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It's Not Over 'til the Fat Lady Sings: Game-Theoretic Analyses of Sports Leagues

Robert Driskill[1] and John Vrooman[1]

Abstract

There have been major disputes about the appropriate game-theoretic analyses of sports leagues. The basic sports-league model assumes a two-team league where each team is seen as a monopolist in its product market and a passive price taker in a duopsony talent market. For simplicity, these models assume talent supply is either perfectly inelastic or perfectly elastic. It has been argued that perfectly inelastic supply is ill suited for duopsony analysis. This article solves the problem of multiple equilibria by proposing a selection criterion that finds a unique equilibrium in a duopsony limit game as talent supply approaches perfect inelasticity.

Keywords

game theory, sports leagues, sports models, duopsony, Nash equilibrium, revenue sharing

> It ain't over 'til it's over.
>
> Yogi Berra

Introduction

Following Fort and Quirk (1995) and Vrooman (1995) (*QFV*), the basic theoretical sports-league model has generally assumed a two-team league. While a two-team

[1] Vanderbilt University, Nashville, TN, USA

Corresponding Author:
John Vrooman, Vanderbilt University, VU Station B #351819, Nashville, TN 37235, USA.
Email: john.vrooman@vanderbilt.edu

league can be analyzed as if each team is a price taker in the market for talent, the underlying motivation for the assumption of price-taking behavior is the existence of many small, atomistic participants.[1] Consequently, a game-theoretic analysis of two-team leagues has seemed desirable since the critiques of Szymanski (2004) and Szymanski and Kesenne (*SK*, 2004). Szymanski (2004, p. 123) summarized the basic problem: "competition between two teams cannot reasonably be treated as analogous to perfect competition."[2]

What has made game-theoretic duopsony analyses problematic for these models has been the use of two simplifying assumptions concerning the elasticity of overall talent supply. In the closed market *American* case, the supply of talent is perfectly inelastic, and if talent is fully employed then one team's talent gain results in another's zero-sum talent loss. If this assumption was true, then there was no way for one team to choose talent independent of the opposing team's choice. Szymanski and Kesenne (2004, p. 240) described the problem this way:

> The choice of one team automatically constrains the other in a two-team model, and so every possible choice of talent is a Nash equilibrium, because the other team has only one feasible response, which is therefore "best." However, this clearly makes little sense as an economic model.

In the open-market *European* model, talent supply is perfectly elastic at an exogenous price and one club's talent choice has no effect on that of its opponent. But if talent is perfectly elastic, then there is no need for a game-theoretic approach to talent choice because the price of talent is exogenous. Despite Szymanski's clear identification of the potential problem of treating two-team leagues as perfectly competitive, two-team models have generally assumed that the teams were price takers in the market for talent.[3]

This article argues that the fundamental problem with the perfectly inelastic supply case is multiple equilibria. The solution proposed in this article follows game theory's usual path taken in the presence of multiple equilibria by using a criterion that chooses one solution from the many possible equilibria.[4] The proposed unique solution is the limit of a solution to a duopsony game with imperfectly inelastic talent supply as the talent supply function approaches perfect inelasticity.

The argument begins with the specification of a conventional team revenue model in duopsony form combined with a one-parameter specification of an inverse supply function for total talent.[5] As this one parameter continuously approaches zero, the talent supply function varies continuously from a smooth upward slope toward the limiting case of a "reverse-L" function. This limiting specification yields an inverse supply function with a perfectly elastic segment up to a fixed amount of talent, where it then becomes perfectly inelastic. Our proposed selection criterion is the limit of the above-specified duopsony game as the talent supply function approaches the "reverse-L" shape.[6]

This approach keeps the customary assumption that the strategy space for the teams is their talent level, rather than their talent expenditure and it allows a comparison of the European perfectly elastic supply and American perfectly inelastic supply as a matter of market size. For markets "large enough" (explained more precisely later), equilibrium in the limiting case occurs with a full employment of a fixed supply of talent, while for smaller markets, equilibrium occurs along the perfectly elastic segment of the inverse talent supply function. This allows a comparison of the differences and similarities between the duopsony approach and the price-taking approach. The findings are that the limiting case is consistent with qualitative results about the relationship between relative market size and team success, and they confirm the paradoxical effects of revenue sharing on competitive balance.[7] In the end, however, the duopsony model leads to major differences in the equilibrium price of talent compared to the price-taking model.

The plan of the article is as follows. First, a brief argument is made that the problem with the perfectly inelastic talent supply is one of multiple equilibria. This is followed by a general formulation of a duopsony model that demonstrates the general differences between this duopsony approach and price-taking approaches. The model is formulated with specific functional forms for revenue and solved for examples of imperfectly elastic talent supply. This illustrates how solutions change as the inverse supply function approaches the reverse-L shape and motivates the analytic conclusions we draw from the limiting case. It is then shown in the limiting case how market size affects whether or not the equilibrium is "European" or "American," and how competitive balance is affected by revenue sharing. The article concludes with the discovery of a previously unnoticed result that revenue sharing with revenue functions that are homogenous of degree zero in team talent levels leads overall to shrinking *league* talent levels.

Multiple Nash Equilibria

The key issue with perfectly inelastic talent is that multiple equilibria exist along identically transposed reaction curves of the two teams. To see this issue, consider an inverse supply function for a league where the cost of hiring more talent is constant over a range of talent T^*, but at T^* the supply of talent becomes perfectly inelastic (vertical). This "reverse L" inverse supply function is shown in Figure 1. T is total talent, c is the cost per unit of talent, and T^* is normalized to unity. For total talent between 0 and 1, the cost of an additional unit of talent is also normalized to 1. For $T = 1$, one unit of talent is supplied for any $c \geq 1$. Now consider the best responses of teams to each other's choices of talent. Let t_1 and t_2 denote the choices of talent by Team 1 and Team 2, respectively. Teams would anticipate that the wage for talent would subsequently adjust to clear the market as follows. If $t_1 + t_2 \leq 1$, then $c = 1$; and if $t_1 + t_2 > 1$, then $c = +\infty$.[8]

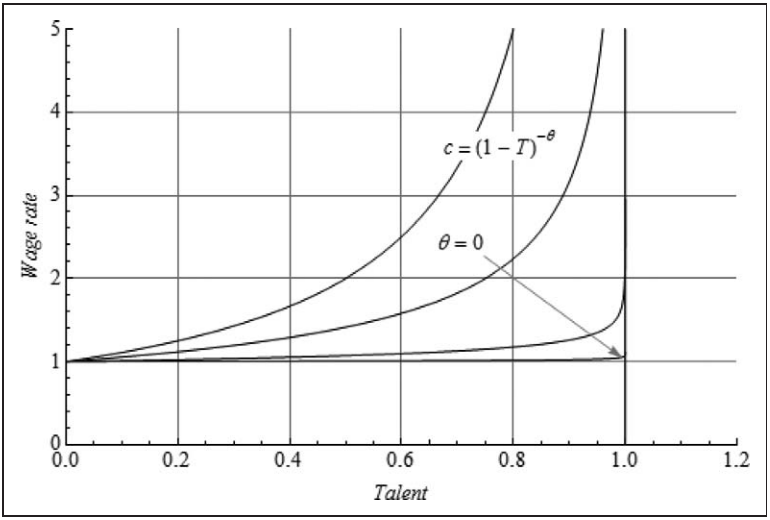

Figure 1. Talent supply.

