Contradiction Example

Question: Show there is no rational number whose square is 2.

Using the proof by contradiction approach, suppose there is a rational number whose square is 2.

Some things to think about before progressing with the problem:

1. What is the formal definition of a rational number?
2. What are some properties of rational numbers?
3. Is there any statement in the definition or properties that could potentially lead to a contradiction?

Define this rational number as $r = \frac{m}{n}$, where m is an integer and n is a natural number, and $\frac{m}{n}$ is written in lowest terms.

Note: by properties of irreducible fractions $r = \frac{m}{n}$ in lowest terms implies that either either m or n is odd

Now $r^2 = 2 \Rightarrow (\frac{m}{n})^2 = 2$

$\Rightarrow m^2 = 2n^2$, this means that m^2 is even

m must be even since any even number times itself is even (m cannot be odd because any odd number times itself is odd).

Now we have $m^2 = 2n^2 \Rightarrow n^2 = (\frac{m}{2})(m)$

Note: $(\frac{m}{2})$ may be even or odd, but we know m is even. An even number times an even number, or an even times an odd number will always be an even number.

Thus n^2 is even, and using the same logic as above n must also be even.

Thus m and n are both even, but by definition of irreducible fractions, either m or n must be odd.

Thus $\frac{m}{n}$ cannot be in lowest terms. $\Rightarrow$ Contradiction!

Through proof by contradiction, we have shown that there is no rational number whose square is 2.