Mathematical Exercises on Daly and Extensions

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Exponential Failures

Let $X \sim Exp(\lambda)$, a random variable for failure inter-arrival time.

$$\mu = \mathbb{E}(X) = \frac{1}{\lambda}$$

μ is the MTBF and λ is the error-rate.

- There is an error exactly at time t with probability:

$$\mathbb{P}(X = t) = \lambda e^{-\lambda t} \quad \text{(pdf)}$$

- There is at least one error before time t with probability:

$$\mathbb{P}(X \leq t) = 1 - e^{-\lambda t} \quad \text{(cdf)}$$

First-Order Approximation and Taylor Series

The Taylor series of a real or complex-valued function $f(x)$, *that is* infinitely differentiable at a real or complex number a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!} + \frac{f''(a)}{2!} + \dots$$

The Taylor series for the exponential function $f(x) = e^x$ at $a = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots$$

Therefore, at first-order $e^{\lambda W} = 1 + \lambda W + o(\lambda)$.
(First-order holds when W is small enough)

$o(\lambda)$: all terms in order of $O(\lambda^x)$, $x > 1$ (think of: *strictly smaller)*
$O(\lambda)$: all terms in order of $O(\lambda^x)$, $x \geq 1$ (think of: *smaller or equal)*

Methodology

Overhead and Waste

W: work of periodic pattern
W_{total}: total work of application
$\mathbb{E}(W)$: expected execution time of a pattern
$\mathbb{E}(W_{\text{total}})$: expected total execution time of application

$$\begin{aligned} \mathbb{E}(W_{\text{total}}) \approx \frac{W_{\text{total}}}{W} \cdot \mathbb{E}(W) &= (1 + \textsc{Overhead}) \cdot W_{\text{total}} \\ &= \frac{1}{1 - \textsc{Waste}} \cdot W_{\text{total}} \end{aligned}$$

where

$$\begin{aligned} \textsc{Overhead} &= \frac{\mathbb{E}(W)}{W} - 1 \\ \textsc{Waste} &= 1 - \frac{W}{\mathbb{E}(W)} \end{aligned}$$

E.x. $W = 100, \mathbb{E}(W) = 125 \Rightarrow \textsc{Overhead} = 25\%, \textsc{Waste} = 20\%$.
When platform MTBF μ is large, overhead and waste have same order.

Methodology

Steps:

- Compute expected execution time of a pattern $\mathbb{E}(W)$
- Derive OVERHEAD or WASTE from $\mathbb{E}(W)$
- Find optimal checkpointing period W (and other parameters)

Parameters

- C: Checkpoint
- R: Recovery
- D: Downtime (for fail-stop errors)
- V: Verification (for silent errors)
- λ^f: Fail-stop error rate
- λ^s: Silent error rate

Fail-stop Errors

- Compute $\mathbb{E}(W)$, assuming C, R are error-free

$$\mathbb{E}(W) = (1 - e^{-\lambda^f W})(\mathbb{E}^{\text{lost}} + D + R + \mathbb{E}(W)) + e^{-\lambda^f W}(W + C)$$

where $\mathbb{E}^{\text{lost}} = \int_0^\infty t\mathbb{P}(X = t | X < W) dt = \frac{\int_0^W t\lambda^f e^{-\lambda^f t} dt}{\mathbb{P}(X<W)}$.
Integrating by parts: $\mathbb{E}^{\text{lost}} = \frac{1}{\lambda^f} - \frac{W}{e^{\lambda^f W} - 1} \approx \frac{W}{2}$.

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$\Rightarrow \mathbb{E}(W) = W + C + \lambda^f W(\frac{W}{2} + D + R) + O((\lambda^f)^2 W^3)$

- Derive $\text{O\scriptsize VERHEAD}$ $\mathbb{H}(W)$

$$\mathbb{H}(W) = \frac{\mathbb{E}(W)}{W} - 1 = \frac{C}{W} + \frac{\lambda^f W}{2} + \lambda^f (D + R) + O((\lambda^f)^2 W^2)$$

- Optimization $W^* = \sqrt{\frac{2C}{\lambda^f}}$, $\mathbb{H}^* = \sqrt{2\lambda^f C} + o(\sqrt{\lambda^f})$

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- Optimization $W^* = \sqrt{\frac{2C}{\lambda^f}}$, $\mathbb{H}^* = \sqrt{2\lambda^f C} + o(\sqrt{\lambda^f})$

- Young's first-order approximation; Daly considered second order
- First-order stays the same when C, R are prone to errors

Silent Errors

Similar to fail-stop except:
- $\lambda^f \rightarrow \lambda^s$
- $\mathbb{E}^{\text{lost}} = W$
- $D = 0$
- V: verification

Silent Errors

Similar to fail-stop except:

- $\lambda^f \to \lambda^s$
- $\mathbb{E}^{\text{lost}} = W$
- $D = 0$
- V: verification

