1

Topology, Quantum Mechanics, and Gauge Theories: The Aharonov-Bohm Effect

Zachary Bednarke
Vanderbilt Univeristy

December 2, 2015

[1]To I owe respect to Reyer Sjamaar, John Baez, and Shlomo Sternberg for writing books and lecture notes that showed me how cool this subject is.

1 Introduction

There are exceedingly many phenomena in that lie at the intersection of the interactions of the geometrical, topological structures of gauge theories and the foundational form of quantum mechanics, (take for example, the beautiful construction of the standard model) but there are few situations where the web of interrelationships are made so apparent as in Aharanov-Bohm Effect. Rarer, still, are the examples that can be experimentally detected, let alone harnessed for engineering applications!

For these reasons, the AB effect has been intensely explored both theoretically and experimentally throughout the five decades that have passed since Aharonov and Bohm first published their findings. Its important results provide the operational basis for superconducting quantum interference devices, or SQUIDs, which are used to measure the magnetic flux of tiny currents to amazing precision. Such a device was employed by Vanderbilt's own John Wikswo[6] when he made his landmark measurement of the action potential of a nerve axon.

Furthermore, the AB effect has a profound presence in condensed matter physics, and especially in non-superconducting nano-rings. For instance, in 1985[5], Webb and Washburn noticed that even though electrons were flowing through a metal, their interaction with the lattice did not disrupt the interference of the AB effect. This is an example of the topological nature of this effect and illustrates its resilience.

2 The role of potentials

In classical electrodynamics, the vector and scalar potentials were first introduced as a convenient mathematical aid for calculating the physical fields E, B, and their duals D and H. While it is true that potentials were needed in order to develop a canonical Lagrangian formalism of dynamics, gauge freedom indicated that the equations of motion relied on the fields alone.

In quantum mechanics, however, the canonical formalism of dynamics requires the potentials. As a result, they cannot be removed from the basic equations, whether they are Schrödinger's equation or Feynman's path integrals. Nevertheless, these equations are gauge invariant, so it seems, at first, as though potentials have no physical significance. This is completely wrong! In this paper, I will show that, despite the gauge invariance of Schrödinger's equation and Feynmans path integrals, the potentials do indeed hold physical significance. I will then discuss the bountiful experimental and practical applications of these results.

2.1 Gauge Invariance In Classical Electromagnetism

In classical electromagnetism, all of the equations of dynamics are contained within the set of equations[4]

$$dF = 0 \tag{1}$$

$$d \star F = J \tag{2}$$

where the electromagnetic field F is a two form on spacetime, and J is a three form on spacetime. If we split spacetime into $\mathbb{S} = \mathbb{R} \times S$, F has the form $F = B + E \wedge dt$ where E and B are 1 and 2 forms, respectively, on S.

Furthermore, we can write

$$F = dA \tag{3}$$

where A is the 1-form vector potential on $\mathbb{S}$. Thus, for a given F, A is only determined up to the addition of a closed 1-form ω. In other words, if we gauge transform A to give $A' = A + \omega$,

$$dA' = dA = F \tag{4}$$

and we obtain the same electromagnetic field.

in the case of 3 spatial dimensions at the nonrelativistic limit of the above equation, we state the requirements of a gauge transformation $(\vec{A}, \phi) \to (\vec{A'}, \phi')$ as

$$\vec{A'} = \vec{A} - \nabla\Lambda \tag{5}$$

$$\phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t} \tag{6}$$

Potentials in Schrödinger Mechanics and Feynman Path Integrals The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial \Psi}{\partial t} = H\Psi \tag{7}$$

which reduces to

$$E\Psi = H\Psi \tag{8}$$

in the time-independent case. This equation governs all non-relativistic quantum mechanics. Since the Hamiltonian is always involved, we see that potentials enter the theory at its most basic level. Likewise, in Feynman's path integral approach, the propogator is given by

$$U(x,t;x',t') = \int_{x'}^{x} e^{iS\left[x(t)\right]/\hbar}\mathcal{D}\left[x(t)\right] \tag{9}$$

where $\mathcal{D}\left[x(t)\right]$ is the Lebesgue measure on the space of all paths from (x',t') to (x,t), and $S\left[x(t)\right]$ is the action:

$$S\left[x(t)\right] = \int_{t'}^{t} \mathcal{L}dt'' = \int_{t'}^{t} (\frac{1}{2}m|\dot{\vec{r}}|^2 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi)dt''. \tag{10}$$

We again see the direct role played by the potentials in this formulation.

2.2 Gauge Invariance of The Two Theories

Note: I have attached a sheet to this that has my proof of the gauge invariance of the Schrödinger equation. I followed along the guidelines of Shankar exercise 18.4.4 to fill it out.

