Topology of the space of metrics with positive scalar curvature

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Notations:

- W is a compact manifold, $\dim W = d$,
- $\mathcal{R}(W)$ is the space of all Riemannian metrics,
- if $\partial W \neq \emptyset$, we assume that a metric $g = h + dt^2$ near ∂W;
- R_g is the scalar curvature for a metric g,
- $\mathcal{R}^+(W)$ is the subspace of metrics with $R_g > 0$;
- if $\partial W \neq \emptyset$, and $h \in \mathcal{R}^+(\partial W)$, we denote

$$\mathcal{R}^+(W)_h := \{g \in \mathcal{R}^+(W) \mid g = h + dt^2 \text{ near } \ W\}.$$

- "psc-metric" = "metric with positive scalar curvature".

Existence Question:

- For which manifolds W the space $\mathcal{R}^+(W)$ is not empty?

It is well-known that for a closed manifold W,

Yamabe invariant

$$\mathcal{R}^+(W) \neq \emptyset \iff Y(W) > 0.$$

Assume $\mathcal{R}^+(W) \neq \emptyset$.

More Questions:

- What is a topology of $\mathcal{R}^+(W)$?
- In particular, what are the **homotopy groups** $\pi_k \mathcal{R}^+(W)$?

Let (W, g) be a spin manifold. Then there is a canonical real spinor bundle $\mathcal{S}_g \to W$ and a Dirac operator D_g acting on the space $L^2(W, S_g)$.

Theorem. (Lichnerowicz '60)
$D_g^2 = \Delta_g^s + \frac{1}{4} R_g$. In particular, if $R_g > 0$ then D_g is invertible.

For a spin manifold W, $\dim W = d$, we obtain a map

$$(W, g) \mapsto \frac{D_g}{\sqrt{D_g^2 + 1}} \in \mathbf{Fred}^{d,0},$$

where $\mathbf{Fred}^{d,0}$ is the space of $\mathcal{C}\ell^d$-linear Fredholm operators.

The space $\mathbf{Fred}^{d,0}$ also classifies the real K-theory, i.e.,

$$\pi_q \mathbf{Fred}^{d,0} = KO_{d+q}$$

It gives the index map

$$\alpha : (W, g) \mapsto \mathrm{ind}(D_g) = [D_g] \in \pi_0 \mathbf{Fred}^{d,0} = KO_d.$$

The index $\mathrm{ind}(D_g)$ does not depend on a metric g.

Index Theory gives a map:

$$\alpha : \Omega_d^{\mathrm{Spin}} \longrightarrow KO_d.$$

Thus $\alpha(M) := \mathrm{ind}(D_g)$ gives a topological obstruction to admitting a psc-metric.

Theorem. (Gromov-Lawson '80, Stolz '93) Let W be a spin simply connected closed manifold with $d = \dim W \geq 5$. Then $\mathcal{R}^+(W) \neq \emptyset$ if and only if $\alpha(W) = 0$ in KO_d.

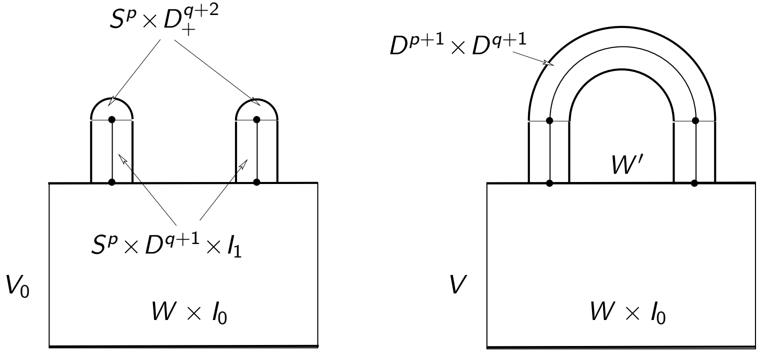
Topology: There are enough examples of psc-manifolds to generate $\mathrm{Ker}\, \alpha \subset \Omega_d^{\mathrm{Spin}}$, then we use **surgery.**

Surgery. Let W be a closed manifold, and $S^p \times D^{q+1} \subset W$.

