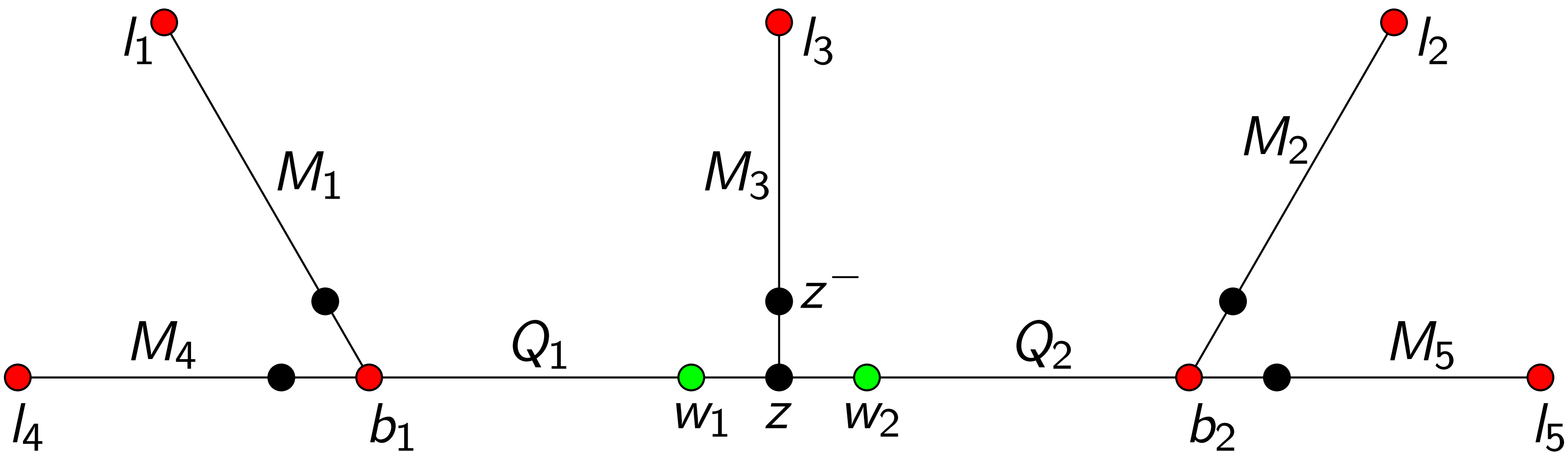

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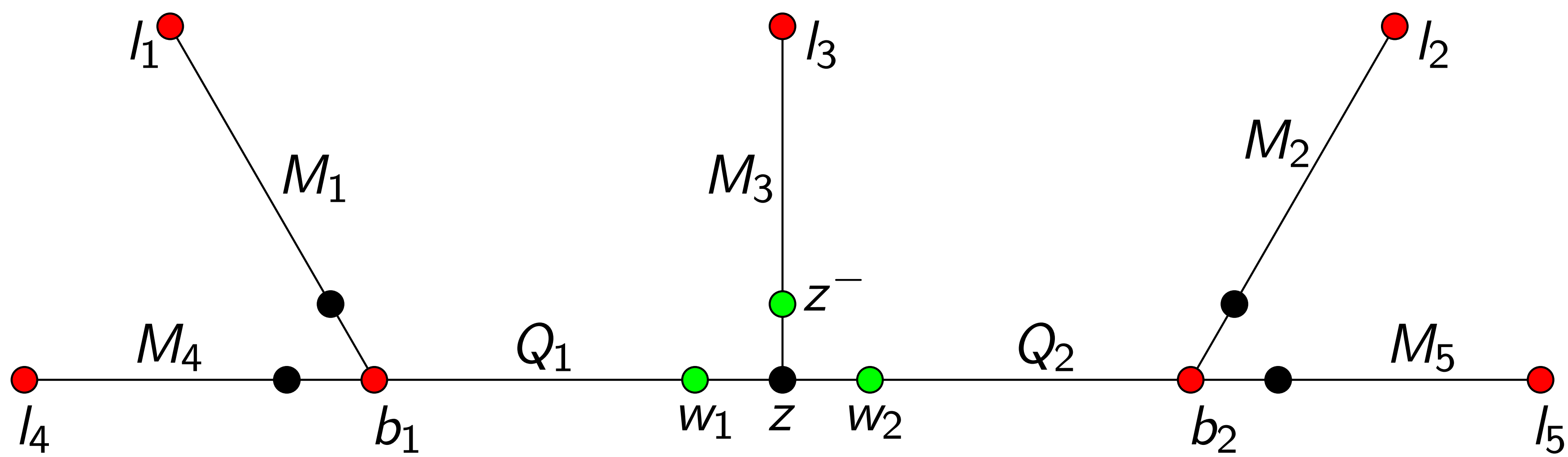

