Differential Forms and a Possible Generalization of Maxwell's Equations

Zachary Bednarke
Vanderbilt Univeristy

December 2, 2015

Abstract

In this lecture I start with a superficial introduction to determinants, exterior derivatives, and their action on differential forms in order to convey arguments about the structure of Maxwell's electromagnetic equation in a way that makes some heuristic sense but doesn't have the aid of rigorous exposition. I start by reconstructing the first two equations, then I talk about splitting spacetime and the problems with last two equations. To construct those, I discuss the role of the metric on spacetime, the concepts of dual spaces and inner products and the energy density and lagrangian. Finally I expose the concepts of orientation and define the Hodge star, discuss "div, grad, and curl", and end with writing down the dual of $F_{\mu\nu}$ and the second set of equations.

Then, I take a detour to fix a few inconsistencies in the equations before discussing the London equations and superconductivity. Then I discuss their fewer restrictions and possible implications.

I owe respect to Reyer Sjamaar, John Baez, and Shlomo Sternberg for writing books and lecture notes that showed me how cool this subject is.

1 Introduction

Throughout this semester Ive had the great fortune of studying differentiable manifolds extracurricularly. The topic is interesting in its own right as an introduction to the mathematics of differential geometry, but it also has incredible utility in the study of geometry in physics. In the small time frame of this presentation, you cant properly discuss the deep geometrical structure of Maxwells equations as a Yang-Mills theory. Instead, I will use the very basics of exterior differentiation and the Hodge star on spacetime to analyze the Maxwell and London equations and derive two possible implications raised by their comparison: a possible generalization of Maxwell equations and the independence of the emergent electromagnetic theory from electrons.

1.1 Determinants

Before we begin talking about physics, their are a few mathematical concepts that we must cover at a superficial level. The first of these is the determinant.

From Linear Algebra and Multivariable calculus, we have a few commonly understood properties of determinants. We begin with the geometrical: if we make up a matrix A whose rows are vectors $\vec{v}_i$ in $\mathbb{R}^n$, the determinant tells you about the generalized volume contained by a parallelpiped formed by those vectors. We write this as:

$$det(A) = det(\vec{a}_1, \vec{a}_2, ..., \vec{a}_n) = det(a_{i,j}) =$$

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix}$$

Next, we can use the structure of the determninant to express the cross product. When we do this, we also use some information about the orientation of the vectors to determine the sign of the output- we call this the right-hand rule. It seems obvious that this rule is arbitrary, and we will see later why that is.

Next up is the curl. This is especially interesting, because the curl isn't a determinant at all- we simply make use of it as a mnemonic to aid in remembering the form of the curl differential.

Notice something- all of these are 3 dimensional cases. We will see later how we can depart from 3 dimensions to higher (or lower) ones.

Finally, I present a few facts that may be well known, but that are not as often taught in introductory math classes. If we define a determinant as a function det which assigns to every $n \times n$ matrix A a number det(A) subject to the following axioms:

$det(I) = 1$ and

If E is an elementary column operation, then $det(E(A)) = k * det(A)$, where

$k = 1$ if E represents the addition of a multiple of a column to another

$k = c$ if E is the multiplication of a column by a number c

$k = -1$ if E is the operation of switching two columns

Then it can be shown that the determinant funtion both exists and is unique. That means that the determinant is more special than we thought, and those 4 properties, when they belong to any function, make it THE determinant.

1.2 Differential Forms

Calculus relies on the geometry of Euclidean space which allows us to integrate objects over paths and spaces and what have you. When you do multivariable calculus, you never really vary more than one variable at a time- you integrate over an interval of the real line for however many dimensions you wish. What about line integrals and surface integrals? They must somehow too be integrated, one coordinate variable at a time, in a manner similar to the euclidean coordinates.

We call these things that you integrate over manifolds, and we loosly define them as spaces which, locally, look like Eucilidean space. We define functions and vectors on them, and we can also define objects called forms on them. We define a 1-form $\omega \in \Lambda^1$(M) as a map from Vect(M) to C^∞(M). In other words, 1-forms take vector fields on M and give back functions on M with the condition of linearity. Later, we consider p-forms.

