# A note on the Partial Trace

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The partial trace is an operation used frequently when we have two or more subsystems $A$ and $B$ and a Hilbert space of the form $\mathcal{H}_A \otimes \mathcal{H}_B$. In particular it's commonly used when we have a system coupled to an environment.

The basic idea is to take an operator $O_{AB}$ on the joint space $\mathcal{H}_A \otimes \mathcal{H}_B$ and 'project out' the part on $\mathcal{H}_B$ to leave an effective operator $O'_A$ on $\mathcal{H}_A$. The way to do this is to average over the possible effects the operator could have when acting on $\mathcal{H}_A$, given we know the operator onto the joint space and also the state on the space $\mathcal{H}_B$.

In other words, assuming the state of the joint system may be written $\rho_A \otimes \rho_B$ the operator $O'_A$ is effectively,

$$O'_A \sim \langle O_{AB} \rangle_{\rho_B} \tag{1}$$

N.B. The fact that the partial trace of an operator depends on the state on $\mathcal{H}_B$ as well as the form of the operator is crucial. In this sense it is unlike a projection to a subspace. This is one of the things that allows you to simplify the effective dynamics when dealing with CP maps, since you can assume forms for the initial environment state (eg high temperature limit) that make the equations more tractable.

Our task is to make Eq(1) more rigorous.

Suppose $\{|a_i\rangle\}$ form a basis for $\mathcal{H}_A$ and $\{|b_i\rangle\}$ form a basis for $\mathcal{H}_B$. Then we can take $\{|a_i\rangle |b_j\rangle\}$ as basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. In this basis we can therefore write $O_{AB}$ as a matrix,

$$(O_{AB})_{ij,i'j'} = \langle a_{i'}| \langle b_{j'}| O_{AB} |b_j\rangle |a_i\rangle \tag{2}$$

Note that the indices look funny here. The pair $ij$ is to be interpreted as a single index.

The partial trace over $\mathcal{H}_B$ is then defined in the obvious way,

$$(O'_A)_{i,i'} = \mathrm{Tr}_{\mathcal{H}_B}((O_{AB})_{ij,i'j'}) = \sum_{j,j'} (O_{AB})_{ij,i'j'} \delta_{j,j'} \tag{3}$$

The object $(O'_A)_{i,i'}$ is a matrix, and can be thought of as specifying the matrix elements of an operator $O'_A$. We can therefore write,

$$O'_A = \mathrm{Tr}_{\mathcal{H}_B}(O_{AB}) = \sum_j \langle b_j| O_{AB} |b_j\rangle \tag{4}$$

which is the true meaning of the funny expression in my notes.

In actual fact we hardly ever need to work at this level of generality. Instead it suffices to consider operators of the form $O_{AB} = O_A \otimes O_B$. The reason for this is that any operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as,

$$O_{AB} = \sum_i O_A^i \otimes O_B^i \tag{5}$$

In other words, while a general operator on the product space cannot be written in product form, it can be written as a sum of operators of product form. (This is similar to the observation that

while states of a composite system cannot necessarily be written in product form, any (pure) state on a product space can be written as a superpostion of product states.)

Let us assume that the initial state of the system factorises as $\rho_{AB} = \rho_A \otimes \rho_B$, but leave the form of the operator as general. This can represent the general case of a CP map where the 'operator' might be in the Heisenberg picture and include some evolution of the joint system. The expected value of $O_{AB}$ acting on they joint system may therefore be expressed as

$$\mathrm{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B}(O_{AB}\rho_{AB}) = \sum_i \mathrm{Tr}_{\mathcal{H}_A}(O_A^i \rho_A) \times \mathrm{Tr}_{\mathcal{H}_B}(O_B^i \rho_B)$$
$$= \mathrm{Tr}_{\mathcal{H}_A}(O_A' \rho_A) \tag{6}$$

where the operator

$$O_A' = \sum_i O_A^i \times \mathrm{Tr}_{\mathcal{H}_B}(O_B^i \rho_B) \tag{7}$$

is what results from taking the partial trace over $\mathcal{H}_B$.

An alternative is instead to consider the operator as being of product form, but the state as being entangled. In particular we will assume the operator is of the form, $O_{AB} = O_A \otimes I_B$, ie we are looking only at observables of the system $A$. This corresponds to the case where we have a state which is entangled with some environmental degree of freedom (perhaps as a result of an earlier interaction), but where we no longer have access to this other degree of freedom.

An important example of this occurs when the initial state is a maximally entangled Bell state, eg

$$\rho_{AB} = \left|\Phi^+\right\rangle \left\langle\Phi^+\right| \tag{8}$$

where $\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle_A \left|0\right\rangle_B + \left|1\right\rangle_A \left|1\right\rangle_B)$. $\rho_{AB}$ is obviously not of product form, however it can be written as,

$$\rho_{AB} = \left|\Phi^+\right\rangle \left\langle\Phi^+\right| = \frac{1}{2}(\left|0\right\rangle \left\langle0\right|_A \otimes \left|0\right\rangle \left\langle0\right|_B + \left|0\right\rangle \left\langle1\right|_A \otimes \left|0\right\rangle \left\langle1\right|_B + \left|1\right\rangle \left\langle0\right|_A \otimes \left|1\right\rangle \left\langle0\right|_B + \left|1\right\rangle \left\langle1\right|_A \otimes \left|1\right\rangle \left\langle1\right|_B)$$
$$= \frac{1}{2} \sum_{ij=0,1} \left|i\right\rangle \left\langle j\right|_A \otimes \left|i\right\rangle \left\langle j\right|_B \tag{9}$$

which is of the form Eq(5).

We now have,

$$\mathrm{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B}(O_{AB}\rho_{AB}) = \frac{1}{2} \sum_{ij} \mathrm{Tr}_{\mathcal{H}_A}(O_A \left|i\right\rangle \left\langle j\right|_A) \times \mathrm{Tr}_{\mathcal{H}_B}(\left|i\right\rangle \left\langle j\right|_B)$$
$$= \frac{1}{2} \sum_{ij} \mathrm{Tr}_{\mathcal{H}_A}(O_A \left|i\right\rangle \left\langle j\right|_A) \times \delta_{ij} \tag{10}$$
$$= \mathrm{Tr}(O_A \rho_A')$$

where the reduced state is given by,

$$\rho_A' = \sum_{i=0,1} \frac{1}{2} \left|i\right\rangle \left\langle i\right|_A = \frac{1}{2} \mathbb{1}_A \tag{11}$$

This leads us to the important result that if we have an initial state which is maximally entangled, but we perform measurements only on one part of the system, that state is indistinguishable from a maximal mixture.

More generally, this is what happens when we entangle states of the system in which we are interested with states of the environment - the system state becomes a mixed state with only classical correlations.