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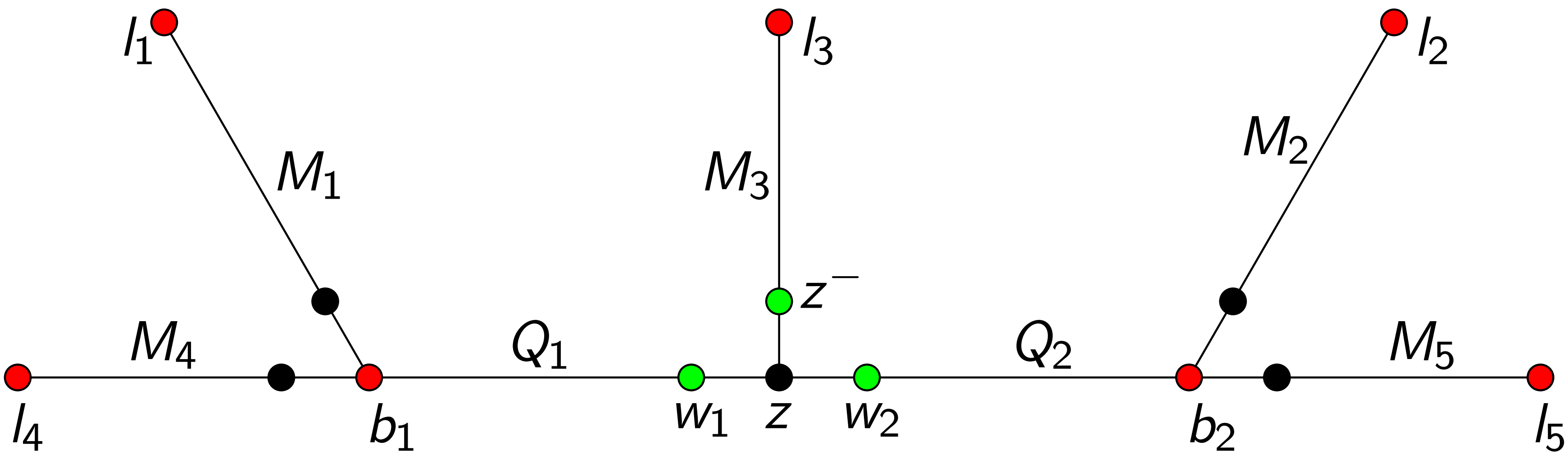