If the marginal revenue to a duopsonist of hiring an additional unit of talent is greater than 1, then there are multiple equilibria associated with this inverse supply. For example, if Team 1 hires t_1 units of talent, Team 2's best response is to hire $1 - t_1$. This leaves the cost of talent at the minimum for both teams. Any attempt to hire more than $(1 - t_1)$ would push the cost of talent to $+\infty$ and hiring less than $(1 - t_1)$ leaves profits on the table. Conversely, if Team 2 hires t_2 units of talent, then Team 1's best response is to hire $(1 - t_2)$ units by the same reasoning. This is therefore a Nash equilibrium. Any allocation that exactly divides the unit of available talent is of course a Nash equilibrium, and so there exists an infinity of equilibria. This creates a problem for the analysis of this game, because theory gives no prediction as to which of these equilibria are expected in actual situations. This requires a "refinement" or selection criterion that chooses one of these many possibilities.[9]

The Model

The exposition of the model begins with a general structural specification that notes the general differences in solutions for the duopsony model compared to solutions with the assumption of price-taking behavior. The revenue model of Dietl, Lang, and Werner (2009) is then combined with a parametric specification of talent supply elasticity that has as a limiting case segments of perfectly elastic and perfectly inelastic talent supply. The model is then solved for a sequence of talent supply parameters to illustrate how solutions change as the talent supply function approaches the limiting inverse-L shape.

The limiting case solution is compared to the price-taking solution, and it is shown that the revenue-sharing paradox holds for both models. More importantly, it is also shown that with the homogenous revenue function used here (and in many other places), revenue sharing shrinks the size of the league in both price-taking and duopsony models. The most striking difference is that for the equilibrium price of talent with "large enough" market size, the price of talent is above the reservation wage in the price-taking case, but equal to the reservation wage in the duopsony case.

General Structural Specification

Revenues for team i are assumed to be a function of the expected win probability w_{ij} for a match with team j,

$$R_i = R_i(w_{ij}).$$

Expected win percentages are in turn determined by a contest success function (CSF) that depends on the levels of talent of both teams:

$$w_{ij} = w_{ij}(t_i, t_j),$$

where t_i is the talent used by team i. This implies that team i's revenue is a function of talent levels of both teams:

$$R_i = R_i(t_i, t_j),$$

which in turn implies marginal revenue product is a function of each team's talent level:

$$MRP_i = MRP_i(t_i, t_j),\ i = 1, 2,\ \ i \neq j.$$

Finally, an inverse talent supply function expresses total talent T that is supplied as a nondecreasing function of the wage c:

$$c = \chi(T),\ \ \chi' \geq 0,\ \ T = t_1 + t_2.$$

The standard (price-taking) model. In the nonduopsony case, as found in virtually all models before Madden (2011), teams are assumed to take c as fixed and exogenous. The first-order conditions are

$$MRP_1(t_1, t_2) = c;\ MRP_2(t_1, t_2) = c. \tag{1}$$

These two first-order conditions then determine t_1 and t_2 as functions of c:

$$t_1 = t_1(c);\ t_2 = t_2(c). \tag{2}$$

If the model is *open* or "European," then these two equations determine the equilibrium because every other endogenous variable can then be solved recursively. If the model is *closed* or "American" (if there is wage responsiveness of talent supply), then equilibrium is determined by adding up the individual demands for talent and equating to total supply:

$$t_1(c) + t_2(c) = T(c). \tag{3}$$

This determines the equilibrium wage, which can then be used to determine t_1 and t_2 and all other endogenous variables.

Whether the talent market is open or closed, the assumption that firms take c as the parametric price of talent insures that in equilibrium:

$$MRP_1(t_1, t_2) = MRP_2(t_1, t_2). \tag{4}$$

This equality of marginal revenue product between the two teams ensures that the determination of $t_{1/}t_2$ is independent of the responsiveness of total talent to its wage. This means that expected win percentages are in turn independent of the responsiveness of total talent to its wage rate. Furthermore, this equality of marginal revenue product is also the starting point in all models for the analysis of the effect of market size on competitive balance and the effect of revenue sharing on competitive balance.[10]

Duopsony. If in fact teams behave strategically with respect to the market for talent, then the first-order conditions are:

$$MR_1(t_1,\ t_2) = \overbrace{\chi(T)}^{c} + t_1\chi'(T); \tag{5.1}$$

$$MR_2(t_1,\ t_2) = \overbrace{\chi(T)}^{c} + t_2\chi'(T). \tag{5.2}$$

These first-order conditions (best-response functions) jointly determine t_1 and t_2, from which values of all other endogenous variables can be computed. In contrast with the nonduopsony approaches, the price of talent is not taken as given by the teams when they make decisions about talent levels. Instead, the effect of talent-level choices on the wage is considered, and thus the determination of t_1 and t_2 is not independent of the talent supply function. Furthermore, there is no implication that marginal revenues will necessarily be equal. In general, any analyses of the effects of market size or the effects of revenue sharing on competitiveness will be more complicated (but realistic) compared to the analyses of models with price-taking behavior. It can be shown that the simplicity of the price-taking analysis for some issues can be maintained in the limiting case of perfectly inelastic talent supply.

Specific Models

For the general specification of the previous section, not much can definitively be said about the questions of interest to sports economics. With this in mind, consider a model with a specific inverse talent supply function that smoothly approaches the reverse-L shape as one parameter approaches zero. A more specific revenue function model is necessary to better illustrate the difference between duopsony and price-taking models.

Unfortunately, even this specific model cannot be solved analytically for arbitrary values of the key parameter of the talent supply function. Consequently, computational solutions are needed to provide a heuristic to see both the possible differences a duopsony approach generates vis-à-vis a price-taking approach and to the concept of the limiting solution as a selection criterion for the perfectly inelastic supply-of-talent case.

A specific talent supply function. The supply of talent is specified by assuming that the price of talent is an increasing function of the total amount of talent, specifically an upward-sloping inverse supply function for talent. This function can parametrically encompass at the limit a perfectly inelastic supply of talent. Let c denote the cost of hiring a unit of talent, and let total talent be denoted as $T = t_1 + t_2$. The increasing inverse supply function is as follows:

$$c(T) = (1 - T)^{-\theta}; \theta > 0. \tag{6}$$

As $\theta \rightarrow 0$ in Figure 1, the inverse supply function gets closer and closer to an inverse-L shape which is perfectly inelastic at $T = 1$ and perfectly elastic for $T \in [0, 1]$. This implies marginal cost function:[11]

$$MC_i = c(T) + t_i c'(T) = (1 - T)^{-\theta-1}(1 - T + \theta t_i). \tag{7}$$

For any nonzero value of θ, this function has elasticities of talent supply that range continuously from infinity at $T = 0$ to zero at $T = 1$. It is also important to note that MC in Equation 7 lies everywhere above talent supply $c(T)$ in Equation 6.