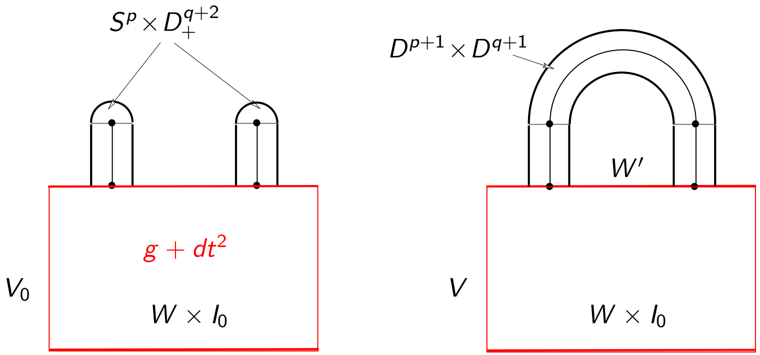
We denote by W' the manifold which is the result of a surgery along the sphere S^p:

$$W' = (W \setminus (S^p \times D^{q+1})) \cup_{S^p \times S^q} (D^{p+1} \times S^q).$$

Codimension of this surgery is $q+1$.



Surgery Lemma. (Gromov-Lawson) Let g be a metric on W with $R_g > 0$, and W' be as above, where $q+1 \geq 3$. Then there exists a metric g' on W' with $R_{g'} > 0$.



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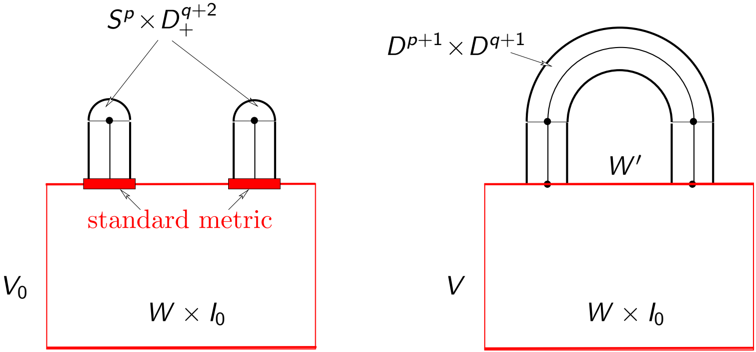

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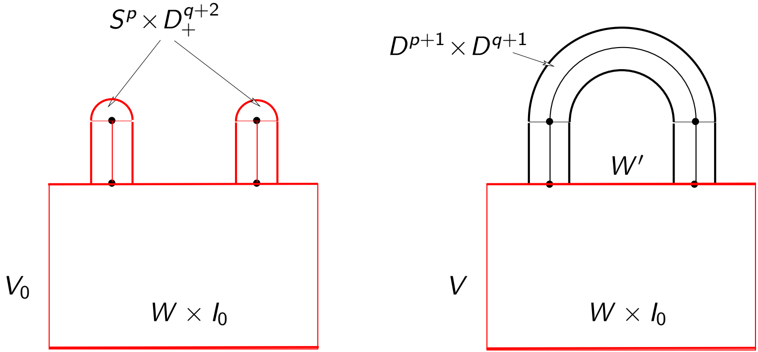

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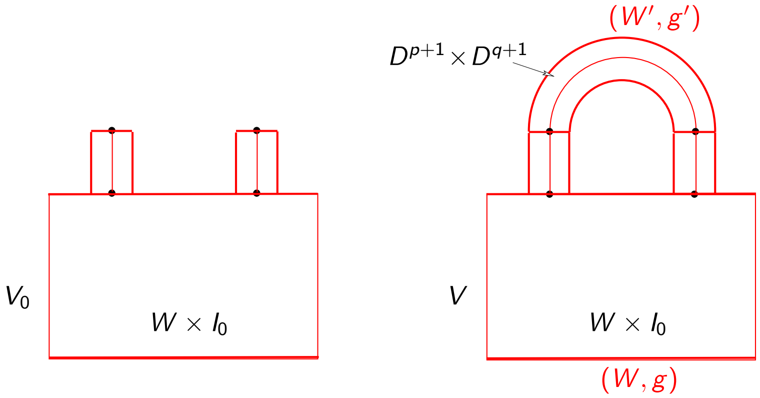


Conclusion: Let W and W' be simply connected cobordant spin manifolds, $\dim W = \dim W' = d \geq 5$. Then

$$\mathcal{R}^+(W) \neq \emptyset \iff \mathcal{R}^+(W') \neq \emptyset.$$

Assume $\mathcal{R}^+(W) \neq \emptyset$.

