Contrapositive Example

Question: Show if $\mid x \mid\leq \frac{1}{n}$ for every positive integer n, then x = 0.

Let "A": $\mid x \mid\leq \frac{1}{n}$ for every positive integer

Let "B": x = 0

We need to show A ⇒ B , but showing this implication for every positive integer means an infinite number of cases.

So define the contrapositive: $\sim B \Rightarrow \sim A$:

$\sim$ A: there exists a positive integer n such that $\frac{1}{n} <\mid x \mid$

$\sim$ B: x $\neq$ 0

Define the contrapositive: If $x \neq 0$ then there exists a positive integer n such that $\frac{1}{n} <\mid x \mid$.

Since the contrapositive statement is equivalent to proving the original statement, let us prove the contrapositive statement. Essentially, we need to find this positive integer n that makes $\frac{1}{n} <\mid x \mid$ true.

We can answer this question using a direct proof

Let $N = \left\lceil \frac{1}{|x|} \right\rceil$

A common place to start thinking about proofs is at the end. We need to find an n, say N, such that $\frac{1}{N} <\mid x \mid$. Why not solve for N and start there? (Yielding the above equation) N needs to be an integer so take the ceiling.

Then for all $n > N$, we have $n > N \geq \frac{1}{|x|}$

Then doing some rearranging we can conclude: $\frac{1}{n} <\mid x \mid$

Thus we have shown that the contrapositive is true.

I initially wanted to solve this problem using cases but, it ends up requiring the need for the direct proof anyway. Here is how you would do it:

We do not know the exact value of x, only that it cannot be zero. Let us look at the cases of x:

1) $x > 1, x < -1$

Any positive integer n would satisfy $\frac{1}{n} <\mid x \mid$, since $\frac{1}{n} \leq 1 < \frac{1}{|x|}$

2) $-1 \leq x \leq 1, x \neq 0$

Due to the density of the real line, given some real number, x, we can always find a smaller rational number. To figure out what the value of the rational number should be, we need the direct proof

Something to also consider: you could potentially use induction to answer the question, without using the contrapositive since you need to show a statement is true for all $n \in N$