Revenue function. A standard model of the demand side in sports economics has become one in which a team owner has a revenue function that depends on the product of own market size m_i and the quality of a match q_{ij} with opponent with market size m_j:

$$R_i = m_i q_{ij,}\ m_i > 0, i, j = 1, 2, i \neq j, \tag{8}$$

where quality is a quadratic function of the probability of winning w_{ij}:[12]

$$q_{ij} = w_{ij} - \frac{1}{2} w_{ij}^2. \tag{9}$$

The probability of winning is determined by a CSF that depends on the amount of talent purchased by team i and team j:

$$w_{ij} = \frac{t_i}{t_i + t_j}. \tag{10}$$

This *CSF* implies that the relative probabilities of winning are identical to relative talent levels $w_1/w_2 = t_1/t_2$.[13] The revenue function for team i in terms of t_i and t_j becomes:[14]

$$R_i = m_i \left[\frac{\frac{1}{2}t_i^2 + t_i t_j}{\left(t_i + t_j\right)^2} \right], i, j = 1, 2, i \neq j, \tag{11}$$

which in turn implies the marginal revenue product for team i in terms of t_i and t_j:[15]

$$MRP_i = \frac{m_i t_j^2}{\left(t_i + t_j\right)^3}. \tag{12}$$

In a price-taking model, the two first-order conditions for each team can be interpreted as demand curves for talent, because each team's choice is conditional on the other team's choice. But because t_j shows up in the first-order condition for team i, these first-order conditions can also be interpreted as best-response functions. For any value of t_j, the first-order condition for Team i tells us the best choice of t_i. In this sense, without an analysis of duopsony, game theory and price-taking behavior have previously led to identical conclusions.

Profit maximization. Profit maximization from Equations 7 and 12 leads to:

$$MRP_i = MC_i = \frac{m_i t_j^2}{T^3} = (1 - T)^{-\theta - 1}(1 - T + \theta t_i). \tag{13}$$

This implicitly defines the ith team's best-response function.

Solution

Examples

The solution of this model is the pair (t_1, t_2) that solves the two best-response functions, and the associated values for the wage, wins for the two teams, and other endogenous variables. For most arbitrary values of θ, a closed-form solution is problematic, but fortunately the model can be solved numerically for a variety of parameter values. These examples display features consistent with the general specification introduced previously and also generate conjectures about limiting behavior. Solutions are shown in Table 1 for $m_1 = 8$ and $m_2 = 4$, as θ approaches zero.

Three features of these solutions emerge. First, the equilibrium values of the endogenous variables clearly depend on θ. This is a feature not shared by the

Table 1. Solutions for $\theta \to 0$.

	$\theta = 1$	$\theta = .1$	$\theta = .01$	$\theta = .001$	$\theta = .0001$
t_1	.2895	.5107	.5785	.5851	.5857
t_2	.1961	.3524	.4072	.4136	.4142
T	.4856	.8631	.9857	.9987	.9999
w_1	.5962	.5917	.5849	.5859	.5858
w_2	.4038	.4083	.4151	.4141	.4142
MR_1	2.7773	1.5744	1.3964	1.3746	1.3727
MC_1	2.7773	1.5740	1.3885	1.3622	1.3720
MR_2	2.8993	1.6133	1.3962	1.3736	1.3749
MC_2	2.8993	1.6134	1.3959	1.3750	1.3739
c	1.9440	1.2200	1.0434	1.0067	1.0007

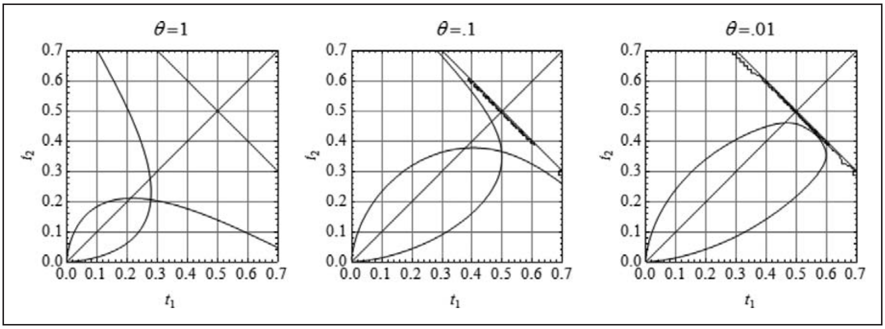

Figure 2. Asymmetric reaction curve solutions for $\theta \to 0$.

price-taking model.[16] Second, as $\theta \to 0$, the talent ratio t_1/t_2 approaches $\sqrt{m_1/m_2} = 1.414$ for specified parameter values. Also, MRP_1, MRP_2, MC_1, and MC_2 all converge to the same value $m_1/\left(1+\sqrt{m_1/m_2}\right)^2 = 1.3726$ for $m_1 = 8$ and $m_2 = 4$ and total talent T approaches unity. The win percentages approach $w_i = \left[t_i/t_j\right]/\left[1+(t_i/t_j)\right]$ ($w_1 = .5858$ and $w_2 = .4142$ for $m_1 = 8$ and $m_2 = 4$). As shown subsequently, this is the same relative talent solution and competitive balance as in the price-taking models, but in contrast to conventional theory the wage relate c approaches unity in the duopsony model.

Reaction curves and their solutions are shown in Figure 2 for these solutions. The upward sloping straight line is $t_1 = t_2$ and the downward-sloping straight line is $t_1 = 1 - t_2$ (and the identical transpose for Team 2, $t_2 = 1 - t_1$). The solution for approaches $\sqrt{m_1/m_2} = \sqrt{1.2} = .586/.414$ for $m_1 = 8$ and $m_2 = 4$. Also notice how the reaction curves get arbitrarily close to the multiple equilibria along straight line $t_1 = 1 - t_2$ as $\theta \to 0$.[17] These simulations suggest two things. First, the values of the endogenous variables depend on the parameter values of the talent supply function. In this example, the relative competitiveness of the large team t_1/t_2 increases directly with value of θ. As θ approaches zero, the price of talent approaches the reservation value of 1. This is a major distinction between this duopsony approach and the price-taking approaches.

