Model theory of abelian C^*-algebras

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Stone duality

Let ZDComp and Bool denote the categories of zero dimensional compact Hausdorff spaces and Boolean algebras respectively.

Stone Duality

- There is a contravariant function ZDComp $\to$ Bool that is an anti-equivalence of categories.
- It sends $X \in$ ZDComp to the Boolean algebra $CL(X)$ of clopen subsets of X.
- The inverse map sends $\mathbb{B} \in$ Bool to the space of ultrafilters on $\mathbb{B}$.
- Given a continuous function $f : X \to Y$ between elements of ZDComp, the functor sends it to the Boolean algebra morphism $\tilde{f} : CL(Y) \to CL(X)$ given by $\tilde{f}(C) := f^{-1}(C)$.

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Gelfand duality

Let Comp and C^*-Alg denote the categories of compact Hausdorff spaces and (unital) C^*-algebras respectively.

Gelfand Duality

- There is a contravariant function Comp $\rightarrow \mathrm{C}^*$-Alg that is an anti-equivalence of categories.
- It sends $X \in$ Comp to the unital C^*-algebra $C(X)$ of continuous functions on X.
- The inverse map sends $A \in \mathrm{C}^*$-Alg to the space $\Sigma(A)$ of maximal ideals on A.
- Given a continuous function $f : X \rightarrow Y$ between elements of ZDComp, the functor sends it to the C^*-algebra morphism $\tilde{f} : C(Y) \rightarrow C(X)$ given by $\tilde{f}(h) := h \circ f$.

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Real rank 0 abelian C^*-algebras

Definition (Cheating)

We say that $C(X)$ is *real rank* 0 if $X \in$ ZDComp.

Let RRZ denote the category of real rank 0 abelian C^*-algebras. We thus have a covariant functor RRZ $\to$ Bool given by composing the restriction of the inverse Gelfand functor to RRZ with the Stone functor; this functor is an equivalence of categories.

More concretely: when $X \in$ ZDComp, $C(X)$ is the closed linear span of its projections and $P(C(X))$ is isomorphic to $CL(X)$ as a Boolean algebra.

Let us call this functor the *forgetful functor*.

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A categorical perspective on ultraproducts

Lemma

Let $(A_i : i \in I)$ be a family of C^*-algebras or Boolean algebras and let $\mathcal{U}$ be a nonprincipal ultrafilter on I. For $J \in \mathcal{U}$, set $A_J := \prod_{i \in J} A_i$. Then:

- *The family (A_J, π_{JK}) is a directed family, where $\pi_{JK} : A_J \to A_K$ is the natural projection map (for $J \supseteq K$).*
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Ultracoproducts of compact spaces

- Suppose that X_i is a compact Hausdorff space for $i \in I$ and that $\mathcal{U}$ is a nonprincipal ultrafilter on I.
- Question: What is $\Sigma(\prod_{\mathcal{U}} C(X_i))$?
- Set $X_J := \coprod_{i \in J} X_i = \beta(\oplus_{i \in J} X_i)$.
- Applying the Gelfand functor to the isomorphism $\prod_{\mathcal{U}} C(X_i) \cong \varinjlim C(X)_J$ yields $\Sigma(\prod_{\mathcal{U}} C(X_i)) \cong \varprojlim X_J$.
- This space is called the *ultracoproduct of the family* (X_i) *with respect to the ultrafilter* $\mathcal{U}$ and is denoted $\coprod_{\mathcal{U}} X_i$.
- If $X_i = X$ for all i, then this is the *ultracopower of* X, denoted $X^{\mathcal{U}}$.
- Another perspective: if $p : X \times I \to I$ is projection onto the second coordinate, then $X^{\mathcal{U}} \cong (\beta p)^{-1}(\mathcal{U})$.

