A simplified derivation of Wang and Busemeyer's Q-test

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Notation: We have two measurements, A and B, to which Ps can respond "yes" or "no". The projection operators corresponding to these measurement outcomes are $P_A, P_{\bar{A}}, P_B, P_{\bar{B}}$, with $P_A + P_{\bar{A}} = P_B + P_{\bar{B}} = 1$.

This derivation makes use of the properties of commutators and projection operators, which together imply,

$$[P_A, P_B] = [P_A, (1 - P_{\bar{B}})] = -[P_A, P_{\bar{B}}] = [P_{\bar{B}}, P_A], \text{ etc.}$$

Note however that the use of commutators is just a mathematical convenience.

Now for the derivation. We begin with,

$$[P_A, P_B] - [P_A, P_B] = 0$$

Inserting two copies of the identity gives,

$$P_A[P_A, P_B] + P_{\bar{A}}[P_A, P_B] - [P_A, P_B]P_B - [P_A, P_B]P_{\bar{B}} = 0$$

Now we use the property of the commutator noted above, to get,

$$P_A[P_{\bar{B}}, P_A] + P_{\bar{A}}[P_B, P_{\bar{A}}] - [P_B, P_{\bar{A}}]P_B - [P_{\bar{B}}, P_A]P_{\bar{B}} = 0$$

Expanding out the commutators gives,

$$P_A P_{\bar{B}} P_A + P_{\bar{A}} P_B P_{\bar{A}} - P_B P_{\bar{A}} P_B - P_{\bar{B}} P_A P_{\bar{B}} = 0$$

Since this operator is identically zero, it follows that for any density matrix ρ,

$$Tr(\{P_A P_{\bar{B}} P_A + P_{\bar{A}} P_B P_{\bar{A}} - P_B P_{\bar{A}} P_B - P_{\bar{B}} P_A P_{\bar{B}}\}\rho) = 0$$

By the linearity and cyclic property of the trace this gives,

$$p(AyBn) + p(AnBy) - p(ByAn) - p(BnAy) = 0$$

Where $p(AyBn) = Tr(P_{\bar{B}} P_A \rho P_A)$ etc. This is Wang and Busemeyer's Q-test.

There are two points worth noting;

1. This derivation does not rely on any property of 'reciprocity' or similar.
2. Instead the properties that are used are actually properties of the operators P_A etc. Specifically we use
   - Completeness, $P_A + P_{\bar{A}} = 1$ etc.
   - Idempotency, $P_A^2 = P_A$ etc.

   The second property in particular holds only if the P's are *projection operators,* which means in theory the Q-test could be violated by POVM type measurements.