Solution of the Limiting Case

Price taking. With price taking, the first-order conditions are

$$MRP_1 = \frac{m_1 t_2^2}{(t_1 + t_2)^3} = c, i = 1, 2, i \neq j, \tag{14}$$

where c is viewed by teams as an exogenous constant. Taking the ratio of the first-order conditions yields:

$$\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}} \text{ which implies: } t_2 = t_1\sqrt{\frac{m_2}{m_1}}. \tag{15}$$

Substituting this relationship between t_1 and t_2 back into Team 1'sfirst-order condition (Equation 14) yields the talent demand curves for both teams:

$$t_1^d = \frac{m_2}{c\left(1 + \sqrt{\frac{m_2}{m_1}}\right)^3} \text{ and } t_2^d = t_1\sqrt{\frac{m_2}{m_1}} = \frac{m_2\sqrt{\frac{m_2}{m_1}}}{c\left(1 + \sqrt{\frac{m_2}{m_1}}\right)^3}. \tag{16}$$

Summing the individual demands creates the downward-sloping market demand for talent in inverse form:

$$c^d = \frac{m_2}{\left(1 + \sqrt{\frac{m_2}{m_1}}\right)^2 \left(t_1^d + t_2^d\right)}. \tag{17}$$

As shown in Figure 3, the inverse talent supply curve has a flat portion where $c = 1$ for $T \in [0, 1]$ and a vertical segment at $T = 1$. The intersection of market demand (MRP_T) and supply curve at A_T determines the equilibrium wage as well as the equilibrium values of t_1 and t_2 at A_1 and A_2. If MRP_T evaluated at $T = 1$ is greater or equal than 1, then talent demand equals supply at $T = 1$ and $c \geq 1$. If MRP_T evaluated at $T = 1$ is less than 1, then $T < 1$ and $c = 1$. From Equation 17, $c > 1$ if and only if:

$$\frac{m_1}{\left(1 + \sqrt{m_1/m_2}\right)^2} \geq 1. \tag{18}$$

This leads to a summary proposition for the price-taking solution:

Proposition 1: The equilibrium of this price-taking model is described by:

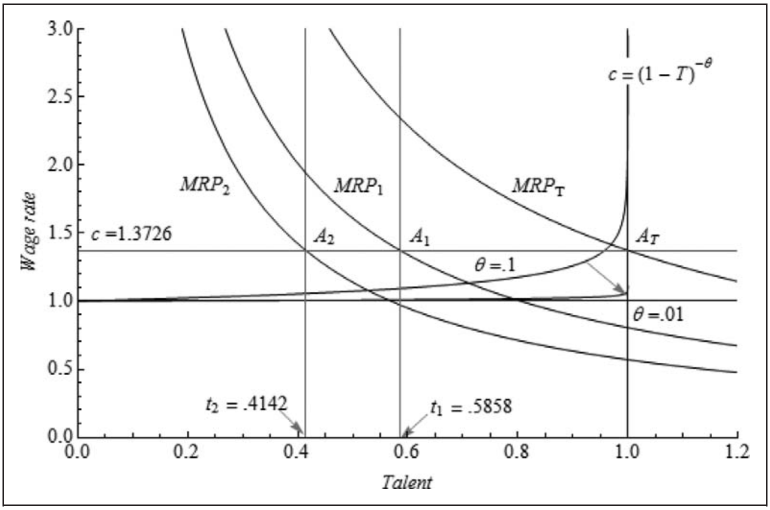

Figure 3. Price-taking solution.

1. Relative talent level: $\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}}$.
2. Wage rate $c = \frac{m_1}{(1+\sqrt{m_1/m_2})^2}$ where total league talent $T = 1$ if $\frac{m_1}{(1+\sqrt{m_1/m_2})^2} \geq 1$.
3. Wage rate $c = 1$ where total league talent $T < 1$ if $\frac{m_1}{(1+\sqrt{m_1/m_2})^2} < 1$.

As shown in Figure 4 (shaded region), the condition $m_1/(1+\sqrt{m_1/m_2})^2 \geq 1$ assures that the market size pair (m_1, m_2) lies sufficiently far northeast of the origin in the positive quadrant. This is what is meant by the markets being "sufficiently large." A numerical example of the price-taking solution is shown in Figure 3 for $m_1 = 8$ and $m_2 = 4$; where $t_1 = .5858$, $t_2 = .4142$, and $c = 8/(1+\sqrt{2})^2 = 1.3726$.

Duopsony. In the duopsony game solution, the ratios of the reaction curves from Equation 13 imply:

$$\frac{m_1 t_2^2}{m_2 t_1^2} = \frac{(1 - T + \theta t_1)}{(1 - T + \theta t_2)}. \tag{19}$$

In the limit $\theta \to 0$, the duopsony talent ratio becomes $\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}}$ which is identical to the talent ratio for the price-taking case. Evaluating Team 1's best-response function for $t_2 = t_1\sqrt{\frac{m_2}{m_1}}$ yields Team 1's reaction curve that is an implicit function solely of its own talent t_1 and exogenous market size parameters:

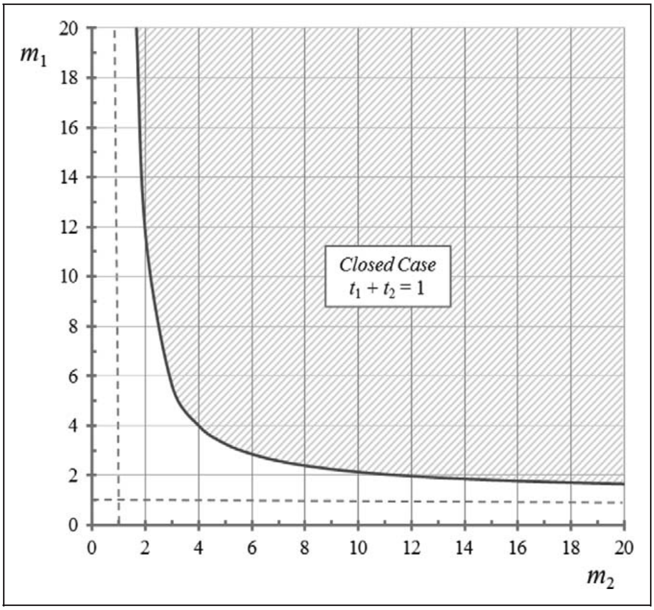

Figure 4. Sufficiently large markets.

$$\frac{m_2}{t_1\left(1+\sqrt{m_2/m_1}\right)^3} = \left[1 - t_1\left(1+\sqrt{\frac{m_2}{m_1}}\right)\right]^{-\theta-1}\left[1 - t_1\left(1 - \theta + \sqrt{\frac{m_2}{m_1}}\right)\right]. \quad (20)$$