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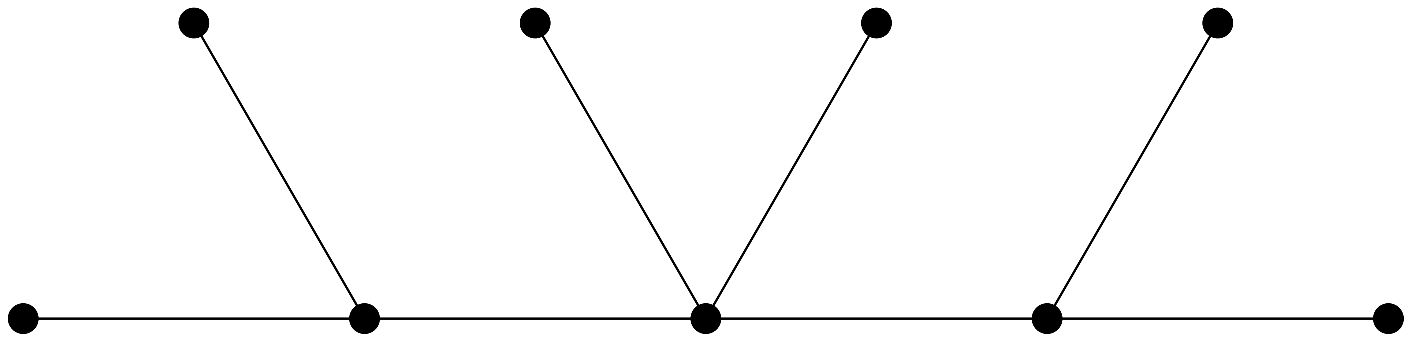
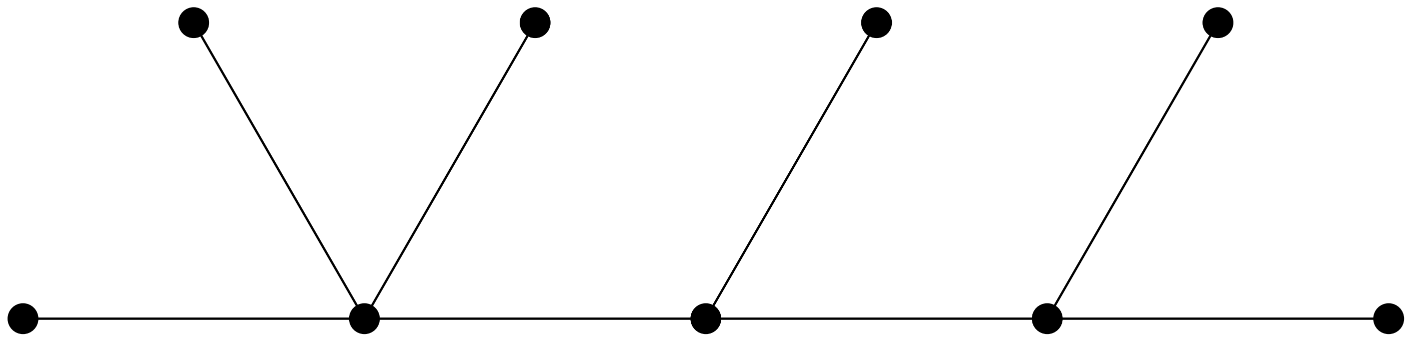
Second and third cases: T has 6 leaves, but only 3 branch vertices.





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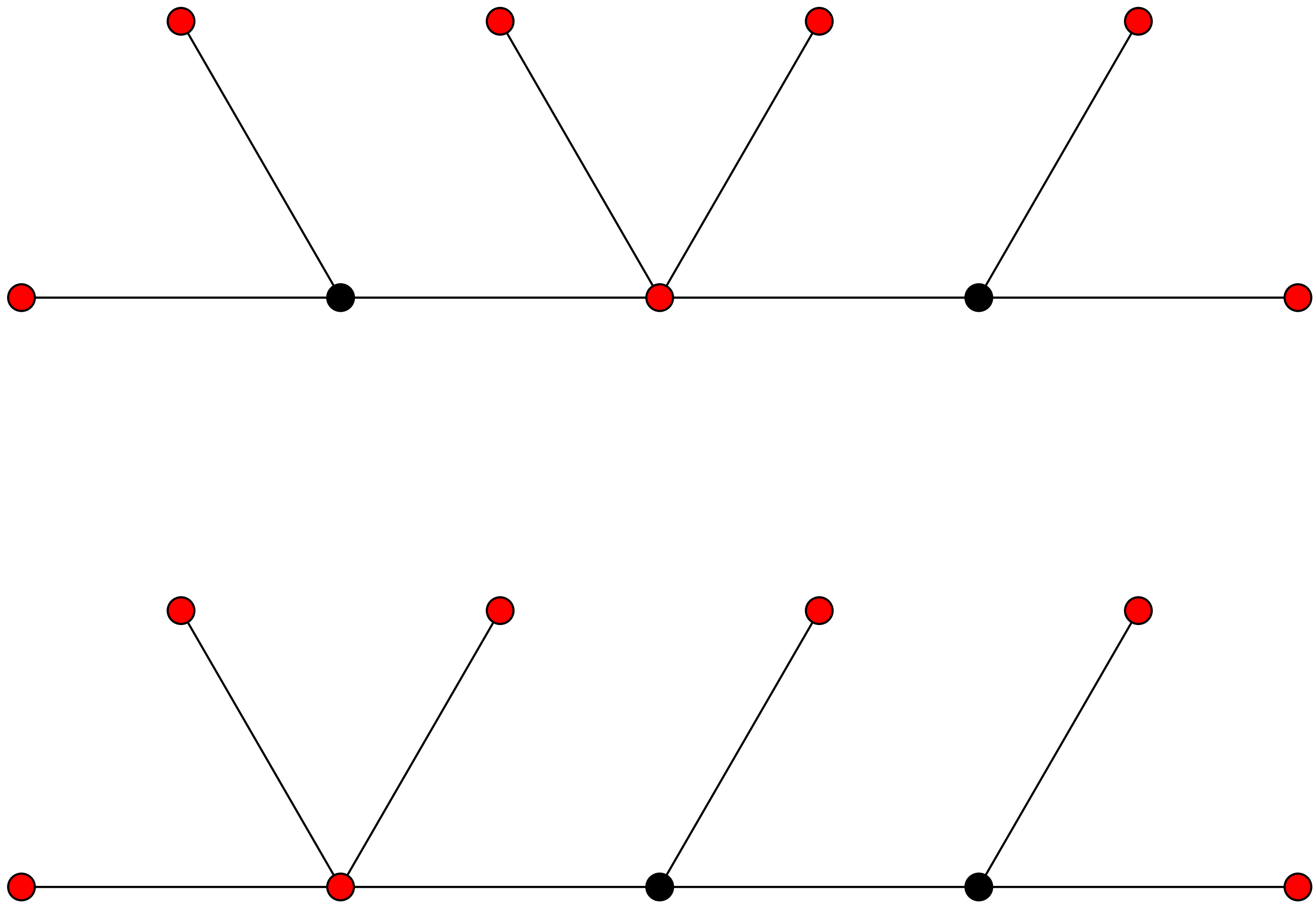