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Ultraproducts of real rank 0 abelian C^*-algebras

- Suppose that each X_i is totally disconnected.
- Since Bool is closed under ultraproducts and ultraroots, it follows from the forgetful functor that so is RRZ, whence RRZ is an axiomatizable class, say axiomatized by T_{rr0}.
- The forgetful functor sends $C(\coprod_{\mathcal{U}} X_i) \cong \prod_{\mathcal{U}} C(X_i) = \varinjlim C(X)_J$ to $CL(\coprod_{\mathcal{U}} X_i) \cong \prod_{\mathcal{U}} CL(X_i) = \varinjlim CL(X)_J$.
- In particular, if X and Y are totally disconnected, then

$$C(X)^{\mathcal{U}} \cong C(Y)^{\mathcal{V}} \Leftrightarrow CL(X)^{\mathcal{U}} \cong CL(Y)^{\mathcal{V}},$$

whence, by Keisler-Shelah, we have

$$C(X) \equiv C(Y) \Leftrightarrow CL(X) \equiv CL(Y).$$

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The general situation

Given a bounded distributive lattice B, one may form a topological space X_B from the maximal filters on B in the same way.

Theorem (Gurevic)

1. *X_B is always a compact T_1-space.*
2. *There is an elementary class of normal bounded distributive lattices such that, if B is normal, then X_B is Hausdorff.*
3. *Conversely, if X is a compact Hausdorff space and B is a closed set base for X (closed under finite union and intersection), then B is normal and X is naturally homeomorphic to X_B.*
4. *If each B_i is a closed set base for X_i, each X_i compact Hausdorff, and $B := \prod_{\mathcal{U}} B_i$, then $\coprod_{\mathcal{U}} X_i$ is naturally homeomorphic to X_B.*

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Model-theoretic reminders

Let T be a (classical or continuous) L-theory.

- T is *model-complete* if every embedding between models is elementary.
- If T' is also an L-theory, then T' is *the model companion of T* if T' is model-complete and $T_\forall = T'_\forall$.
- If $M \subseteq N$ are L-structures, then M is *existentially closed (e.c.) in N* if there is an embedding $N \hookrightarrow M^{\mathcal{U}}$ that is the identity on M.
- $M \models T$ is an *existentially closed model of T* if M is e.c. in N whenever $M \subseteq N \models T$.
- T has a model companion if and only if the class of e.c. models of T is elementary, in which case this elementary theory is the model companion of T.
- The model companion of T has QE (and is called the *model completion of T*) if and only if $T_\forall$ has the *amalgamation property*.

Model theory of T_{bool}

Let T_{bool} denote the (classical) theory of boolean algebras.

Theorem

1. *T_{bool} has a model completion, namely the theory of atomless Boolean algebras.*
2. *This model completion is $\aleph_0$-categorical with unique countable model $CL(2^{\mathbb{N}})$.*

Corollary

T_{rr0} is $\forall\exists$-axiomatizable.

Proof.

T_{bool} is $\forall\exists$-axiomatizable (as it is model-complete), so closed under limits of chains. Applying the forgetful functor, so is T_{rr0}. □

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$\mathrm{Th}(C(2^{\mathbb{N}}))$

Theorem

$\mathrm{Th}(C(2^{\mathbb{N}}))$ *is* $\aleph_0$*-categorical and is the model completion of* T_{ab}*.*

Proof.

- If $C(X) \equiv C(2^{\mathbb{N}})$, then $C(X)$ has real rank 0, so $CL(X) \equiv CL(2^{\mathbb{N}})$, whence $CL(X) \cong CL(2^{\mathbb{N}})$ and thus $C(X) \cong C(2^{\mathbb{N}})$.
- Since real rank 0 is $\forall\exists$-axiomatizable and $(T_{\mathrm{ab}})_{\forall} = \mathrm{Th}_{\forall}(C(2^{\mathbb{N}}))$, we have that e.c. models of T_{ab} are real rank 0. Thus, if $C(X)$ is an e.c. model of T_{comp}, then $CL(X)$ is an e.c. model of T_{bool}, whence $CL(X) \equiv CL(2^{\mathbb{N}})$ and thus $C(X) \equiv C(2^{\mathbb{N}})$. The converse is similar, so $\mathrm{Th}(C(2^{\mathbb{N}}))$ is the model companion of T_{ab}.
- Since T_{ab} has the amalgamation property (fiber products), $\mathrm{Th}(C(2^{\mathbb{N}}))$ has QE and is the model completion of T_{ab}.

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Projectionless abelian C^*-algebras and continua

Exercise

$C(X)$ is projectionless if and only if X is connected.

Recall that a *continuum* is a connected, compact Hausdorff space. So the Gelfand functor sends the subcategory of projectionless abelian C^*-algebras onto the category of continua.