This is shown for Team 1 in Figure 5 at A_1 and for Team 2 A_2. The *MRP* side of this equation is a rectangular hyperbola that is independent of θ and the equilibrium *MC* function has an asymptote where $\theta \to 0$ at:

$$t_1^a = \frac{1}{1+\sqrt{\frac{m_2}{m_1}}}. \quad (21)$$

As $\theta \to 0$, the equilibrium marginal cost curve approaches an inverse-L with the vertical segment positioned at the asymptote, and the horizontal segment extending from 0 to $t_1^a = \frac{1}{1+\sqrt{\frac{m_2}{m_1}}}$. Furthermore, the *MC* function lies everywhere above the supply function. If the *MRP* curve for each club crosses this limiting marginal cost function on the horizontal segment, then $MC = 1$ and $T < 1$. If *MRP* intersects the *MC* curve on the vertical segment, then $MC > 1$, and talent is fully employed at $T = 1$. If $\frac{m_2}{\left(1+\sqrt{m_2/m_1}\right)^2} > 1$, then MRP_1 intersects the MC_1 curve on its vertical segment, but if $\frac{m_2}{\left(1+\sqrt{m_2/m_1}\right)^2} \leq 1$ then MR_1 intersects MC_1 on its horizontal segment and

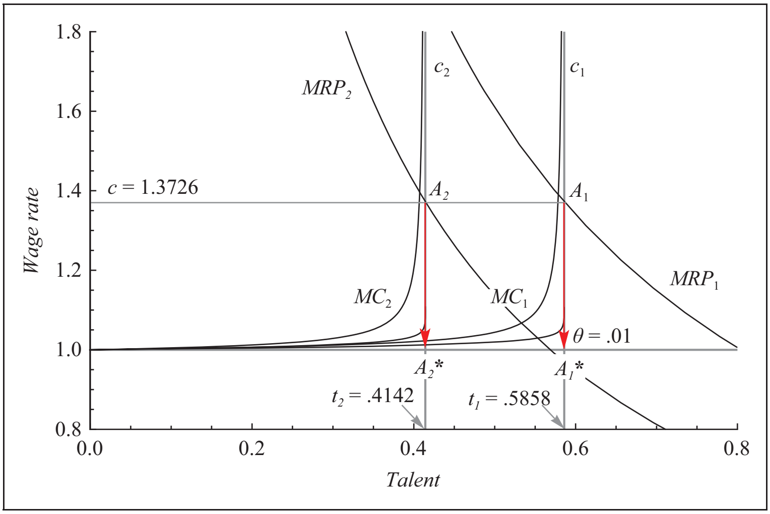

Figure 5. Asymmetric duopsony solution.

$t_1 \leq \frac{1}{\left(1+\sqrt{m_2/m_1}\right)}$. The major discovery is that $c = 1$ in both duopsony cases for both clubs at A_1* and A_2* as shown in Figure 5.

The following duopsony proposition is immediate:

Proposition 2: The equilibrium of the duopsony model is characterized by:

1. Relative talent is equal to the square root of the ratio of market size: $\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}}$.
2. The duopsony price of talent is the reservation wage $c = 1$ and level of total league talent $T = 1$, if $\frac{m_1}{\left(1+\sqrt{m_1/m_2}\right)^2} > 1$.
3. The duopsony price of talent is the reservation wage $c = 1$ and level of total league talent $T < 1$, if $\frac{m_1}{(1+\sqrt{m_1/m_2})^2} \leq 1$.

The important difference between the equilibrium under the assumption of duopsony and the equilibrium under the assumption of price-taking behavior is obviously the price of talent. The nonlimiting duopsony case yields multiple equilibria all of which imply a price of talent equal to the reservation wage (in this case $c = 1$). The limiting model preserves this outcome but picks the unique talent ratio $\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}}$, which is identical to the relative talent ratio for the price-taking case. The actual level of talent for each team (and for the league) depends on whether the markets are "sufficiently large." If the markets are relatively small, total league talent is

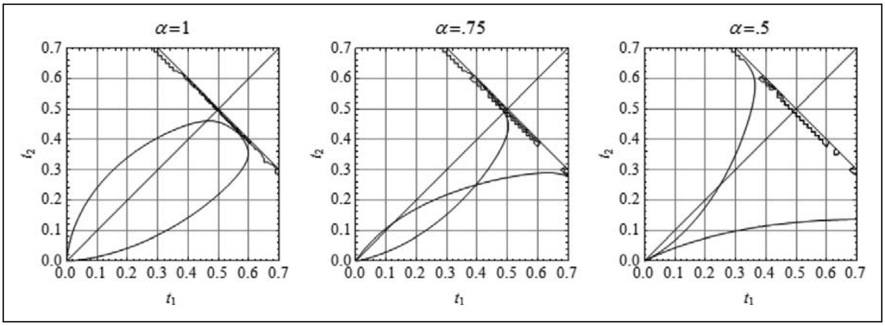

Figure 6. Revenue-sharing paradox.

less than 1. If the markets are relatively large, then talent is fully employed and league talent $T = 1$.

Revenue-Sharing Paradox

A major controversy in the modeling sports leagues concerns the effects of revenue sharing on competitive balance. Intuition suggests that revenue sharing should improve competitive balance between a large and a small market team. The existence of a counterintuitive *revenue-sharing paradox* arises from virtually all models with profit-maximizing teams. This result is robust in the limiting case of this duopsony model as $\theta \to 0$. It is also shown that as revenue-sharing approaches the equal sharing cartel level, then relative talent t_1/t_2 approaches the league revenue maximizing ratio m_1/m_2 and talent surprisingly approaches zero for both clubs.

To demonstrate the effect of revenue sharing on competitive balance consider a hybrid revenue function where α is the home team revenue share, and $(1-\alpha)$ is the visiting team share for $\alpha \in [.5, 1]$. The revenue-sharing profit function becomes:

$$\pi_i' = \alpha R_i + (1-\alpha)R_j - t_i\chi(T). \tag{22}$$

The associated best-response functions are:

$$\frac{\alpha m_1 t_2^2 - (1-\alpha)m_2 t_1 t_2}{(t_1+t_2)^3} = (1-t_1-t_2)^{-\theta-1}(1-t_1-t_2+\theta t_1). \tag{23.1}$$

$$\frac{\alpha m_2 t_1^2 - (1-\alpha)m_1 t_1 t_2}{(t_1+t_2)^3} = (1-t_1-t_2)^{-\theta-1}(1-t_1-t_2+\theta t_2). \tag{23.2}$$

Solution pairs for Equations 23.1 and 23.2 are shown in Figure 6 for $\theta = .01$ from left to right for $\alpha = 1$, $\alpha = .75$, and $\alpha = .5$, where $\alpha = .5$ is the pure syndicate. As $\alpha \rightarrow .5$, the talent levels for both teams *clearly approach zero.* The equilibrium talent ratios t_1/t_2 for different revenue-sharing splits (values of α) can be found from the ratio of Equations 23.1 to 23.2:

$$\frac{\alpha m_1 t_2^2 - (1-\alpha) m_2 t_1 t_2}{\alpha m_2 t_1^2 - (1-\alpha) m_1 t_1 t_{2_i}} = \frac{(1 - t_1 - t_2 + \theta t_1)}{(1 - t_1 - t_2 + \theta t_2)}. \tag{24}$$