Theorem. (Chernysh, Walsh) Let W and W' be simply connected cobordant spin manifolds, $\dim W = \dim W' \geq 5$. Then

$$\mathcal{R}^+(W) \cong \mathcal{R}^+(W').$$

Let $\partial W = \partial W' \neq \emptyset$, and $h \in \mathcal{R}^+(\partial W)$, then

$$\mathcal{R}^+(W)_h \cong \mathcal{R}^+(W')_h.$$

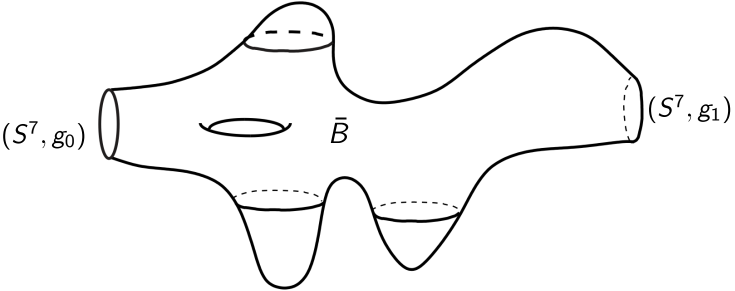
Questions:

- What is a topology of $\mathcal{R}^+(W)$?
- In particular, what are the **homotopy groups** $\pi_k \mathcal{R}^+(W)$?

Example. Let us show that $\mathbf{Z} \subset \pi_0 \mathcal{R}^+(S^7)$.

Let B be a Bott manifold, i.e. B is a simply connected spin manifold, $\dim B = 8$, with $\alpha(B^8) = \hat{A}(B) = 1$.

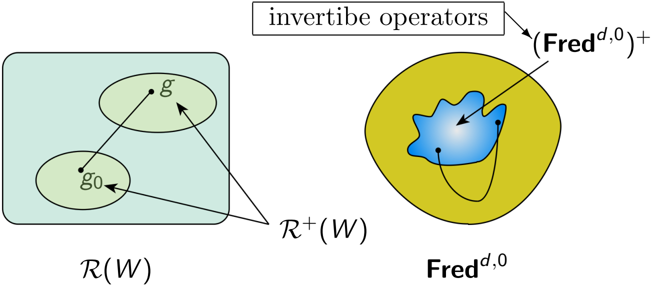
Let $\bar{B} := B \setminus (D^8_1 \sqcup D^8_2)$:



Thus $\mathbf{Z} \subset \pi_0 \mathcal{R}^+(S^7)$.

Index-difference construction (N.Hitchin):

Let $g_0 \in \mathcal{R}^+(W) \neq \emptyset$ be a base point, and $g \in \mathcal{R}^+(W)$.
Let $g_t = (1-t)g_0 + tg$.



Fact: The space $(\mathbf{Fred}^{d,0})^+$ is contractible.

The index-difference map: $\mathsf{A}_{g_0} : \mathcal{R}^+(W) \longrightarrow \Omega\mathbf{Fred}^{d,0}$.
We obtain a homomorphism:

$$\mathsf{A}_{g_0} : \pi_k\mathcal{R}^+(W) \longrightarrow \pi_k\Omega\mathbf{Fred}^{d,0} = KO_{k+d+1}$$

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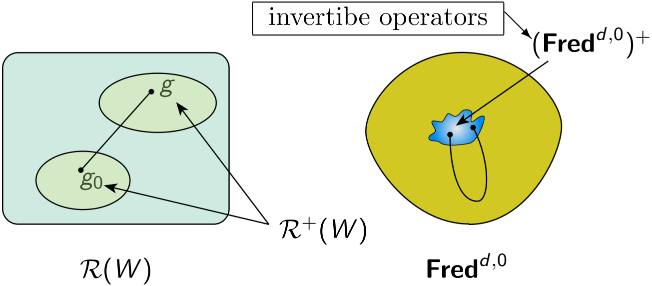

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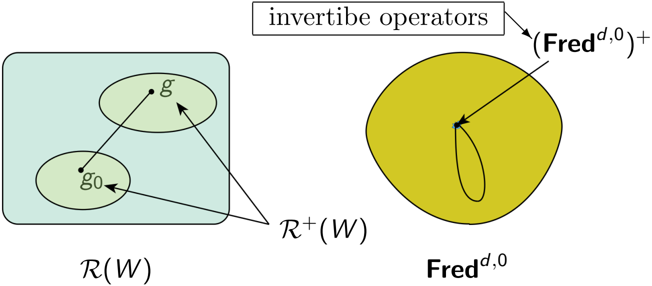

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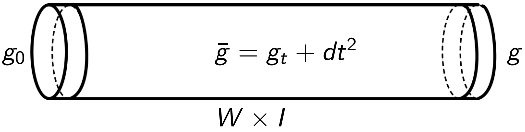
There is another way to construct the index-difference map.

