Early and Often

How Voting Systems Affect Democracy
and
Math Affects Voting Systems

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What is an electoral system?

The phrase "electoral system" refers to the rules governing how voters express their preferences as to who governs, together with rules for how those votes determine who is seated in office.

Example: In the United States, the electoral system is quite simple. The nation is broken into states, which are the electoral districts for Senate elections, and states into smaller districts for House elections. At each election, voters in each district select exactly one candidate for office; the candidate with the most votes cast for him/her is seated.

A motivating example

In the 2000 U.S. Presidential election in Florida, 5,963,110 votes were cast, distributed

$$\begin{pmatrix} \text{Bush} & \text{Gore} & \text{Nader} & \text{Other} \\ 2,912,790 & 2,912,253 & 97,488 & 40,579 \end{pmatrix}$$

Mr. Bush received the state's electoral votes and went on to win the Presidency. What made this result interesting is that most Nader voters actually preferred Gore to Bush, while not many Bush voters preferred Nader to Gore: that is, if voters were presented with a series of two-way choices, the outcomes would probably have been approximately

$$\begin{pmatrix} \text{Bush/Gore} & \text{Bush/Nader} & \text{Gore/Nader} \\ 2,912,790 \ / \ 3,009,741 & \text{Bush} \gg \text{Nader} & \text{Gore} \gg \text{Nader} \end{pmatrix}$$

Thus Gore wins every individual match, but loses the tournament!

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2. What different electoral systems are possible?
3. What effects, if any, would changes have on U.S. public life?

Borda count

A first attempt (Jean-Charles de Borda, ca. 1800):

Instead of simply picking one favorite, the voter ranks their top n candidates. A voter's most favored candidate receives $n-1$ weighted votes, their next favored $n-2$, etc.

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Advantage: Borda count ($n = 3$) resolves Bush v. Gore v. Nader without paradoxes.
Disadvantage?

Borda's shortcomings

Example: Suppose we have one right-wing candidate R and two left-wing candidates L and K. 65 right-wing voters prefer $R > L > K$, but a total of 50 left-wing voters may prefer either $K > L > R$ or $L > K > R$.
Predict: With all 50 voting hard-left ($K > L$), who wins the election?
What happens as voters move towards the center (preferring L over K)?

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R	130
L	115
K	110
Winner	R

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$K > L$	50	40
R	130	130
L	115	125
K	110	100
Winner	R	R

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$K > L$	50	40	30
R	130	130	130
L	115	125	135
K	110	100	90
Winner	R	R	L!

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$K > L$	50	40	30	0
R	130	130	130	130
L	115	125	135	165
K	110	100	90	50
Winner	R	R	L!	L by a landslide!

Irrelevant Alternatives and Strategic Voting

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The choice between L and K is what is known as an "irrelevant alternative": a voter changing his or her relative ranking of two candidates should never affect the fates of candidates above both or below both! But this is precisely what happens, because Borda count forces the right-wing voters to choose one of the *left-wing* candidates to "throw their second vote" away on!

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> which proceeded to adopt Borda count for its elections.

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Or informally

There is no perfect electoral system.

Things are even worse if we just look at weighted-preferences systems:

Theorem (cf. [Simon & Blume 94])

Let $n \geq 3$. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ different possible one-on-one matchups in this election: for each one, choose a winner at random. Additionally, choose any ranking of the n candidates at random. Then we can find a population of voters whose overall preferences combine to realize all the head-to-head matchups and the overall ranking.

Or informally again:

There is *definitely* no perfect weighted or plurality voting system.

First Past The Post

The U.S. actually uses a weighted-preference system, assigning weight 1 to a voter's top choice and weight 0 to all other choices. This system is sometimes called "Plurality vote", or **First Past The Post**, by analogy with horseracing.

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At last count, only the U.S. and 10 other countries use FPTP to elect their legislatures. In fact, this is the only weighted-preference system in use anywhere in the world for national elections.

Two-Round voting

While the French Academy might be honest enough to use Borda count, the remainder of France today uses a two-round system to elect its legislators.