In the limit where $\theta \rightarrow 0$ (Equation 24) becomes a quadratic in terms of the talent ratio $\frac{t_1}{t_2}$:

$$m_2 \left(\frac{t_1}{t_2}\right)^2 - \frac{(1-\alpha)}{\alpha}(m_1 - m_2)\frac{t_1}{t_2} - m_2 = 0. \tag{25}$$

Define $x \equiv \frac{(1-\alpha)}{\alpha}$ and the quadratic solution of Equation 25 becomes:

$$\frac{t_1}{t_2} = \frac{\sqrt{(m_1 x - m_2 x)^2 + 4 m_1 m_2} + m_1 x - m_2 x}{2 m_2}. \tag{26}$$

It is straightforward now to show that increased revenue sharing ($\alpha \rightarrow .5$) decreases competitive balance (increases t_1/t_2) and $\frac{d}{d\alpha}\left(\frac{t_1}{t_2}\right) < 0$. For $m_1 > m_2$ the derivative of Equation 26 is positive with respect to x:

$$\frac{d}{dx}\left(\frac{t_1}{t_2}\right) = \frac{(m_1 - m_2)(\sqrt{(m_1 x - m_2 x)^2 + 4 m_1 m_2} + m_1 x - m_2 x)}{2 m_2 \sqrt{(m_1 x - m_2 x)^2 + 4 m_1 m_2}} > 0, \tag{27}$$

and the derivative of $x \equiv \frac{(1-\alpha)}{\alpha}$ with respect to α is negative: $\frac{\mathrm{d}x}{\mathrm{d}\alpha} = -\frac{1}{\alpha^2} < 0$. This implies that the revenue-sharing paradox still holds and that there is an inverse relationship between revenue sharing and competitive balance, that is, $\frac{d}{d\alpha}\left(\frac{t_1}{t_2}\right) = \left[\frac{\mathrm{d}x}{\mathrm{d}\alpha}\right]\left[\frac{d}{dx}\left(\frac{t_1}{t_2}\right)\right] < 0$. More precisely, as $\alpha \rightarrow .5$ in Equation 25 competitive balance moves from the duopoly solution $\frac{t_1}{t_2} = \sqrt{\frac{m_1}{m_2}}$ toward the less balanced relative talent solution $\frac{t_1}{t_2} = \frac{m_1}{m_2}$ at the limit.

It can also be shown that the revenue-sharing limit solution $\frac{t_1}{t_2} = \frac{m_1}{m_2}$ is the cooperative cartel profit-maximizing talent ratio. Total cartel profit Π_C is the sum of individual club revenues minus total talent cost:

$$\pi_C = R_1 + R_2 - cT = \frac{m_1\left(\frac{1}{2}t_1^2 + t_1 t_2\right) + m_2\left(\frac{1}{2}t_2^2 + t_1 t_2\right)}{(t_1 + t_2)^2} - c(1 - T)^{-\theta}. \tag{28}$$

Cartel profit and revenue each reach a maximum when $\frac{\partial \pi_C}{\partial t_1} = \frac{\partial \pi_C}{\partial t_2}$:

$$\frac{t_2(m_1 t_2 - m_2 t_1)}{(t_1 + t_2)^3} = \frac{t_1(m_2 t_1 - m_1 t_2)}{(t_1 + t_2)^3} = (1 - T)^{-\theta - 1}(1 - T + \theta T). \tag{29}$$

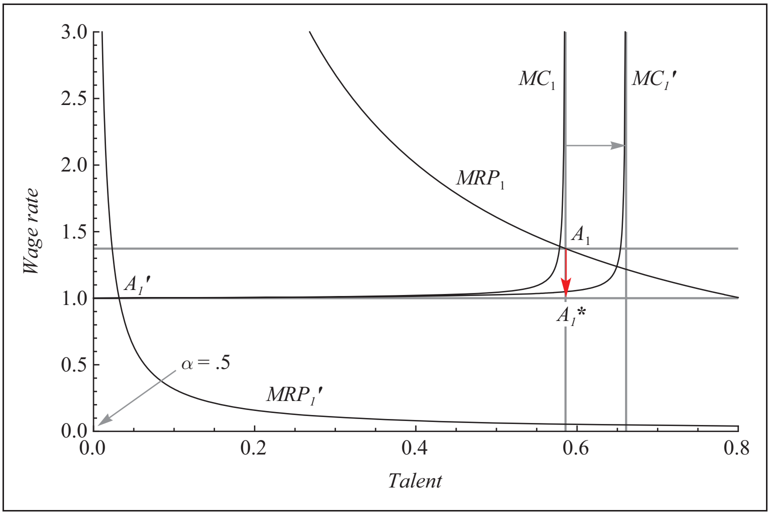

Figure 7. Revenue-sharing large market.

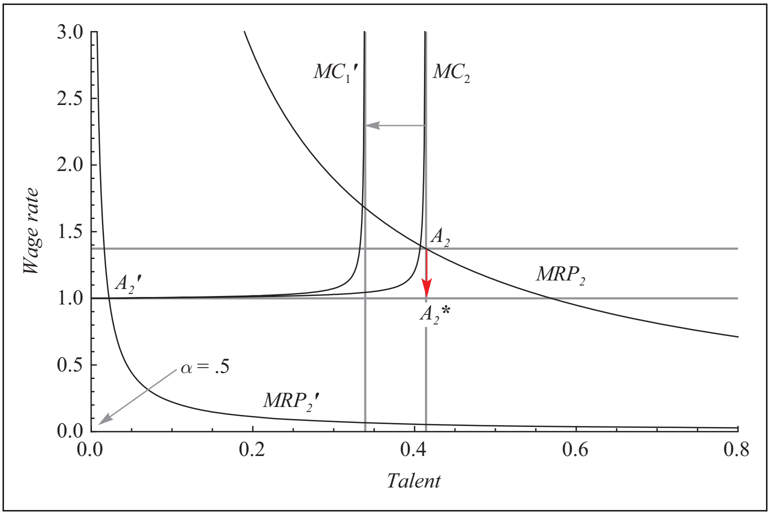

Figure 8. Revenue-sharing small market.

This implies that $t_2(m_1t_2 - m_2t_1) = t_1(m_2t_1 - m_1t_2)$ and for $m_1 > m_2$ the cartel profit maximum obtains when the relative talent ratio is the same as the ratio of market size:

$$\frac{t_1}{t_2} = \frac{m_1}{m_2} \tag{30}$$

This all leads to the following proposition about revenue sharing in profit-max leagues:

Proposition 3: For α between 0 and .5 increased revenue sharing ($\alpha \rightarrow .5$):

1. Generally decreases competitive balance (increases t_1/t_2): $\frac{d}{d\alpha}\left(\frac{t_1}{t_2}\right) < 0$.
2. Specifically decreases competitive balance at the limit toward the cartel profit maximum $\frac{t_1}{t_2} = \frac{m_1}{m_2}$.
3. Leads to a reduction in talent for each team that approaches zero at the limit.

