Iff Example

Let x be a real number. Let $\lceil x \rceil$ be the smallest integer greater than x (ceiling), and $\lfloor x \rfloor$ be the largest integer less than x (floor). Prove that $\lceil x \rceil = \lfloor x \rfloor$ if and only if x is an integer.

Note: the "iff" in the question, meaning that you have to prove both conditional statements:

1) if $\lceil x \rceil = \lfloor x \rfloor$ then x is an integer (forward)

2) if x is an integer then $\lceil x \rceil = \lfloor x \rfloor$ (backward)

Start with showing the forward conditional statement is true (i.e. - Show (1) is true).

By the definition of ceiling and floor, we can say: $\lceil x \rceil \geq \lfloor x \rfloor$

We know that $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ for any real number. If $\lceil x \rceil = \lfloor x \rfloor$ then $x = \lfloor x \rfloor$ which is an integer.

Thus we have shown the forward conditional statement is true.

Now show the backwards conditional statement is true (i.e. - Show (2) is true).

We know that x is an integer. Thus x is the smallest integer greater than x, that is $\lceil x \rceil = x$, and x is the largest integer smaller than x, that is $\lfloor x \rfloor = x$.

Thus $\lceil x \rceil = \lfloor x \rfloor$, and we have shown the backwards conditional statement is true.

We have thus shown that $\lceil x \rceil = \lfloor x \rfloor$ iff x is an integer.

Note: Each of the conditional statements were shown to be true using a direct proof method.