Gauge Invariance of the Feynman Path Integral Suppose we perform a gauge transformation $(\vec{A}, \phi) \to (\vec{A'}, \phi')$ to get $\vec{A'} = \vec{A} - \nabla\Lambda$ and $\phi' = \phi + \frac{1}{c}\frac{\partial\Lambda}{\partial t}$. Then

$$S \to S_\Lambda = S - \int_{t}^{\prime t} \frac{q}{c}(\vec{v}\cdot\nabla\Lambda + \frac{\partial\Lambda}{\partial t''}dt'' \tag{11}$$

But

$$\vec{v}\cdot\nabla\Lambda + \frac{\partial\Lambda}{\partial t''} = \frac{\partial\Lambda}{\partial t''} \tag{12}$$

is the total derivative along the trajectory. So

$$S_\Lambda = S + \frac{q}{c}\big[\Lambda(\vec{r}', t') - \Lambda(\vec{r}, t)\big] \tag{13}$$

But since the endpoints of the paths are held fixed, varying the action will lead to the same dynamics.

The propagator thus picks up only an exponential term:

$$U \to U_\Lambda = U \cdot exp(\frac{iq}{\hbar c}\big[\Lambda(\vec{r}', t') - \Lambda(\vec{r}, t)\big]) \tag{14}$$

Because

$$U(\vec{r}, t; \vec{r}', t') = \langle\vec{r}|U(t, t')|\vec{r}'\rangle \tag{15}$$

we see that the effect of the gauge transformation is equivalent to a change in coordinate basis:

$$|\vec{r}\rangle \to |\vec{r}_\Lambda\rangle = e^{(iq\Lambda/\hbar c)}|\,|\vec{r}\rangle \tag{16}$$

Thus, the state vector is unchanged after a gauge transformation.

3 The Effect

Now I will describe the experimental setup that demonstrates this effect[1]. Consider a beam of coherent electrons that is split into two parts, each of which are allowed to enter a long, conducting, cylindrincal tube, as shown in attached figure 1.

After they go through the tubes, the beams are combined coherently at F. We intend to chop the beams into wave packets that long compared to the wavelength λ, but short compared with the length of the tubes. The potential then grows as a function of time, but differently in each tube. Funally, it falls back to zero before the electron comes near the other edge of the tube. Thus, the potential is nonzero only while the electrons are well inside the tube (region II). When the electron is in region III, there is again no potential. The purpose here is to ensure that the electron is in a time varying potential, but not in an electric field (the field does not penetrate far from the edge of the tube, and is nonzero only when the electron is far from the edges).

Now let $\psi(x,t) = \psi_1^0(x,t) + \psi_2^0(x,t)$ be the wave function of an electron when the potential is absent, where ψ_i represents the part going through the ith tube. The solution is thus

$$\psi = \psi_1^0 e^{-iS_1/\hbar} + \psi_2^0 e^{-iS_2/\hbar}, \tag{17}$$

where

$$S_i = e\int \phi_i dt \qquad (i = 1, 2). \tag{18}$$

Thus, the interference of the two beams will depend on the phase difference $(S_1 - S_2)/\hbar$, and there is a measureable effect done by the potential independently of the fields. This effect is essentially quantum mechanical since it is due to wave interference.

3.1 The Dual Case

Now, the structure of Maxwell's equations in vaccua,

$$dF = 0 \qquad d \star F = 0, \tag{19}$$

indicate that, due to relativistic considerations, we should find a similar result with the magnetic field and vector potential.

Notice that the phase difference, $(S_1 - S_2)/\hbar$, can be expressed as the integral $\frac{e}{\hbar} \oint \phi dt$ around a cycle in spacetime, with ϕ evaluated at the center of the wave packet. The relativistic generalization of the above integral is

$$\frac{e}{\hbar} \oint (\phi dt - \frac{\vec{A}}{c} \cdot dx), \tag{20}$$

where the integration path is any circuit in spacetime.

Let's now look at a time independent path. The above integral says that

$$\Delta S/\hbar = -\frac{e}{c\hbar} \oint \vec{A} \cdot dx, \tag{21}$$

where$\oint \vec{A} \cdot dx = \int \vec{H} \cdot ds = \Phi$ is the magnetic flux though the circuit. This corresponds to a second experiemntal setup. Imagine that we, again, spit the beams coherently. In the middle of the two beams, be place a very long solenoid. We shield the electrons from it by putting a supercoductor around it, but the vector can still have an effect.

Since all of the magnetic field comes through a small surface and the regions through which the electrons pass have no fields, the flux is a constant.

Due to the vector potential, the phase difference between the paths is given by (21). The phase difference will always be π along the line that goes through the original beam direction[1], and this is because the beams destructively interfere completely there.

To quote a result from Shankar, whenever $(q\Phi/\hbar c) = 2n\pi$ where $n = 1, 2...$, the interference pattern moves a full cycle. In otherwords, an integer multiple of the flux quantum

$$\Phi_0 = \frac{2\pi\hbar c}{q} \tag{22}$$

makes no difference in the physics of the beam.