Let $g_0 \in \mathcal{R}^+(W) \neq \emptyset$ be a base point, and $g \in \mathcal{R}^+(W)$, and

$$g_t = (1-t)g_0 + tg.$$

Then we have a cylinder $W \times I$ with the metric $\bar{g} = g_t + dt^2$:



It gives the Dirac operator $D_{\bar{g}}$ with the Atyiah-Singer-Patodi boundary condition. We obtain the second map

$$\mathbf{ind}_{g_0} : \mathcal{R}^+(W) \longrightarrow \Omega\mathbf{Fred}^{d,0}, \quad g \mapsto \frac{D_{\bar{g}}}{\sqrt{D_{\bar{g}}^2+1}} \in \mathbf{Fred}^{d+1,0} \sim \Omega\mathbf{Fred}^{d,0}.$$

Index Theory: $\mathbf{ind}_{g_0} \sim A_{g_0}$.

The classifying space $\mathbf{B}\mathrm{Diff}^{\partial}(W)$. Let W be a connected spin manifold with boundary $\partial W \neq \emptyset$. Fix a collar $\partial W \times (-\varepsilon_0, 0] \hookrightarrow W$. Let

$$\mathrm{Diff}^{\partial}(W) := \{\varphi \in \mathrm{Diff}(W) \mid \varphi = Id \ \text{ near } \ \partial W\}.$$

We fix an embedding $\iota^{\partial} : \partial W \times (-\varepsilon_0, 0] \hookrightarrow \mathbf{R}^m$ and consider the space of embeddings

$$\mathbf{Emb}^{\partial}(W, \mathbf{R}^{m+\infty}) = \{\iota : W \hookrightarrow \mathbf{R}^{m+\infty} \mid \iota|_{\partial W \times (-\varepsilon_0, 0]} = \iota^{\partial} \}$$

The group $\mathrm{Diff}^{\partial}(W)$ acts freely on $\mathbf{Emb}^{\partial}(W, \mathbf{R}^{m+\infty})$ by re-parametrization: $(\varphi, \iota) \mapsto (W \xrightarrow{\varphi} W \xhookrightarrow{\iota} \mathbf{R}^{m+\infty})$. Then

$$\mathbf{B}\mathrm{Diff}^{\partial}(W) = \mathbf{Emb}^{\partial}(W, \mathbf{R}^{m+\infty})/\mathrm{Diff}^{\partial}(W).$$

The space $\mathbf{B}\mathrm{Diff}^{\partial}(W)$ classifies smooth fibre bundles with the fibre W.

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$$\begin{array}{c} E(W) \\ \Big\downarrow W \\ \mathbf{B}\mathrm{Diff}^{\partial}(W) \end{array} \qquad E(W) = \mathbf{Emb}^{\partial}(W, \mathbf{R}^{m+\infty}) \times_{\mathrm{Diff}^{\partial}(W)} W$$

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$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "W"] & E(W) \arrow[d, "W"] \\ B \arrow[r, "f"] & \mathbf{B}\mathrm{Diff}^{\partial}(W) \end{tikzcd}$$

Moduli spaces of metrics. Let W be a connected spin manifold with boundary $\partial W \neq \emptyset$, $h_0 \in \mathcal{R}^+(\partial W)$. Recall:

$$\begin{aligned} \mathcal{R}(W)_{h_0} &:= \{g \in \mathcal{R}(W) \mid g = h_0 + dt^2 \ \text{ near } \ \partial W\}, \\ \mathrm{Diff}^{\partial}(W) &:= \{\varphi \in \mathrm{Diff}(W) \mid \varphi = Id \ \text{ near } \ \partial W\}. \end{aligned}$$

The group $\mathrm{Diff}^{\partial}(W)$ acts freely on $\mathcal{R}(W)_{h_0}$ and $\mathcal{R}^+(W)_{h_0}$:

$$\begin{aligned} \mathcal{M}(W)_{h_0} &= \mathcal{R}(W)_{h_0}/\mathrm{Diff}^{\partial}(W) = \mathbf{B}\mathrm{Diff}^{\partial}(W), \\ \mathcal{M}^+(W)_{h_0} &= \mathcal{R}^+(W)_{h_0}/\mathrm{Diff}^{\partial}(W). \end{aligned}$$