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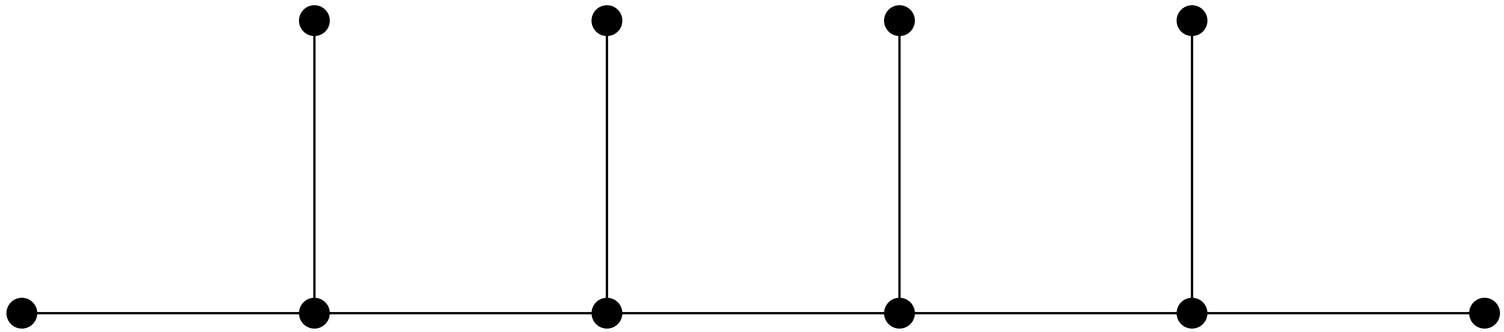

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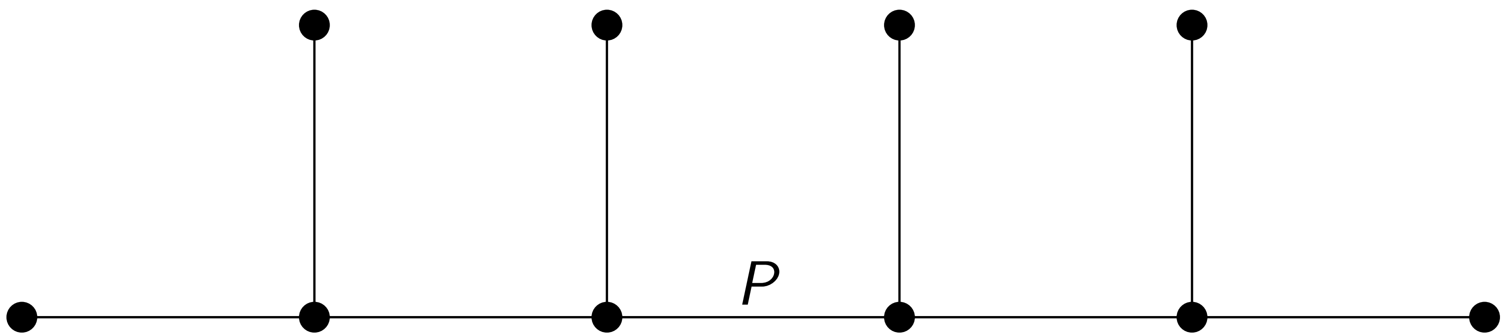