- Compute $\mathbb{E}(W)$, assuming C, R, V are error-free

$$\mathbb{E}(W) = W + V + (1 - e^{-\lambda^s W})(R + \mathbb{E}(W)) + e^{-\lambda^s W} C$$

$$\Rightarrow \mathbb{E}(W) = W + V + C + \lambda^s W(W + V + R) + O((\lambda^s)^2 W^3)$$

- Derive $\textsc{Overhead}$ $\mathbb{H}(W)$

$$\mathbb{H}(W) = \frac{\mathbb{E}(W)}{W} - 1 = \frac{V + C}{W} + \lambda^s W + \lambda^s(V + R) + O((\lambda^s)^2 W^2)$$

- Optimization $W^* = \sqrt{\frac{V+C}{\lambda^s}}$, $\mathbb{H}^* = 2\sqrt{\lambda^s(V + C)} + o(\sqrt{\lambda^s})$

Fail-stop + Silent

- Compute $\mathbb{E}(W)$, assuming C, R, V are error-free

$$\begin{aligned}\mathbb{E}(W) = (1 - e^{-\lambda^f W})(&\mathbb{E}^{\text{lost}} + D + R + \mathbb{E}(W)) \\ &+ e^{-\lambda^f W}(W + V + (1 - e^{-\lambda^s W})(R + \mathbb{E}(W)) \\ &\qquad\qquad\qquad + e^{-\lambda^s W} C)\end{aligned}$$

where $\mathbb{E}^{\text{lost}} = \frac{1}{\lambda^f} - \frac{W}{e^{\lambda^f W} - 1} \approx \frac{W}{2}$.

Fail-stop + Silent

- Compute $\mathbb{E}(W)$, assuming C, R, V are error-free

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where $\mathbb{E}^{\text{lost}} = \frac{1}{\lambda^f} - \frac{W}{e^{\lambda^f W} - 1} \approx \frac{W}{2}$.

$$\begin{aligned}\Rightarrow \mathbb{E}(W) = W + V + C + \lambda^f W(\frac{W}{2} + D + R) \\ + \lambda^s W(W + V + R) + O(\lambda^2 W^3)\end{aligned}$$

- Derive OVERHEAD $\mathbb{H}(W)$

$$\mathbb{H}(W) = \frac{\mathbb{E}(W)}{W} - 1 = \frac{V + C}{W} + (\frac{\lambda^f}{2} + \lambda^s)W + O(\lambda)$$

- Optimal $W^* = \sqrt{\frac{V+C}{\frac{\lambda^f}{2}+\lambda^s}}$, $\mathbb{H}^* = 2\sqrt{(\frac{\lambda^f}{2} + \lambda^s)(V + C)} + o(\sqrt{\lambda})$

Summary

First-order approximation:

	Fail-stop errors	Silent errors	Both errors
Pattern	$W+C$	$W+V+C$	$W+V+C$
Optimal W^*	$\sqrt{\frac{C}{\frac{\lambda^f}{2}}}$	$\sqrt{\frac{V+C}{\lambda^s}}$	$\sqrt{\frac{V+C}{\lambda^s+\frac{\lambda^f}{2}}}$
Optimal $\mathbb{H}^*$	$2\sqrt{\frac{\lambda^f}{2}C}$	$2\sqrt{\lambda^s(V+C)}$	$2\sqrt{\left(\lambda^s+\frac{\lambda^f}{2}\right)(V+C)}$

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Optimal $\mathbb{H}^*$	$2\sqrt{\frac{\lambda^f}{2}C}$	$2\sqrt{\lambda^s(V+C)}$	$2\sqrt{\left(\lambda^s+\frac{\lambda^f}{2}\right)(V+C)}$

Extensions to hierarchical checkpointing

- Disk checkpoint for fail-stop, in-memory checkpoint for silent [Benoit et al., IPDPS'16]
- Buddy/double checkpointing algorithm for fail-stop [Dongarra, Herault, Robert, IPDPS'13]

Observations

Observation 1

For a set $\mathcal{X}$ of independent error sources:

$$\mathbb{E}(W) = \underbrace{W + o_{\mathrm{ff}}}_{\text{error-free time}} + \sum_{x \in \mathcal{X}} \underbrace{\lambda^x W}_{\substack{\text{expected} \\ \#\text{ errors} \\ \text{of type } x}} \cdot \underbrace{\left(f_{\mathrm{re}}^x \cdot W + Constant \right)}_{\text{expected re-execution time}} + O(\lambda)$$

- o_{ff}: total overhead in a fault-free execution, i.e., $\sum$resilience ops.
- f_{re}^x: fraction of re-executed work in case of an type-x error.