The basic example of a 1-form is the directional derivative: for any smooth function $f \in C^\infty$(M), there exists a 1-form df defined by

$$df(v) = \nabla f \cdot v$$

which we call the exterior derivative of f. 1-forms look like

$$\omega_\mu dx^\mu$$

and p forms look like

$$\nu_I dx^I$$

where I is the multi index that keeps track of what coordinate differentials are part of the form.

It is defined that $dx_\mu dx_\mu = 0$, so the maximum degree of forms that live on a manifold is dictated by its dimension. We define $dx^\mu \wedge dx^\nu = dx^\nu \wedge dx^\mu$

The Action of the Exterior Derivatve on Differential Forms We have to move fast to get through everything, so here's the formula for an external derivative:

$$d\omega = (\partial_\mu \omega_I) dx^\mu \wedge dx^I$$

The Leibniz law applied to $dd\omega$ makes it obvious that $dd\omega = 0$ It is also obvious that

$$d : \Lambda^p(M) \mapsto \Lambda^{p+1}(M)$$

Thus, the exterior derivative raises the degree of a form by 1.

Let me remind you of a few things that share similar properties.

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times v) = 0$$

I would love to work out the details of what I'm about to say, but, in $\mathbb{R}^{\nVDash}$, d acting on a function is like the gradient, d acting on a 1-form is like a curl, d acting on a 2 form is like the divergence.

2 (Not quite)Maxwell's Equations

Now, it is time to begin to reconstruct maxwell's equations.

$$div(E) = \rho \qquad curl(B) - \frac{\partial E}{\partial t} = \vec{j} \tag{1}$$

$$div(B) = 0 \qquad curl(E) + \frac{\partial B}{\partial t} = 0 \tag{2}$$

Notice that I have labeled two pairs- we will soon see why. Let's consider the second pair. Knowing what we now know, we can start thinking of E and B not as vector fields, but as 1-forms! Since we seem to want the curl of E and the divergence of B, lets try

$$E = E_i dx^i \qquad B = B_i dx^j dx^k \tag{3}$$

And define the electromagnetic field F, a 2-form on $\mathbb{R}^{\ntrianglerighteq}$, as

$$F = B + E \wedge dt \tag{4}$$

Or in its components,

$$F = 1/2 F_{\mu\nu} dx^\mu dx^\nu \tag{5}$$

So, what is dF?

It is NOT like the div since we are in 4 dimensions now.

$$dF = d(B + E \wedge dt) = dB + dE \wedge dt$$

By linearity of d, we can break it up into spatial and time derivatives:

$$dF = d_s B + dt \wedge \partial_t B + (d_s E + dt \wedge \partial_t E) \wedge dt$$

$$df = d_s B + (\partial_t B + d_s E) \wedge dt$$

This vanishes only when both parts vanish, or when

$$d_s B = 0$$

$$\partial_t B + d_s E = 0$$

Which are just Maxwell equations, so $dF = 0$ conveys all the info!

2.1 But why do we care?

You might say "Let's assume you're right, Zach, then whats the point of doing this if they say the same thing???" The answer is that it is general. If we take spacetime to be any dimensional manifold and define the electormagnetic field to be the 2-form F on M, and all we need to say is

$$dF = 0$$

Sometimes, not always, spacetime can be decomposed into space and time, which which splits F into E and B fields. This is extremely important! While we climbed up the mountain of theory with E and B as our guide, we must recognize that they are a construction that corresponds to the nonrelativistic environment we live in.

An interesting side note is that E and B are linear combinations of all the coordinate 1 and 2 forms, respectively, so with a topological splitting of spacetime comes a unique representation of E and B. Also, realize that Maxwell's equations only come out in the form that we know when spacetime is split.

If we have an arbitrary M, it may be differmorphic to our 3x1 spacetime in many ways or none- so there is no "best" way to do so.

3 The Metric

We're halfway done, and we still have a glaring problem- if you noticed this right away, kudos. The second set of Max's equations seems to mix the roles of d on E and B! Why are we taking the curl of a 2 form? How perverse! And those are equal signs, for God's sake- if we have $div(E)$ on one side (and assume it is a 2+1 form) then ρ must also be! Clearly, something is wrong here.