The implications of duopsony Proposition 2 and revenue-sharing Proposition 3 are shown for the large market club ($m_1 = 8$) in Figure 7 and the small market club ($m_2 = 4$; both for $\theta = .01$) in Figure 8. In the absence of revenue-sharing, Proposition 2 states that the profit-maximizing talent level obtains at A_1 and A_2 for the large and small market clubs and that each duopsony club then sets its wage at $c = 1$ for that talent at A_1* and A_2*, respectively. At the revenue-sharing limit $\alpha \rightarrow .5$ Proposition 3 holds that the profit-maximizing level of talent approaches zero. Solving Equation 26 for t_2 in terms of t_1 and t_1 in terms of t_2 for $\alpha = .48$ and then substituting both into Equation 23 yields the revenue-sharing solutions A_1' and A_2' for the large and small market clubs with the wage set at the duopsony reservation rate $c = 1$.

Increased revenue-sharing amounts to progressive collusion as MRP_1 and MRP_2 both shift toward MRP_1' and MRP_2'. As $\alpha \rightarrow .5$, the marginal revenue product curves for both clubs collapse toward the origin with the vertical and horizontal axes as asymptotes. Given these market size parameters ($m_1 = 8$; $m_2 = 4$) duopsony talent equilibrium shifts from $t_1/t_2 = \sqrt{m_1/m_2} = 1.414$ with $t_1 = .5858$ at A_1 and $t_2 = .4142$ at A_2 toward $t_1/t_2 = m_1/m_2 = 2$ with $t_1 = .0318$ and $t_2 = .0222$ at A_1' and A_2'. The most interesting result is that while revenue sharing increases the talent ratio t_1/t_2 from $\sqrt{m_1/m_2}$ toward m_1/m_2, the absolute levels of talent are approaching zero for both clubs as $\alpha \rightarrow .5$.[18]

This all derives from a revenue function that depends only on the ratio of talent, and not individual talent levels. With homogenous-of-degree-zero league revenue functions, maximization of revenue occurs at the talent ratio $t_1/t_2 = m_1/m_2$. Therefore, for any such ratio of talent, t_1/t_2, the cartel would maximize profits by scaling down operations to zero.[19] What is even more remarkable is that this vanishing talent effect occurs for any model in which revenues depend on wins defined by talent ratios, such as the commonly used logistic CSF: $w_i = t_i^{\gamma}/\left(t_i + t_j\right)^{\gamma}$ and *this is true for any* γ.

The cartelization problem from revenue sharing only emerges when competitive balance is viewed in actual talent rather than win percentages, and when the strategic choice variable is talent rather than talent expenditure. The solution to the problem is not to further obscure underlying duopsony behavior and avoid the problem altogether by using payroll as a convenient strategic choice variable. Rather, the immediate solution in duopsony theory is to introduce absolute talent as well as relative talent into the measure of the quality of a game.

Conclusion

Since its modern origins in QFV theory, there has been a major confusion in the economic modeling of professional sports leagues. The controversy is about how to model duopsony in a two-team league with perfectly inelastic aggregate talent supply. SK attempted to reconcile the talent supply-side issue by making a distinction in theory and reality between open talent markets with a perfectly elastic talent supply and closed markets with a perfectly inelastic talent supply. The open markets are found in the talent markets of the European football leagues, whereas the closed markets characterize the more provincial North American sports. Open and closed markets lead to different conclusions about competitive balance and the paradoxical effects of revenue sharing on competitive balance and player compensation.

In the closed market case (QFV), there is a simple linear relationship between talent and winning with no formal *CSF*. The only nonlinearity appears in the revenue function that shows the diminishing marginal returns to wins based on fan preferences. In the closed case, the ratio of talent is equivalent to the ratio of home market size of the two clubs ($w_1/w_2 = t_1/t_2 = m_1/m_2$) and players' *MRP* is simply the marginal revenue of a win. Revenue sharing will not affect the relative demand for wins or talent between the clubs and competitive balance will remain the same. The difference is that revenue sharing will proportionately reduce the marginal revenue of a win for both clubs and lead to player exploitation.

In the open-market case (SK), the nonlinear demand-side assumptions are left intact, but a second nonlinearity is introduced with a *CSF* on the supply side. The *CSF* is usually specified in the logistic form where success depends on the proportion of talent that a team has in each game. The result of increasing the nonlinearity in production of wins is to dampen the market power of the larger market club.[20] As a result, the open league equilibrium is more competitive and the talent ratios are proportional to the square root of the ratios of home market size for the two clubs ($w_1/w_2 = t_1/t_2 = \sqrt{m_1/m_2}$). Revenue sharing at the limit would reduce competitive balance in an open league to that of a closed league and team payrolls would drop to the reservation level. Revenue sharing would yield the same equilibrium in open and closed markets at the limit (Vrooman, 2007, 2009).

In the course of the open league critique, there has also been major confusion about the use of duopoly game theory in the two-team league model and specification of an inconsistent duopsonist as a scheming monopolist on the demand side who willingly accepts the market price for talent on the supply side, where he wields considerable labor market power. As a result of the confusion, there seems to be a building consensus to resort to a nonstandard strategy space of talent expenditure instead of talent when analyzing the game between members of a two-team league. Talent expenditure is an attractive alternative to talent because it is a quick fix to a misdiagnosed problem. Madden (2011, p. 419) agrees with the schizophrenic problems in price-taking theory:

> A problem with the existing approaches is that they treat the two-club talent market as perfectly competitive overlooking the market power clubs might be expected to have in such a setting. . . .

He sees the problem as one of game specification and proposes a backward induction solution of a new strategic market game (SMG) where:

> (T)he two clubs will anticipate correctly the (non-negligible) impact that their decisions will have on the market clearing wage, thus capturing their talent market power. If the decision variables were quantities of talent, the new model would simply and exactly be a Cournot duopsony model. However, in the most basic league, with its perfectly inelastic supply of talent, the market clearing wages are not then well defined. So we insist now on talent expenditures as the club strategic choice variables, an assumption that is anyway probably more realistic for the context of a sports league. (Madden, 2011, p. 419)

Madden unfortunately abandons this well-defined duopsony game and insists on talent expenditure as a strategic choice variable (originally proposed by and later retracted by Szymanski, 2013) rather than the conventional game theory choice of talent level. This is because the inelastic wage problem was really a solvable problem of multiple equilibria, rather than being undefined. The appropriate game-theoretic remedy is to find the unique limit of the duopsony game as total talent supply approaches perfect inelasticity. Moreover, the use of talent expenditure as a strategic variable yields exactly the same results as the price-taking theory that Madden criticizes.