Lemma

The class of projectionless abelian C^*-algebras forms a universally axiomatizable class.

Proof.

Immediate from the semantic test. Or here is the projectionless axiom:

$$\sup_{\|f\|=1} \min(2\|1 - ff^*\| \dot{-} 1, 1 \dot{-} 4\|ff^* - (ff^*)^2\| = 0.$$

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A unique universal theory of continua

Let T_{cont} denote the theory axiomatizing $C(X)$ for X a *nondegenerate* continuum (an $\forall\exists$-theory).

Theorem (K.P. Hart)

If $C(X), C(Y) \models T_{\mathrm{cont}}$, then $\mathrm{Th}_\forall(C(X)) = \mathrm{Th}_\forall(C(Y))$.

The proof proceeds by constructing a function $C \to B^{\mathcal{U}}$, where B and C are lattice bases of closed sets for X and Y respectively, that satisfies a criteria that ensures that there is a continuous surjection $X^{\mathcal{U}} = X_{B^{\mathcal{U}}} \to X_C = Y$.

Bankston's work on co-existentially closed continua

- Bankston, in a series of papers, has studied the model theory of continua in a dual form, sometimes relying on the first-order theory of lattice bases.
- He thus prefixes usual model theoretic jargon with "co-". For example, he talks about *co-existentially closed continua*. Here is a sample result:

Theorem (Bankston)

Co-existentially closed continua are hereditarily indecomposable.

Proof.

In our terminology, he proves that there is an $\forall\exists$-theory $T \supseteq T_{\text{cont}}$ such that $C(X) \models T$ if and only if X is hereditarily indecomposable. Now use a result of Bellamy: every metrizable continuum is a surjective image of a metrizable hereditarily indecomposable continuum. □

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In our terminology, he proves that there is an $\forall\exists$-theory $T \supseteq T_{\text{cont}}$ such that $C(X) \models T$ if and only if X is hereditarily indecomposable. Now use a result of Bellamy: every metrizable continuum is a surjective image of a metrizable hereditarily indecomposable continuum. $\square$

Bankston's work on co-existentially closed continua

- Bankston, in a series of papers, has studied the model theory of continua in a dual form, sometimes relying on the first-order theory of lattice bases.
- He thus prefixes usual model theoretic jargon with "co-". For example, he talks about *co-existentially closed continua*. Here is a sample result:

Theorem (Bankston)

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The pseudo-arc

Definition

A continuum X is *chainable* if, given any open cover $\mathcal{V}$ of X, there is an open cover $(U_1, \ldots, U_m)$ of X such that:

- each U_i is contained in an element of $\mathcal{V}$, and
- $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Theorem (Bing)

There is a unique (up to homeomorphism) continuum that is both hereditarily indecomposable and chainable. This continuum is called the pseudo-arc, denoted $\mathbb{P}$.

Question (Bankston)

Is $\mathbb{P}$ a co-existentially closed continuum? I.e. is $C(\mathbb{P})$ an e.c. model of T_{cont}?

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Infinitary axiomatizability of chainability

Theorem (Eagle-G.-Vignati)

The class of models $C(X)$ of T_{cont} such that X is chainable is uniformly definable by a sequence of universal types.

This means that there is a family $(\varphi_{k,m}(\vec{x}_k))$ of existential formulae, where $|\vec{x}_k| = k$, such that, given $C(X) \models T_{\text{cont}}$, we have that X is chainable if and only if $C(X) \models \sup_{\vec{x}_k} \inf_m \varphi_{k,m}(\vec{x}_k) = 0$ for all k.

We could also say that chainability is a sup $\bigvee$ inf-property of continua.

The pseudo-arc is a co-existentially closed continuum

Corollary (Eagle-G.-Vignati)

There is an e.c. model $C(X)$ of T_{cont} such that X is chainable. Consequently, $X \cong \mathbb{P}$.

Proof.

By the previous theorem and the Hart result, one can use Robinson forcing to obtain an e.c. model of T_{cont} that is chainable. $\square$

The pseudo-arc is descriptive set-theoretically generic (the subset of subcontinua of $[0,1]^{\mathbb{N}}$ homeomorphic to $\mathbb{P}$ is dense G_δ). This result shows that $\mathbb{P}$ is is also model-theoretically generic (namely generic for Robinson forcing).

