Quantitative regularity estimates on manifolds with boundary under integral curvature bounds.

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- This talk is about *quantitative regularity estimates*.
- That is to say, what conditions on M imply that there exist coordinate charts of a definite size and in which the metric has good regularity in these charts?
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Standard comparison geometry can be thought of as an early example of this.

- If the second fundamental form II of ∂M is positive, then there is a collar neighborhood about ∂M of a definite size that depends on the norm of II and the sectional curvature $\sec(M)$.
- Moreover, one can control the volume distortion of the normal exponential map. (cf. Heintze-Karcher (1978)).
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However, there exist more sophisticated systems of equations defined on M that should (and do) give better results. Interior results (when $\partial M = \emptyset$) have been known for a long time.

- If M satisfies

$$\mathrm{Vol}(M) > v_0 > 0,\ \mathrm{Diam}\, M \le D,\ |\sec(M)| \le K,$$

  then M admits an atlas of charts of a definite size, in which the metric is uniformly bounded in $C^{1,\alpha}$, any $\alpha < 1$. The size of the charts and the regularity only depend on $\dim(M)$, v_0, D, and K.
- The "definite size" comes from Cheeger's Lemma: the hypotheses imply that $\mathrm{inj}(M)$ is bounded below.
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In 1990, Anderson showed (more than) the following:

- If M satisfies

$$\mathrm{Diam}\, M \leq D,\ \mathrm{inj}(M) \geq i_0 > 0 \text{ (size conditions)}$$

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$$|\mathrm{Ric}(M)| \leq K \text{ (regularity condition)}$$

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Key idea: In a harmonic coordinate system, the laplacian of the metric has the form

$$\Delta g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\,\mathrm{Ric}(g)_{\alpha\beta},$$

where $B_{\alpha\beta}$ is quadratic in the first derivatives of g_{ij}.
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Let us return to the case where ∂M is nonempty.

- In appropriate boundary harmonic coordinates, we have

$$\partial_\nu g^{nn} = -2(n-1)Hg^{nn},$$

  where ∂_ν is the normal derivative, H is the mean curvature. So one might hope that boundary regularity can be extracted from the mean curvature.
- In order to apply boundary elliptic $L^{2,p}$ regularity, one would then need H to be bounded the Besov space $B^{1-1/p,p}(\partial M)$, but this is a rather unnatural requirement!

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Theorem (Anderson-Katsuda-Kurylev-Lassas-Taylor 2005)

Suppose M satisfies

- *Intrinsic boundary conditions: Diameter* $\leq D$, *|Ricci|* $\leq K$, *injectivity radius* $\geq i_0 > 0$.
- *Bulk conditions (on* $M \backslash \partial M$*): Diameter* $\leq D$, *|Ricci|* $\leq K$, *injectivity radius* $\geq i_0 > 0$
- *Extrinsic conditions:* $|H|_{C^{0,\alpha}(\partial M)} < H_0$, *normal injectivity radius* $\geq i_0 > 0$.

Then about each point of M there is coordinate chart (ϕ, U) *in which the* $C^{1,\alpha}(U)$ *norm of g is bounded in terms of* n, D, K, i_0, H_0. *The diameter of U is bounded from below in terms of the same constants.*

- If the mean curvature H is bounded in the stronger Lipschitz norm, then g is bounded in the Zygmund space C^2_*.

The regularity comes from a careful analysis of the following interdependent system of equations.

$$\Delta g^{in} = B^{in}(g, \partial g) - 2\,\mathrm{Ric}(g)^{in} \tag{1}$$

$$\partial_\nu g^{nn} = -2(n-1)Hg^{nn} \tag{2}$$

$$\partial_\nu g^{\alpha n} = -(n-1)Hg^{\alpha n} + \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} \tag{3}$$

$$\Delta g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\,\mathrm{Ric}(g)_{\alpha\beta} \tag{4}$$

$$g_{\alpha\beta}\big|_{\partial M} = h_{\alpha\beta}, \tag{5}$$

where $1 \leq i, j \leq n$, $1 \leq \alpha, \beta \leq n-1$.

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Jeff Cheeger explained in his thesis that quantitative regularity results can be turned into so-called (pre)compactness or convergence theorems.

Definition

Write that $(M_i, g_i) \to (M_\infty, g_\infty)$ in the $C^{k,\alpha}$ ($0 < \alpha < 1$, $k \in \mathbb{Z}_{\geq 0}$) topology if there exist diffeomorphisms

$$f_i : M_\infty \to M_i$$

such that $f_i^* g_i \to g_\infty$ in the $C^{k,\alpha}$ topology on M_∞.