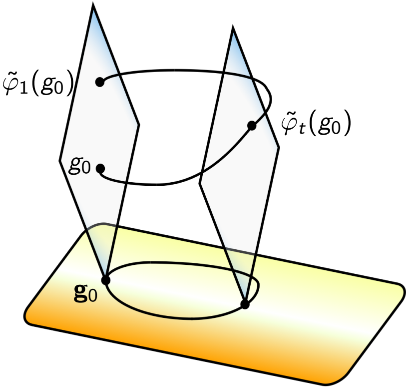
Consider the map $\mathcal{M}^+(W)_{h_0} \to \mathbf{B}\mathrm{Diff}^{\partial}(W)$ as a fibre bundle:

$$\mathcal{R}^+(W)_{h_0} \to \mathcal{M}^+(W)_{h_0} \to \mathbf{B}\mathrm{Diff}^{\partial}(W)$$

Let $g_0 \in \mathcal{R}^+(W)_{h_0}$ be a "base point".

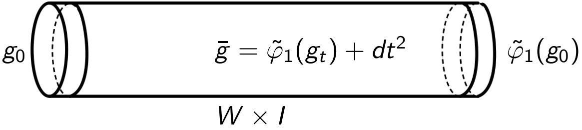
We have the fibre bundle:

$$\begin{array}{c} \mathcal{M}^+(W)_{h_0} \\ \downarrow \scriptstyle{\mathcal{R}^+(W)_{h_0}} \\ \mathbf{B}\mathrm{Diff}^{\partial}(W) \end{array}$$



Let $\varphi : I \to \mathbf{B}\mathrm{Diff}^{\partial}(W)$ be a loop with $\varphi(0) = \varphi(1) = \mathbf{g}_0$, and $\tilde{\varphi} : I \to \mathcal{M}^+(W)_{h_0}$ its lift.

We obtain:



$$\Omega\mathbf{B}\mathrm{Diff}^{\partial}(W) \xrightarrow{e} \mathcal{R}^+(W)_{h_0} \xrightarrow{\mathbf{ind}_{g_0}} \Omega\mathbf{Fred}^{d,0}$$

Let W be a spin manifold, $\dim W = d$. Consider again the index-difference map:

$$\mathbf{ind}_{g_0} : \mathcal{R}^+(W) \longrightarrow \Omega\mathbf{Fred}^{d,0},$$

where $g_0 \in \mathcal{R}^+(W)$ is a "base-point". In homotopy groups:

$$(\mathbf{ind}_{g_0})_* : \pi_k \mathcal{R}^+(W) \longrightarrow \pi_k \Omega\mathbf{Fred}^{d,0} = KO_{k+d+1}.$$

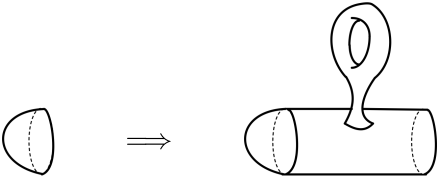
Theorem. (BB, J. Ebert, O.Randal-Williams '14) Let W be a spin manifold with $\dim W = d \geq 6$ and $g_0 \in \mathcal{R}^+(W)$. Then

$$\pi_k \mathcal{R}^+(W) \xrightarrow{(\mathbf{ind}_{g_0})_*} KO_{k+d+1} = \begin{cases} \mathbf{Z} & k+d+1 \equiv 0,4\ (8) \\ \mathbf{Z}_2 & k+d+1 \equiv 1,2\ (8) \\ 0 & \text{else} \end{cases}$$

is non-zero whenever the target group is non-zero.

Remark. This extends and includes results by Hitchin ('75), by Crowley-Schick ('12), by Hanke-Schick-Steimle ('13).

Let $\dim W = d = 2n$. Assume W is a manifold with boundary $\partial W \neq \emptyset$, and W' is the result of an admissible surgery on W. For example:



$W = D^{2n}$ $\qquad$ $W' = D^{2n} \cup ((S^{2n-1} \times I)\#(S^n \times S^n))$

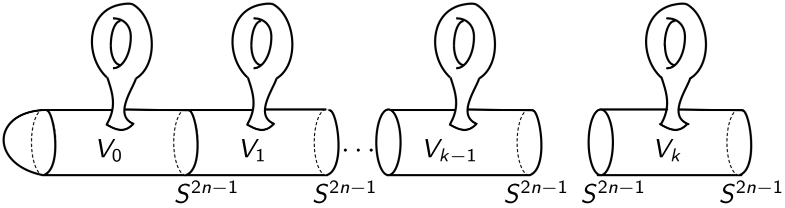
Then we have:

$$\mathcal{R}^+(D^{2n})_{h_0} \cong \mathcal{R}^+(W')_{h_0},$$

where h_0 is the round metric on S^{2n-1}.