- Round 1: All candidates are shown on the ballot. Each voter chooses one candidate. Any candidate whose vote share in this round exceeds 12.5% of the number of registered voters in the district moves on to Round 2.

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Disadvantages: it is inconvenient for voters and expensive for the state to have voting occur twice, a week or two apart. More importantly, the delay between the rounds invites parties to prepare strategic manipulation of their members' votes, especially if the party's own candidate has been eliminated in the first round.

Ranked-ballot Alternative Vote

Possibly due to the same unique evolutionary pressures that gave us the platypus, Australia developed a unique electoral system: the Alternative Vote. In this system, voters rank all n candidates in order of preference. To determine a winner,

AV Algorithm

- Count all first-place votes. IF one candidate has more than 50% of the first-place votes, s/he is the winner.
- ELSE find the candidate with the fewest first-place votes. Eliminate this candidate from all ballots. Run AV ALGORITHM on the revised ballots, which are now ranked lists of size $n - 1$.

This voting system has the advantage that the eventual winner was in fact preferred by a majority of voters. (One disadvantage, however, is that rates of ballot spoilage – i.e. ballots which are invalid because they are improperly filled out – are notoriously high in AV elections.)

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AV3	40	E	60 wins	E

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FPTP	33 wins	14	24	29
2nd Round	49	*E*	*E*	51 wins
AV1	33	14	24	29
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AV3	40	*E*	60 wins	*E*
Borda	120	173	184 wins	123

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In **Proportional Representation** (PR)

- voters usually vote for parties rather than individuals;
- fill several seats in the legislature;
- parties are allotted seats in proportion to the number of votes they receive

District Size

In the literature on voting systems, a crucial variable ends up being the number of representatives elected by the same body of voters, always denoted M. In the U.S. and other FPTP systems, $M = 1$. In PR, $M > 1$. This quantity is often called “District Size” (though this should not be confused with the unrelated but important question of how many *voters* are included in each voting district).

If M is very large, smaller parties have a good chance of netting a few seats, since the electoral system pools all the votes they receive nationwide. The smaller the value of M, the higher the threshhold for winning seats.

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Another way of looking at this distinction is

FPTP is just the limit of PR as $M \to 1$.

District Size: the Tradeoff

Large M

- Advantage: Nearly every vote goes toward the election of a winner. "Emotional disenfranchisement" is nearly zero.

Small M

- Advantage: Each district is a constitutency – the representatives owe loyalty to the specific voters who sent them to office, not simply to the national party.

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Large M

- Advantage: Nearly every vote goes toward the election of a winner. "Emotional disenfranchisement" is nearly zero.
- Disadvantage: National diversity may not be reflected in the party's choice of representatives to fill its allotted seats. Small extremist parties can gain a foothold in the legislature.

Small M

- Advantage: Each district is a constitutency – the representatives owe loyalty to the specific voters who sent them to office, not simply to the national party.
- Disadvantage: Voters who do not vote for winners are unrepresented and may become emotionally disenfranchised.

How small can a party be and still survive?

If a party's support is below $\frac{2}{3}\frac{1}{M+1}$ of a district, it has basically no chance to win any of that district's seats.

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Corollary

In the U.S., where $M = 1$, third parties are toast.

Duverger's Law

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Principle (Duverger's Hypothesis)

PR systems with $M > 1$ *will tend to accompany political scenes with more than two major parties.*

There are no known major exceptions to Duverger's Law. Research into better and better quantitative statements of these and related principles continues to be an active area of research in political science, sometimes called "The Duvergerian Agenda".

Citations

- Taagepara, R., *Predicting Party Sizes: The Logic of Simple Electoral Systems*; Oxford University Press 2007
- Farrell, D., *Electoral Systems: A Comparative Introduction*; Palgrave 2001
- Robbin, J., “Ultrafilters and Arrow’s Theorem on the Impossibility of A Fair Election”, unpublished manuscript available at Robbin’s faculty page at the University of Wisconsin
- Simon, C. and Blume, L., *Mathematics for Economists*; Norton 1994
- Duverger, M., *Political Parties: Their Organization and Activity in the Modern State*; Methuen 1954

Thank you!