- Here $C^{k,\alpha}$ is the usual Hölder space.
- A similar definition can be made for $L^{k,p}$ ($1 < p < \infty$, $k \in \mathbb{Z}_{\geq 1}$), where $L^{k,p}$ is the usual Sobolev space.

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Definition

A class $\mathcal{M}$ of smooth Riemannian manifolds is *precompact* in the $L^{k,p}$ (resp. $C^{k,\alpha}$) topology if every sequence $\{(M_i, g_i)\} \subset \mathcal{M}$ admits an $L^{k,p}$ (resp. $C^{k,\alpha}$) convergent subsequence.
The class is *weakly precompact* if every sequence admits a weakly convergent subsequence.

In this language, the theorem reads

Theorem (Anderson-Katsuda-Kurylev-Lassas-Taylor 2005)

Let $\mathcal{M}$ be the class of Riemannian manifolds with boundary satisfying

- *Intrinsic boundary conditions: Diameter $\leq D$, |Ricci| $\leq K$, injectivity radius $\geq i_0 > 0$.*
- *Bulk conditions (on $M \backslash \partial M$): Diameter $\leq D$, |Ricci| $\leq K$, injectivity radius $\geq i_0 > 0$*
- *Extrinsic conditions: $|H|_{Lip} < H_0$, normal injectivity radius $\geq i_0 > 0$.*

Then $\mathcal{M}$ is precompact in $C^{1,\alpha}$, any $0 < \alpha < 1$. In fact $\mathcal{M}$ is weakly precompact in the Zygmund space C^2_*.

- This still leaves open some fundamental questions. For instance, what kind of boundary regularity should we expect, even in the special case of convex manifolds with uniformly bounded curvature quantities?
- Since the mean curvature involves only one derivative of the metric, the best one could hope for is $L^{1,p}$ or $C^{0,\alpha}$ regularity.

Write *II* for the second fundamental form.

Theorem (Kodani 1990)

Suppose $\mathcal{M}$ is the class of Riemannian manifolds with boundary satisfying

- *Diameter (and boundary Diameter)* $\leq D$
- *Volume* $\geq v_0 > 0$
- *|Sectional Curvature|* $\leq K$
- $0 \leq II < K$

Then $\mathcal{M}$ is precompact in the Lipschitz topology on metric spaces.

Remark: This conclusion is quite a bit weaker than, say, $C^{0,\alpha}$ precompactness, since it does not make any reference to the pointwise convergence of the Riemannian metrics.

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The regularity part of Kodani's theorem comes from the estimates of Heintze-Karcher, and therefore ultimately boils down to ODE comparison theory.

Theorem (K 2013)

Let $\mathcal{M}$ be the class of Riemannian manifolds with boundary satisfying

- *Intrinsic boundary conditions: |Sectional Curvature| $\leq K$, Area $\geq A_0$*
- *Bulk conditions (on $M \backslash \partial M$): Diameter $\leq D$, |Sectional Curvature| $\leq K$*
- *Extrinsic conditions:* $0 < 1/H_0 < H < H_0$

Then $\mathcal{M}$ is precompact in the $C^{0,\alpha}$ topology, any $0 < \alpha < 1$. In fact, $\mathcal{M}$ is precompact in the $L^{1,p}$ topology, $p < \infty$.

- The lower area bound on the intrinsic boundary is used to show that there can be no ‘local volume collapse’ in the interior. This helps to obtain charts of a definite size
- To get the $L^{1,p}$ metric regularity, one might hope to utilize elliptic regularity for the laplacian as a map

$$\Delta : L^{1,p} \to L^{-1,p},$$

but an estimate of this type, in contrast to the $L^{2,p}$ case, requries a priori C^1 control over the metric tensor– a condition that is even stronger than our desired conclusion!

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Theorem (K)

Under the same hypothesis, $\mathcal{M}$ is precompact in the $C^{0,\alpha}$ topology, any $0 < \alpha < 1$, and precompact in the $L^{1,p}$ topology, any $1 < p < \infty$.

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We also obtain an analogue of the following theorem, when the mean curvature is merely pointwise bounded.

Theorem (Anderson-Katsuda-Kurylev-Lassas-Taylor 2005)

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There are other compactness theorems that depend on L^p bounds of curvature quantities instead of pointwise bounds. Once again, the regularity comes from linear $L^{2,p}$ elliptic estimates. Analogues of these theorems should be expected when $H \in B^{1-1/p,p}(\partial M)$ (but one would like to do even better).