The game-theoretic analysis developed in this article meets the usual expectation in a duopsony model that the wages paid to talent are always less than the marginal revenue product of talent. In fact, this model is the only analysis that actually determines a duopsony wage rate. With fairly standard assumptions on revenue functions and a well-defined supply functions, the revenue-sharing paradox still holds that revenue sharing does not increase balance. This analysis also isolates strategic talent reductions as the previously hidden result of cartelization from increased revenue sharing. This result is true for any revenue function (including the widely used logistic *CSF*) where wins are solely a function of relative talent. Of course, there are other assumptions, such as win-maximizing teams (Vrooman, 2007, 2009) or revenue functions that depend on total talent (Madden, 2011) that can overturn this now familiar paradox.

Authors' Note

"The opera ain't over until the fat lady sings" was used as a catch phrase by Washington Bullets coach Dick Motta to caution against overconfidence as the Bullets won the 1978 NBA

Championship. "The Fat Lady sings" refers to Brünnhilde's final immolation scene in Wagner's *Götterdammerung*.

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Notes

1. Vrooman (1995) used a nonlinear model where *QF* is treated as a special linear case. "If the marginal product of playing talent is diminishing . . . the actual competitive balance solution under profit maximization will be more competitive than that predicted by league revenue maximization solution" (p. 976).
2. See Madden (2010) for an attempt to motivate price-taking behavior in a two-team model by treating the two "teams" as two continua of different team types.
3. See Falconieri, Palomino, and Sakovics (2004), Szymanski (2004), Szymanski and Kesenne (2004), and Dietl et al. (2009). Madden (2011) is the exception, but in contrast to the approach used in this article, he used talent expenditure as the teams' strategic choice variable instead of the level of talent.
4. Most widely known selection criteria are associated with dynamic games such as backward induction, forward induction, and subgame perfection. In static games like the one here, refinements such as Pareto optimality are sometimes suggested. The search for refinements in other contexts is widespread, such as e-stability in learning models (Evans & Honkapodja, 2001), limits of a finite-horizon equilibrium in infinite-horizon models (Driskill, 1997, 2006; Driskill & McCafferty, 2001, Fershtman & Kamien, 1990).
5. The revenue function is from the model of Dietl, Lang, and Werner (2009), which was derived from an explicit model of consumer choice. This revenue model has some unusual properties not discussed previously.
6. This seems a natural selection criterion because it captures the notion that a perfectly inelastic inverse supply function is really an abstraction designed to be close to a "very" inelastic function. The notion that a parametric change from "very" inelastic to perfectly inelastic should lead to a change in the number of equilibria from one to infinity suggests that these multiple equilibria are simply an artifact of this perfectly inelastic abstraction. The only one of these equilibria that are reflective of the underlying economic forces is the one associated with the limit as the supply moves from *very inelastic* to arbitrarily close to *perfectly elastic*.
7. Much of the literature is focused on whether revenue sharing left competitive balance unchanged (the "invariance proposition") or made it worse. What seems more interesting

is the more general counterintuitive result that competitive balance does not increase. Vrooman (1995) called this "the revenue-sharing paradox."

8. This avoids the problem posed by Madden (2011) that if $t_1 + t_2 = 1$, the wage is indeterminate or undefined. One response to his view is to see talent in discrete (but very fine) units and then consider best responses as being such that $t_1 + t_2$ is always one (very small) unit less than 1.
9. Note here that for all of these possible Nash equilibria, the price of talent is reduced to the minimum reservation wage of unity. This is one important feature that differentiates this strategic talent choice model from the price-taking approach.
10. See, for example, a general specification of this equality in Winfree and Fort (2011). Also note that in the duopsony model of Madden (2011), the choice of talent expenditure as the strategic choice variable implies this equality of marginal revenue. Without the imposition of more structure on the model, for example, a specific talent supply function, or a homogeneity restriction on the revenue functions with respect to team talent levels, nothing definitive can be derived about key issues, such as the effect of revenue sharing on competitive balance.
11. The focus here is on the limiting case, but this functional form is flexible for nonzero values of θ.
12. This concave revenue function reflects the *uncertainty of outcome hypothesis* that fans prefer winning close contests. Vrooman calls this the Yankee paradox where $q_{ij} = \varphi w_{ij} + (1 - \varphi) w_{ij} w_{ji}$ and fan preference for competitive balance is an empirical question $(1 - \varphi)$. In the quadratic case, $\varphi = .5$ for $w_{ij} = 1 - w_{ji}$.
13. See Fort and Winfree (2009) for the implications of other specifications of contest success functions.
14. One important feature of this *MRP* function is zero-degree homogeneity in team talent levels. Changing both teams talent level in the same proportion leaves revenue unaffected. Madden (2011) argues that this is an undesirable feature. This feature is responsible later for important effects of revenue sharing on league talent.
15. Alternatively

$$\begin{aligned}\mathrm{MRP}_i &= \mathrm{MR}_i\mathrm{MP}_i = \partial R_i/\partial t_i = (\partial R_i/\partial w_i)(\partial w_i/\partial t_i)\\ &= [m_i t_j/(t_i + t_j)][t_j/(t_i + t_j)^2] = m_i t_j^2/(t_i + t_j)^3.\end{aligned}$$

16. This feature is not shared by Madden (2011) where talent expenditure is the strategic choice variable.
17. For "smaller" markets, the convergence of best-response curves may not get closer to 0.5, 0.5. This suggests that as $\theta \to 0$, the unique solution pair becomes arbitrarily close to the $t_1 = 1 - t_2$ locus as long as m_1 and m_2 are sufficiently large, while the solution pair approaches the constant marginal cost case when m_1 and m_2 are sufficiently small. See Figure 4 for a description of the precise meaning of "sufficiently large."
18. An example of the talent drain occurred when French Ligue1 was ranked last of the Big 5 leagues in Europe by Union of European Football Associations (UEFA) in 2001, while FIFA had the French national team ranked first in the world. The quality of Ligue 1 was

declining because of a talent drain, while the French national team was strong, because of the development of French grassroots talent and French talent playing for higher wages throughout Europe. Before 2005, French Ligue 1 TV revenue-sharing formula allocated 83% for solidarity, 10% merit, and 7% appearances. Increased merit sharing under *Charte 2002 des clubs de football* was justified on the premise that large market Ligue 1 clubs were at a disadvantage in international competition (Champions League) because of solidarity sharing. Beginning in 2005, Ligue 1 changed its formula to 50% solidarity, 30% league finish, and 20% appearances.

19. Similar results of decreasing competitive balance from increased revenue sharing is found in Vrooman (2007, 2009), but the price-taking model could not separate the effects on talent from the cost per unit of talent.
20. Vrooman (1995) first used exponential revenue and cost functions for equilibrium $w_1/w_2 = [m_1/m_2]^{\alpha/\beta}$.

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Author Biographies

Robert Driskill is an economics professor at Vanderbilt University.

John Vrooman is an economics professor at Vanderbilt University.