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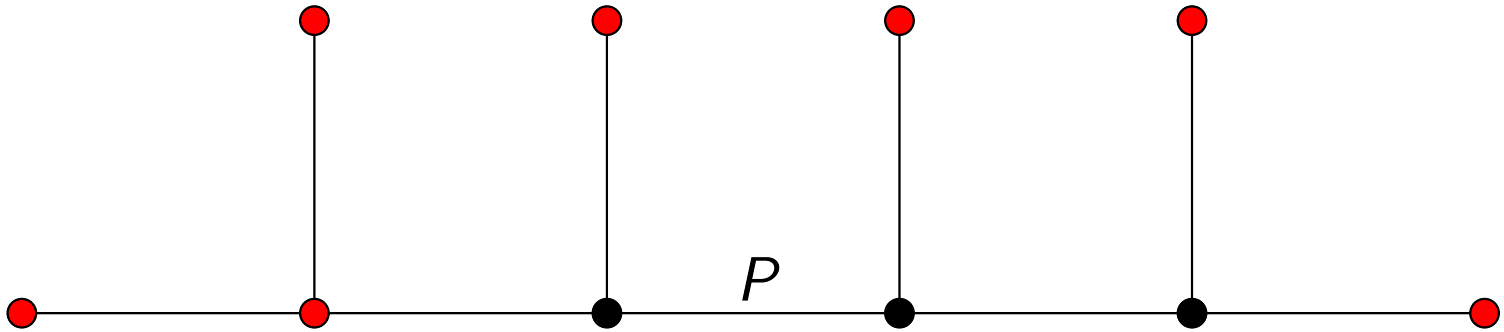

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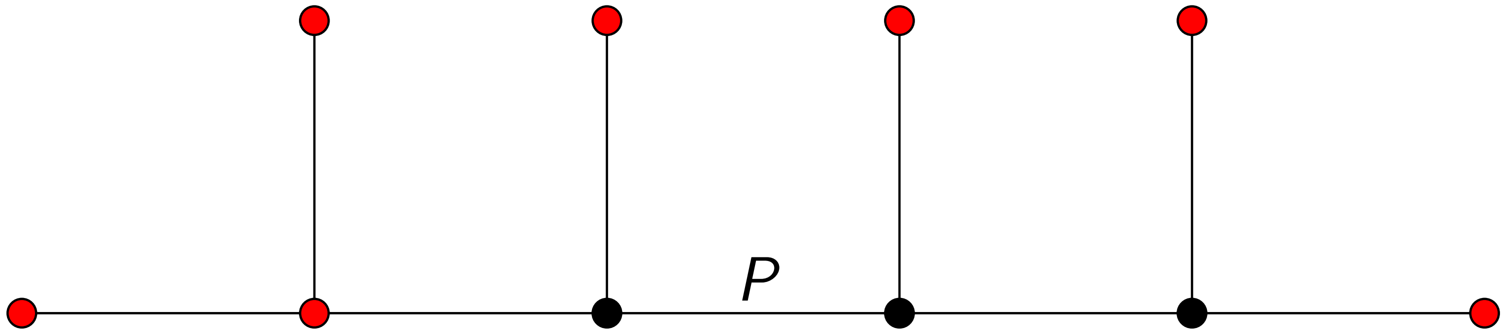