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T_{cont} does not have the amalgamation property

Observation (Hoehn)

T_{cont} does not have the amalgamation property.

Proof.

- Let $f, g : [0,1] \to \mathbb{S}^1$ be given by $f(x) = e^{2x\pi i}$ and $g(y) = e^{2(y+1)\pi i}$.
- Suppose W is compact and $r, s : W \to [0,1]$ are such that $f \circ r = g = \circ s$.
- Let $A = r^{-1}([0, \frac{1}{2}]) \cap s^{-1}([\frac{1}{2}, 1])$ and $B = r^{-1}([\frac{1}{2}, 1]) \cap s^{-1}([0, \frac{1}{2}])$.
- Suppose that $w \in A \cap B$. Then $r(w) = s(w) = \frac{1}{2}$, so $f(r(w)) = e^{\pi i} \neq e^{2\pi i} = g(s(w))$, a contradiction.
- It follows that $A \cap B = \emptyset$, so W is disconnected. □

Failure of quantifier elimination

Corollary

If X is a nondegenerate continuum, then $C(X)$ does not have quantifier elimination.

Proof.

Since there is a unique universal theory of nondegenerate continua, if $C(X)$ had QE, then $\mathrm{Th}(C(X))$ would be the model completion of T_{cont}, whence T_{cont} would have the amalgamation property. □

It was known that if X is compact and $C(X)$ has QE, then either $X \cong 2^{\mathbb{N}}$ or X is connected. Thus:

Corollary

$C(2^{\mathbb{N}})$ is the only abelian C^*-algebra with QE.

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Is there a model companion?

Theorem

$C(\mathbb{P})$ is the only possible model of T_{cont} that is model complete. If this is indeed the case, then $\text{Th}(C(\mathbb{P}))$ *is the model companion of T_{cont}.*

Proof.

If $C(X)$ is model-complete, then $\text{Th}(C(X))$ is the model-companion of T_{cont}. But $C(\mathbb{P})$ is an e.c. model of T_{cont}, so $C(\mathbb{P}) \models \text{Th}(C(X))$. $\square$

Further observations about $C(\mathbb{P})$

Theorem (G.-last week, so grain of salt please)

1. $C(\mathbb{P})$ *is an enforceable model of* T_{cont}*, that is, it is the only model of* T_{cont} *that can be produced by Robinson forcing.*
2. $C(\mathbb{P})$ *has every* $\sup\bigvee\inf$*-property of continua.*
3. $C(\mathbb{P})$ *is the prime model of its theory.*

Remark

A consequence of (1) is that $\mathbb{P}$ is a continuous image of every co-e.c. continuum. This follows from a result of Bellamy: $\mathbb{P}$ is a continuos image of every hereditarily indecomposable continuum. Can one give a model-theoretic proof of this more general result?

A digression on $\mathcal{R}$

Let $\mathcal{R}$ denote the unique separable hyperfinite II_1-factor. The *Connes Embedding Problem* (CEP) asks whether or not every II_1-factor is $\mathcal{R}^{\mathcal{U}}$-embeddable.

Theorem (G.)

1. *$\mathcal{R}$ is an enforceable model of* $\text{Th}_\forall(\mathcal{R})$*.*
2. *$\mathcal{R}$ possess every every* $\sup\bigvee\inf$*-property of embeddable II_1-factors.*
3. *CEP is equivalent to the statement that $\mathcal{R}$ possess every every* $\sup\bigvee\inf$*-property of II_1-factors.*

Corollary

Many properties of II_1-factors are not $\sup\bigvee\inf$*-axiomatizable, e.g. property (T), proper fundamental group, non-*(Γ)*...*

Model companions in operator algebras: a summary

Summary

The following classes do not have a model companion:

- Tracial von Neumann algebras (G.-Hart-Sinclair)
- C^*-algebras (Eagle-Farah-Kirchberg-Vignati)
- Operator systems (G.-Lupini)
- Operator spaces (Lupini)

The existence of a model companion is unknown for the following classes:

- Projectionless abelian C^*-algebras
- Projectionless C^*-algebras
- Stably finite C^*-algebras

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