3.1 Hodge Star

Let's define some sort of function that takes $\Lambda^p(M) \mapsto \Lambda^{n-p}$ where n is the number of dimensions of M. It's not obvious that such a thing exists. But, we know about the things it acts on, so we know some things about it. We know that it has something antisymmetric about it, since forms have that property. We know from hindsight that we want $\star 1 = -dtdxdydz$, and it is trivially multininear. It smells like a determinant is coming. In fact, the full definition of the Hodge star is, for all $\omega, \mu \in \Lambda^p(M)$,

$$\omega \wedge \star\mu = \langle \omega, \mu \rangle dt \wedge dx \wedge dy \wedge dz$$

Where the wedge of μ and ν is the inner product defined by a metric on M, and $-dtdxdydz$ is a chosen positive orientation.

3.2 Minkowski and Lorentz

We need a metric on our spacetime M that varies smoothly from point to point, and we want it to have signature (3,1) since we know how our spacetime works from experience. We call a manifold with such a quality a Lorentz manifold

Let's start by splitting M into $\mathbb{R} \times S$ and defining

$$g = -dt^2 + g^3$$

Then we have

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g^3 & \\ 0 & & & \end{bmatrix}$$

which is the Lorentz metric.

Remember, spacetime is split, so we are dealing with special relativity.

Our fancy new metric lets us do some wild things! Forget about integrating arc lengths, we can use it to raise and lower operators and even to convert between vector fields and 1-forms! This brings up an absolutely fundamental subtelty that I did not touch on- tangent spaces. If we properly define ou metric, it takes vector fields from the tangent space and shoots out 1-forms. It does this by equipping the vector fields with the metric, almost like a wearable "Real-number midas-touch" because vector fields that wear it can now act on other vectors nd produce a real number, since they are now 1-forms. We can write it like

$$g : \nu \mapsto g(\nu, \bullet)$$

Now, to take an inner product of two vectors, we do

$$\langle \mu, \nu \rangle = g_{\alpha\beta}\mu^\alpha \nu^\beta$$

Examine

$$1/2(\langle E, E\rangle + \langle B, B\rangle)$$

This should smell like something familiar. Look at it with epsilons and mus where you expect them- its the energy density. Similarly, the electromagnetic Lagrangian is

$$1/2(\langle E, E\rangle - \langle B, B\rangle) = -1/2F_{\mu\nu}F^{\mu\nu}$$

4 The Second Pair of Equations

Hopefully $F_{\mu\nu}$ is still on the chalkboard. Let's compare it to its dual, $\star F_{\mu\nu}$:

$$\star F_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}$$

So this amounts to a transformation of $E_i \mapsto -B_i, B_i \mapsto E_i$ This duality accounts for the main difference between the two sets of equations! The other differences are the source terms. But wait, we have a fancy metric. so we can convert $\vec{j}$ to a 1-form

$$j = j_1 dx^1 + j_2 dx^2 + j_3 dx^3$$

Similarly, we can define

$$J = j - \rho dx^0$$

and call it the current. Now we can take the other 2 equations, and, realizing that we need to take the dual forms of E and B with the Hodge star, we can get rid of the problem we had identified earlier! Let's hit them both with the Hodge star at the same time:

$$\star|curl(B)| \wedge dt + \star(\partial^i E_i dx \wedge dy \wedge dz + \star\partial^0 E_i dx^j \wedge dx^k$$

$$\star d \star F = \star_s d_s \star_s E \wedge dt + \star_s d_s \star_s B - \partial^0 E_i dx^i$$

We get out

$$\star_s d_s \star_s Edt = \rho dt$$

$$\star_s d_s \star_s B - \partial^0 E_i dx^i = j$$

Now we have our new equations of electromagnetism:

$$dF = 0$$

$$\star d \star F = J$$

4.1 But things get better

This is where I'd like to take the chance to clear up some confusion brough on by the conventions we use while learning EM in school. Remember D, the electric displacement field, and H, the magnetic inductance field? We never really used them, and I was confused by them. But Maxwell used them independently of E and B in his formulae, and they are defined as dual fields:

$$D = \star E \qquad H = \star B$$

What's more, we can conveniently use these to get rid of some cumbersome Hodge stars, and they allow you to make more sense of the quantities used in calculations. For instance- we integrate E over surfaces all the time, but this would make more sense if we were integrating a 2-form over a surface. Thus, we now write Gauss's Law as

$$dD = \rho$$

Following this train of thought, we realize that we should identify ρ and J as 3-forms on spacetime (with 2 spatial differentials and one of time).