Observation: It is enough to prove the result for $\mathcal{R}^+(D^{2n})_{h_0}$ or any manifold obtained by admissible surgeries from D^{2n}.

We need a particular sequence of **surgeries:**



Here $V_0 = (S^n \times S^n) \setminus D^{2n}$,

$V_1 = (S^n \times S^n) \setminus (D^{2n}_- \sqcup D^{2n}_+), \ldots, V_k = (S^n \times S^n) \setminus (D^{2n}_- \sqcup D^{2n}_+)$.

Then $W_k := V_0 \cup V_1 \cup \cdots \cup V_k = \#_k(S^n \times S^n) \setminus D^{2n}$.

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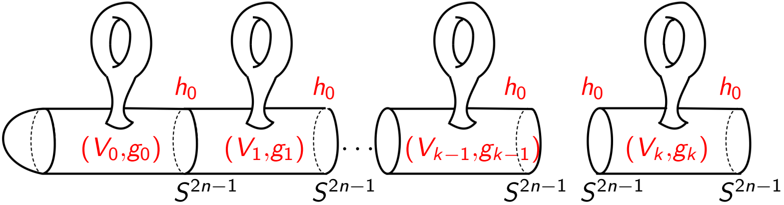

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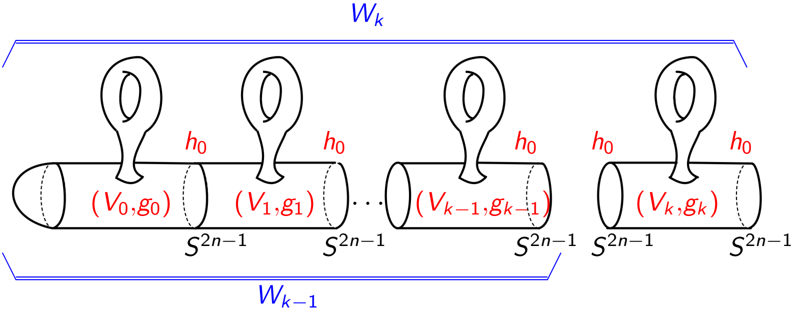
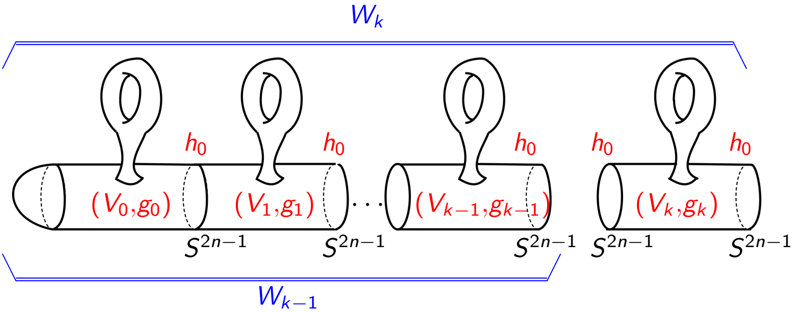


We have the **composition map**

$$\mathcal{R}^+(W_{k-1})_{h_0} \times \mathcal{R}^+(V_k)_{h_0,h_0} \longrightarrow \mathcal{R}^+(W_k)_{h_0}.$$

Gluing metrics along the boundary gives the map:

$$\mathfrak{m} : \mathcal{R}^+(W_{k-1})_{h_0} \longrightarrow \mathcal{R}^+(W_k)_{h_0}, \quad g \mapsto g \cup g_k$$