- Boundary estimates under integral curvature bounds.
- Controlled volume growth:
- For fixed $C > 0$ Consider the largest r so that for any $s < r$ there holds

$$\frac{\mathrm{Vol}(B_p(s))}{s^n} \geq C, \tag{6}$$

where $B_p(s)$ is the ball of radius s centered at p.

- Call r the *volume radius at p* and write $r := r_\nu(p)$.

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Definition

Fix $p > n/2$. Let $\mathcal{M}$ be the class of Riemannian manifolds with boundary satisfying

- Intrinsic boundary conditions:

$$\mathrm{Diam}(\partial M) \leq D, \; ||\mathrm{sec}(\partial M)||_{L^p(\partial M)} \leq K, r_\nu(\partial M) \geq r_0 > 0.$$

- Bulk conditions (on $M \backslash \partial M$):

$$\mathrm{Diam}(M) \leq D, \; ||\mathrm{sec}(M)||_{L^p(M)} \leq K. \tag{7}$$

- Extrinsic conditions: Assume that $r_\nu(p) \geq r_0$ for each $p \in M$, and that the mean curvature H satisfies

$$||H||_{B^{1-1/p,p}(\partial M)} \leq H_0. \tag{8}$$

Here $B^{1-1/p,p}(\partial M)$ is the space of restrictions of functions in $L^{1,p}(M)$.

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The class $\mathcal{M}$ is precompact in the weak $L^{2,p}$ topology.

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This is nice, but the condition on H seems unnatural. Namely, it is contrived in order to be able to use the $L^{2,p}$ regularity theory.

Theorem (K, 2016)

Fix $p > n$ and $q > 2p(n-1)$. Let $\mathcal{M}$ be the class of Riemannian manifolds with boundary satisfying

- *Intrinsic boundary conditions:*

$$\mathit{Diam}(\partial M) \leq D,\ ||\sec(\partial M)||_{L^p(\partial M)} \leq K, r_\nu(\partial M) \geq r_0 > 0.$$

- *Bulk conditions (on $M \backslash \partial M$):*

$$\mathit{Diam}(M) \leq D,\ ||\sec(M)||_{L^p(M)} \leq K. \tag{9}$$

- *Extrinsic conditions: Assume that $r_\nu(p) \geq r_0$ for each $p \in M$, and that the mean curvature H satisfies*

$$||H||_{L^q(\partial M)} \leq H_0. \tag{10}$$

Theorem

The class $\mathcal{M}$ is precompact in the $L^{1,p}$ topology.

- As before, we cannot obtain the $L^{1,p}$ bounds on the metric from the regularity theory of $\Delta : L^{1,p} \to L^{-1,p}$. Instead, we consider the mapping $\Delta : L^{1+\epsilon,p} \to L^{\epsilon-1,p}$, $0 < \epsilon < 1/p$.
- This requires a priori control of the metric in C^β,

$$1 - n/p < \beta < 1,$$

control that we do not have!
- The L^q bound on H allows one to initially boost the regularity of g^{nn} up to C^β, then a careful bootstrap argument gives regularity for the other metric components.

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$$1 - n/p < \beta < 1,$$

  control that we do not have!
- The L^q bound on H allows one to initially boost the regularity of g^{nn} up to C^β, then a careful bootstrap argument gives regularity for the other metric components.

The L^q bound on H can be subtituted into any of the previous theorems (one then modifies the conclusion accordingly.) This produces a reasonably satisfying convergence theory for manifolds with boundary.

Conjecture

Fix $n > p > n/2$ and $q > \frac{2(n-1)np}{n-p}$. Let $\mathcal{M}$ be the class of Riemannian manifolds with boundary satisfying

- *Intrinsic boundary conditions:*

$$\mathit{Diam}(\partial M) \leq D,\ ||\sec(\partial M)||_{L^p(\partial M)} \leq K, r_\nu(\partial M) \geq r_0 > 0.$$

- *Bulk conditions (on $M \backslash \partial M$):*

$$\mathit{Diam}(M) \leq D,\ ||\sec(M)||_{L^p(M)} \leq K. \tag{11}$$

- *Extrinsic conditions: Assume that $r_\nu(p) \geq r_0$ for each $p \in M$, and that the mean curvature H satisfies*

$$||H||_{L^q(\partial M)} \leq H_0. \tag{12}$$

Theorem

The class $\mathcal{M}$ is precompact in the $L^{1,\frac{np}{n-p}}$ topology.

Thank you!