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Observation 2

The optimal pattern satisfies:

$$\begin{aligned} W^* &= \sqrt{\frac{o_{\text{ff}}}{\sum_{x \in \mathcal{X}} \lambda^x f_{\text{re}}^x}} \\ \mathbb{H}^* &= 2\sqrt{o_{\text{ff}} \sum_{x \in \mathcal{X}} (\lambda^x f_{\text{re}}^x)} + O(\lambda) \end{aligned}$$

Observations

Example: Fail-Stop + Silent

$$\mathbb{E}(W) = W + \underbrace{V + C}_{o_{\text{ff}}} + \lambda^f W(\underbrace{\frac{1}{2}}_{f_{\text{re}}^f} W + D + R) + \lambda^s W(\underbrace{1}_{f_{\text{re}}^s} W + V + R) + O(\lambda)$$

$$W^* = \sqrt{\frac{o_{\text{ff}}}{\sum_{x \in \mathcal{X}} \lambda^x f_{\text{re}}^x}} = \sqrt{\frac{V + C}{\lambda^s + \frac{\lambda^f}{2}}}$$

$$\mathbb{H}^* = 2\sqrt{o_{\text{ff}} \sum_{x \in \mathcal{X}} (\lambda^x f_{\text{re}}^x)} + O(\lambda) = 2\sqrt{\left(\lambda^s + \frac{\lambda^f}{2}\right)(V + C)} + O(\lambda)$$

Observations

Exercise: Silent Error with Intermediate Verifications

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$$\mathbb{E}(W) = W + \underbrace{nV + C}_{o_{\text{ff}}} + \lambda^s W \Big(\underbrace{\frac{1}{2}\big(1 + \frac{1}{n}\big)}_{f^s_{\text{re}}} W + \frac{n+1}{2} V + R \Big) + O(\lambda)$$

$$W^* = \sqrt{\frac{o_{\text{ff}}}{\lambda^s f^s_{\text{re}}}} = \sqrt{\frac{nV + C}{\frac{1}{2}\left(1 + \frac{1}{n}\right)\lambda^s}}$$

$$\mathbb{H}^* = 2\sqrt{o_{\text{ff}} \lambda^s f^s_{\text{re}}} + O(\lambda) = 2\sqrt{\lambda^s \frac{1}{2}\left(1 + \frac{1}{n}\right)(nV + C)} + O(\lambda)$$

$$n^* = \sqrt{\frac{C}{V}}$$

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Exercise: Silent Error with Intermediate Verifications

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$$W^* = \sqrt{\frac{o_{\text{ff}}}{\lambda^s f^s_{\text{re}}}} = \sqrt{\frac{nV + C}{\frac{1}{2}\left(1 + \frac{1}{n}\right)\lambda^s}}$$

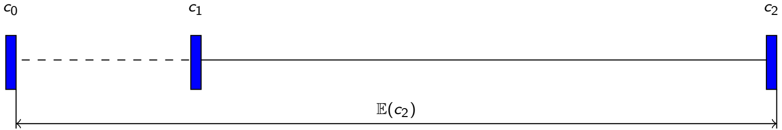
$$\mathbb{H}^* = 2\sqrt{o_{\text{ff}} \lambda^s f^s_{\text{re}}} + O(\lambda) = 2\sqrt{\lambda^s \frac{1}{2}\left(1 + \frac{1}{n}\right)(nV + C)} + O(\lambda)$$

$$n^* = \sqrt{\frac{C}{V}}$$

Extensions

- Using partial/inaccurate verifications to detect silent errors [Bautista-Gomez, HiPC'15]
- (Almost) optimal multi-level checkpointing for fail-stop errors [Presented at JLESC on Tuesday]

Dynamic Programming for Chains



How many intermediate checkpoints? What positions?

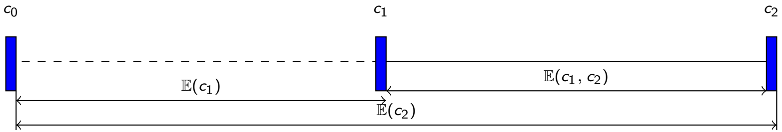
1. Find reusable sub-problem (and its optimal solution)
2. Find initialization case

Let $W_{c_1,c_2} = \sum_{i=c_1}^{c_2} W_i$

Objective

Compute optimal $\mathbb{E}(c_2)$

Dynamic Programming for Chains



$$\mathbb{E}(c_2) = \min_{0 \le c_1 < c_2} \{\mathbb{E}(c_1) + \mathbb{E}(c_1, c_2) + C\}$$

Initialization: $\mathbb{E}(0) = 0$

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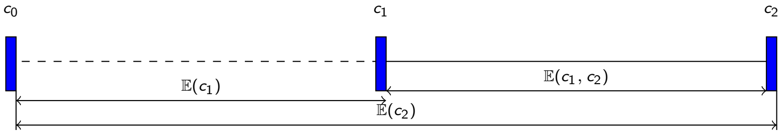

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Initialization: $\mathbb{E}(0) = 0$

$$\mathbb{E}(c_1, c_2) = (1 - e^{-\lambda W_{c_1,c_2}})\Big(\mathbb{E}^{\text{lost}}_{c_1,c_2} + R + \mathbb{E}(c_1) + \mathbb{E}(c_1, c_2)\Big) + e^{-\lambda W_{c_1,c_2}} W_{c_1,c_2}$$