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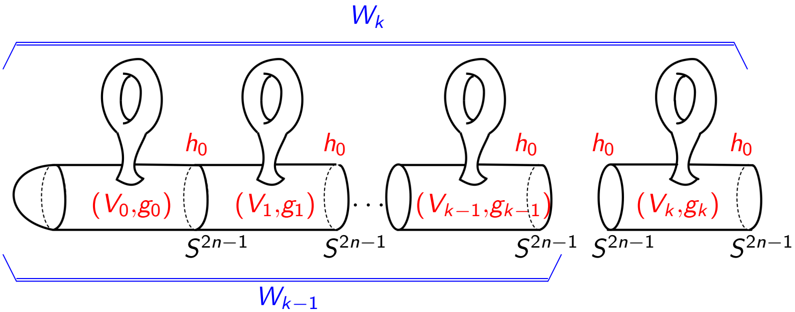
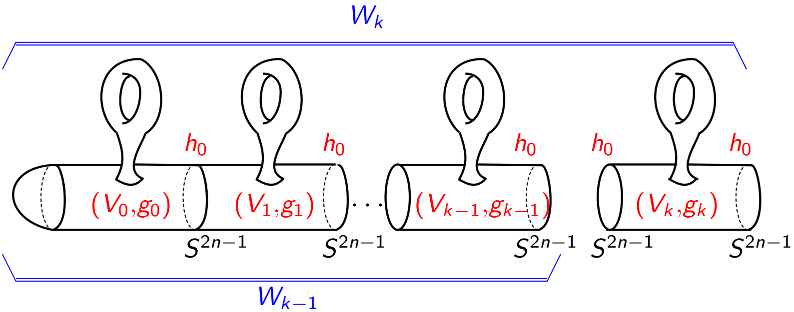
Geometry: The map $\mathfrak{m} : \mathcal{R}^+(W_{k-1})_{h_0} \longrightarrow \mathcal{R}^+(W_k)_{h_0}$ is homotopy equivalence.



Let and $s : W_k \hookrightarrow W_{k+1}$ be the inclusion.
It induces the stabilization maps

$$\mathrm{Diff}^{\partial}(W_0) \to \cdots \to \mathrm{Diff}^{\partial}(W_k) \to \mathrm{Diff}^{\partial}(W_{k+1}) \to \cdots$$

$$\mathbf{B}\mathrm{Diff}^{\partial}(W_0) \to \cdots \to \mathbf{B}\mathrm{Diff}^{\partial}(W_k) \to \mathbf{B}\mathrm{Diff}^{\partial}(W_{k+1}) \to \cdots$$



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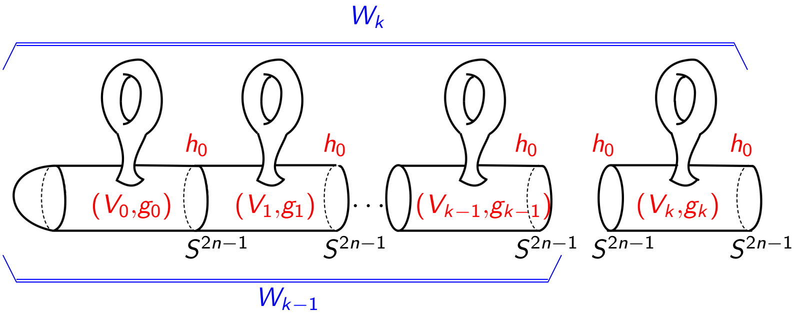
Topology-Geometry: the space $\mathbf{B}\mathrm{Diff}^{\partial}(W_k)$ is the moduli space of all Riemannian metrics on W_k which restrict to $h_0 + dt^2$ near the boundary ∂W_k.



Let and $s : W_k \hookrightarrow W_{k+1}$ be the inclusion.
It gives the fiber bundles:

$$
\begin{tikzcd}
\mathcal{M}^+(W_0)_{h_0} \arrow[r] \arrow[d, "\mathcal{R}^+(W_0)_{h_0}"'] & \mathcal{M}^+(W_1)_{h_0} \arrow[r] \arrow[d, "\mathcal{R}^+(W_0)_{h_0}"'] & \cdots \arrow[r] & \mathcal{M}^+(W_k)_{h_0} \arrow[r] \arrow[d, "\mathcal{R}^+(W_k)_{h_0}"'] & \cdots \\
\mathbf{B}\mathrm{Diff}^\partial(W_0) \arrow[r] & \mathbf{B}\mathrm{Diff}^\partial(W_1) \arrow[r] & \cdots \arrow[r] & \mathbf{B}\mathrm{Diff}^\partial(W_k) \arrow[r] & \cdots
\end{tikzcd}
$$

with homotopy equivalent fibers $\mathcal{R}^+(W_0)_{h_0} \cong \cdots \cong \mathcal{R}^+(W_k)_{h_0}$

We take a limit to get a fiber bundle:

$$\begin{array}{c} \mathbf{M}^+_\infty \\ {\scriptstyle \mathbf{R}^+_\infty}\big\downarrow \\ \mathbf{B}_\infty \end{array} = \lim_{k\to\infty} \left(\begin{array}{c} \mathcal{M}^+(W_k)_{h_0} \\ {\scriptstyle \mathcal{R}^+(W_k)_{h_0}}\big\downarrow \\ \mathbf{B}\mathrm{Diff}^\partial(W_0) \end{array} \right)$$

where $\mathbf{R}^+_\infty$ is a space homotopy equivalent to $\mathcal{R}^+(W_k)_{h_0}$.