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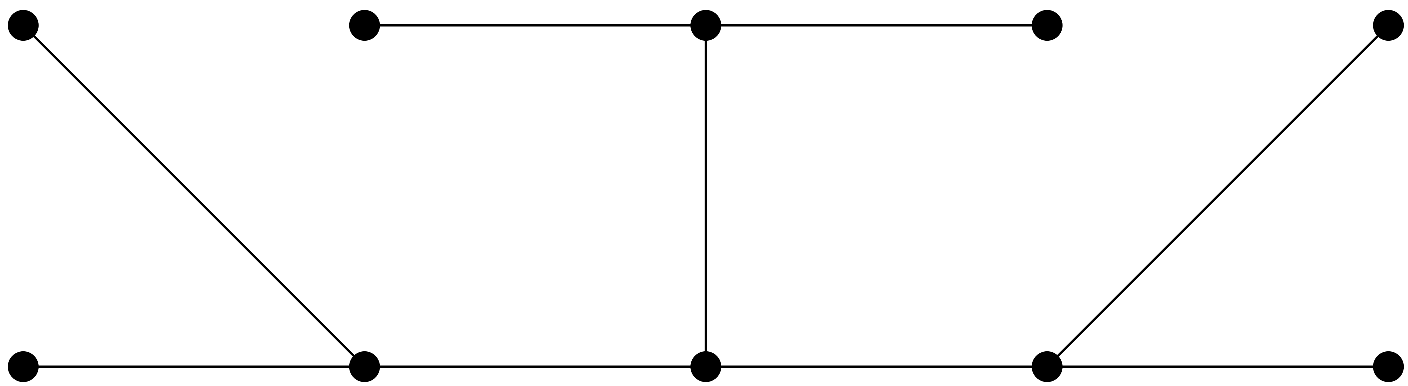

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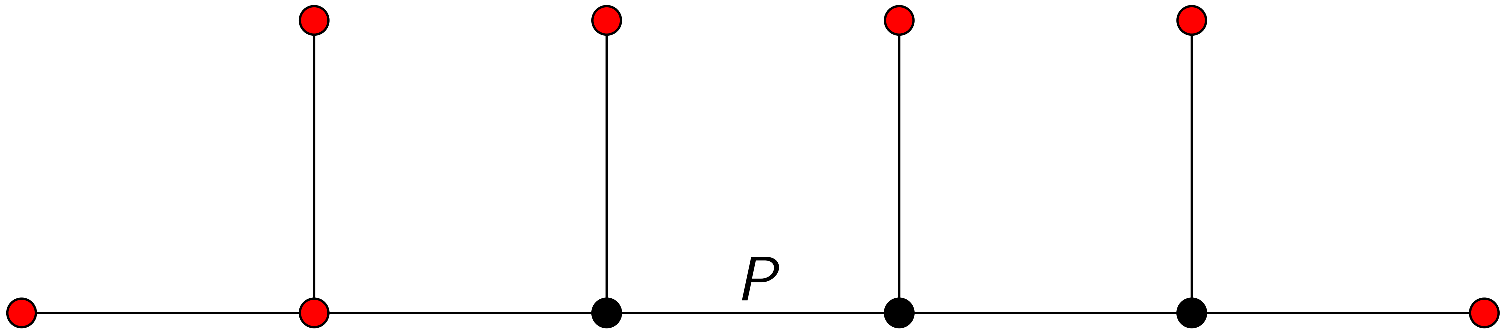

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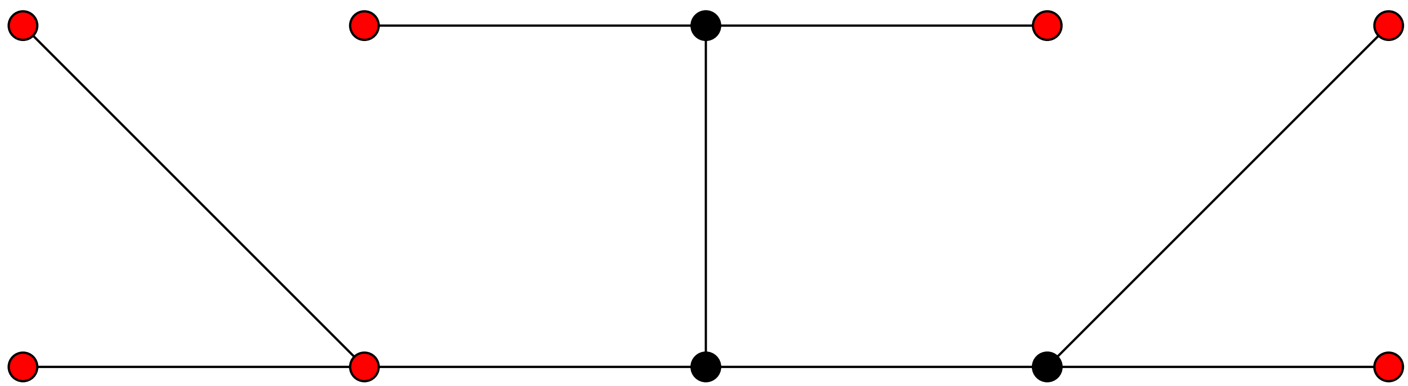

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Thank you for your attention!