$$J = J_i dx^j \wedge dx^k \wedge dt$$

Maxwell's Equations We write his equations in a final state now.

$$\star F = H + D \wedge dt \tag{6}$$

and

$$dF = 0 \tag{7}$$

$$d \star F = J \tag{8}$$

And the differential equations are

$$d_s D = \rho \qquad d_s B = 0 \tag{9}$$

$$d_s E + \partial^0 H = 0 \qquad d_s H - \partial^0 D = j \tag{10}$$

5 A Possible Generalization of Maxwell's Equations

Now, let's take a look at the theories that must be in place in order for us to use Maxwell's formulae in the real world. We get from them that $j = \sigma \star E$ but we have no idea what the souce of this current actually is! For all we know, "electrons could be a particle or a fluid!" (The fluid case is even more interesting, as certain fluid motions correspond to equations of motion with conformal symmetry- i.e. they look just like Maxwell's equations... but thats a story for another day.)

5.1 Superconductivity and the London Equations

So now we take a detour into supercoductivity. In this domain, it makes sense to mimic a network inductor that satisfies $V = L\frac{\partial t}{\partial I}$. So in 1933, the Londons wrote down the "first London equation

$$E = \Lambda \star \frac{\partial t}{\partial J}, \tag{11}$$

where Λ is a constant. From our relations, $J = \star dH$, so $\Lambda^{-1}E = \star\frac{\partial t}{\partial J} = \frac{\partial t}{\partial} \star dH = \mu^{-1}\frac{\partial t}{\partial} \star d \star B$. With Faraday's law, $\frac{\partial t}{\partial}(d \star d \star B + (\mu/\Lambda)B = 0$. I won't show this, but $d \star d \star B = -\nabla^2 B$

Thus we have, for a superconducting slab infinite in x and y and with edges and $y = |a|$, the solution

$$\frac{\partial B}{\partial t} = C(t)\cosh(\frac{y}{\lambda}$$

where $\lambda = \sqrt{\frac{\Lambda}{\mu}}$. This says that magnetic fields have a "pentration depth" with which they fall off exponentially.

But, the Meissner effect emperically shows that there is NO B penetration. So the Londons insisted that quantity in the parentheses of the above differential equations vanishes. This is the second London equation:

$$d \star J = -\frac{1}{\Lambda}B \tag{12}$$

Now I wave my hands and claim that the London equations yield a modified Maxwell equation:

$$\Lambda d \star J = -F$$

Notice that superconductivity leads to modified Maxwell equations, whereas ordinary conductivity is supplementary to them.

5.2 Modification to the Lagrangian

On a simply connected spacetime, we can use the vector potential, a 1-form, to obtain the electromagnetic field: $dA = F$. Then, Maxwell's equations become the variational equations for the Lagrangian with density

$$\mathcal{L}_M = 1/2dA \wedge \star dA - \mu A \wedge J \tag{13}$$

But the London equations yield the "Proca" equations, after we drop the requirement that $J = 0$ (although we still have conservation of charge)

$$\mathcal{L}_P = 1/2(dA \wedge \star dA - \frac{1}{\lambda^2}A \wedge \star A) \tag{14}$$

You have to trust me, but λ^2 has units of mass squared, and we no longer require an abscence of current J. If you do the calculation, Maxwell's equations pop out of these in the mass zero limit. This implies that the Maxwell equations are the mass-zero limit of the Proca equations!

6 Ending Remarks

A few comments are in order. Maybe the world is actually superconducting and we live in a mass zero limit in which photons are massless and, in vaccua, the equations of electromagnetism attain conformal symmetry in the vaccum.

However, if we firmly believe that electromagnetism is a gauge theory like the standar model (this is the aspect of the geometrical stucture that I didn't have time to cover), we could take mass acquistion as a symmetry spontaneously broken by a Higgs mechanism. In the standard treatment one gets the Higgs field as the spin zero field given by a Cooper pair of electrons in a superconductor.

But since the electrons are not needed for charge transport, as no external source term occurs in the London equation formalism, one might imagine an entirely different origin for the Higgs field. Do we need electrons for superconductivity?