Remark. We still have the map:

$$\Omega\mathbf{B}_\infty \xrightarrow{e} \mathbf{R}^+_\infty \xrightarrow{\mathbf{ind}} \Omega\mathbf{Fred}^{d,0}$$

which is consistent with the maps

$$\Omega\mathbf{B}\mathrm{Diff}^\partial(W_k) \xrightarrow{e} \mathcal{R}^+(W_k)_{h_0,h_0} \xrightarrow{\mathbf{ind}_{g_0}} \Omega\mathbf{Fred}^{d,0}$$

Topology: the limiting space $\mathbf{B}_\infty := \lim\limits_{k\to\infty} \mathbf{B}\mathrm{Diff}^\partial(W_k)$ has been understood.

About 10 years ago, **Ib Madsen, Michael Weiss** introduced new technique, parametrized surgery, which allows to describe various

Moduli Spaces of Manifolds.

Theorem. (S. Galatius, O. Randal-Williams) There is a map

$$\mathbf{B}_\infty \xrightarrow{\eta} \Omega_0^\infty \mathrm{MT}\theta_n$$

inducing isomorphism in homology groups.

This gives the fibre bundles:

$$\begin{tikzcd} \mathbf{M}_\infty^+ \arrow[r] \arrow[d, "\mathbf{R}_\infty^+"'] & \hat{\mathbf{M}}_\infty^+ \arrow[d, "\mathbf{R}_\infty^+"'] \\ \mathbf{B}_\infty \arrow[r, "\eta"] & \Omega_0^\infty \mathrm{MT}\Theta_n \end{tikzcd}$$

Again, it gives a holonomy map

$$\mathbf{e} : \Omega\Omega_0^\infty \mathrm{MT}\Theta_n \longrightarrow \mathbf{R}_\infty^+$$

The space $\Omega_0^\infty \mathsf{MT}\Theta_n$ is the **moduli space of $(n-1)$-connected $2n$-dimensional manifolds**.

In particular, there is a map (spin orientation)

$$\hat{\alpha} : \Omega_0^\infty \mathsf{MT}\Theta_n \longrightarrow \mathbf{Fred}^{2n,0}$$

sending a manifold W to the corresponding Dirac operator.

$$\begin{tikzcd}
\Omega\Omega_0^\infty \mathsf{MT}\Theta_n \arrow[rr, "\Omega\hat{\alpha}"] \arrow[dr, "\mathbf{e}"'] & & \Omega\mathbf{Fred}^{2n,0} \\
& \mathbf{R}_\infty^+ \arrow[ur, "\mathbf{ind}"'] &
\end{tikzcd}$$

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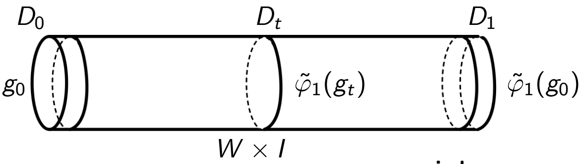
\end{tikzcd}$$



$$\Omega\mathbf{B}\mathrm{Diff}^\partial(W) \xrightarrow{e} \mathcal{R}^+(W)_{h_0} \xrightarrow{\mathbf{ind}_{g_0}} \Omega\mathbf{Fred}^{d,0}$$

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Index Theory

gives commutative diagram

$$\begin{tikzcd}
\Omega\Omega_0^\infty \mathsf{MT}\Theta_n \arrow[rr, "\Omega\hat{\alpha}"] \arrow[dr, "\mathbf{e}"'] & & \Omega\mathbf{Fred}^{2n,0} \\
& \mathbf{R}_\infty^+ \arrow[ur, "\mathbf{ind}"'] &
\end{tikzcd}$$

Then we use algebraic topology to compute the homomorphism

$$(\Omega\hat{\alpha})_* : \pi_k(\Omega\Omega_0^\infty \mathsf{MT}\Theta_n) \longrightarrow \pi_k(\Omega\mathbf{Fred}^{2n,0}) = KO_{k+2n+1}$$

to show that it is nontrivial when the target group is non-trivial.

THANK YOU!