A Brief Topological Discussion As we saw, the path a particle took as it went around the solenoid had physical consequences, and, had a particle gone around more than once, it would have picked up an even greater phase factor. This could be seen more directly from Feynman's sum over paths, although the two approaches yield the same result. Also, notice that this experimental situation effectively introduces a point in spacetime that is no longer accessible to the electrons: a singularity in the manifold of spacetime. In other words, space is no longer simply connected. We may as well have redefined spacetime to be

$$\mathbb{S} = \mathbb{R} \times S \setminus \{B(r < R)\} \tag{23}$$

Here, S is space, and $B(r < R)$ is the open ball with radius less than R, the radius of the solenoid.

This indicates that not all integrals of the vector potential over paths will be the same. This idea has even deeper consequences in differential topology and has been studied intensively.

4 Applications of the Aharonov-Bohm Effect

As I described at the outset, this effect involves theoretical interplay of geometry, topology, ideas from quantum mechanics and yet is still an effect that can be readily probed with experiment and practically applied. Moreover, the precision with which one can measure miniscule amounts of flux in a circuit by looking at interference patterns in an electron beam allows for SQUIDs to play an important role as tools to this day.

4.1 SQUIDs

SQUIDs are impressive pieces of human engineering. Following the Aharonov-Bohm effect's exposition in 1959, they were invented in 1964 by Robert Jaklevic, John J. Lambe, James Mercereau, and Arnold Silver of Ford Research after Brian David Josephson postulated the Josephson effect in 1962. They utilize the quantum mechanical Aharonov-Bohm effect to its full potential (mind that pun) and allow us to measure tiny currents wherever they occur- even in living things! SQUIDS are actively used at Vanderbilt in the Biophysics department for everything from measuring the electromagnetic fluctuations of the human gut to measureing the action potential of nerves, as I mentioned earlier. Outside of Biology, they can be used to detect tiny movements- for this reason, their is a SQUID in each of the four gyroscopes on Gravity Probe B that is currently observing the limits of general relativity[3].

However, they are even more impressive than I had let on, as they rely on several other effects of superconductors as well! For instance, without the discovery of Josephson curents, we would not have Josephson junctions, and there would be lacking an extremely effective way to set up a device to measure the Aharonov-Bohm effect.

4.2 Carbon Nanotubes

Another curious place in which we find miniscule amounts of magnetic flux is within carbon nanotubes. In the case of these systems, a handful of electrons moving through the tubes' walls replace the beams of electrons and the paths they travel are around the nanotube enclosing the flux. Thus, when electrons pass through a cylindrical electrical conductor aligned in a magnetic field, their wave-like nature manifests itself as a periodic oscillation in the electrical resistance as a function of the enclosed magnetic flux[2].

Just as in the case of non-superconducting nano-rings, we find that the interference pattern resulting from the Aharonov-Bohm effect is strongly preserved. Again, this is due to the topological nature of the effect that makes it resilient.

5 Conclusion and Future Directions

I've tried my absolute best to convey how interesting this effect is, both theoretically and experimentally. It is impossible to fully grasp its implications without having knowledge of gauge theories, topology, and the foundations of quantum mechanics. At a higher, more pedagogical level, this effect is important because it instructs us to take nothing as a certainty: while potentials had played no physical role in dynamics ever before, two guys had the brains to figure out a revolutionary deviation resulting from quantum mechanics. A similar mindset can be taken when considering the future directions of this endeavor. One possible future direction is in the engineering of quantum computers. Leading theorists of today like Frank Wilzcek and Max Tegmark posit that quantum computing will rely heavily upon topological invariants in qunatum states, as they are the most resilient! And if there is any way to make a cubit state more resilient, the people who implement it will be made rich for doing so.

Outside of quantum computing, some experimentalists have considered utilizing the measurement of flux quanta to maesure gravitational waves. Another proposed area is that of fine measurements of the casimir effect. To me this would be incredibly exciting, as it would combine the already vast number of concepts within the Aharonov-Bohm effect with those of quantum field theory.

References

[1] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. 115(3):485–491.

[2] Adrian Bachtold, Christoph Strunk, Jean-Paul Salvetat, Jean-Marc Bonard, Laszl Forr, Thomas Nussbaumer, and Christian Schnenberger. Aharonovbohm oscillations in carbon nanotubes. 397(6721):673–675.

[3] Herman Batelaan and Akira Tonomura. The aharonovbohm effects: Variations on a subtle theme. 62(9):38–43.

[4] Zachary Bednarke. Differential forms and a possible generalization of maxwell's equations.

[5] Jrg Schelter, Patrik Recher, and Bjrn Trauzettel. The aharonov-bohm effect in graphene rings. 152(15):1411–1419.

[6] J. P. Wikswo, J. P. Barach, and J. A. Freeman. Magnetic field of a nerve impulse: first measurements. 208(4439):53–55.