On Bases of Cardinal Functions and Their Role in Approximate Sampling Methods

Keaton Hamm
Department of Mathematics
Vanderbilt University
Nashville, TN 37240
Email: keaton.hamm@vanderbilt.edu

Jeff Ledford
Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284
Email: jpledford@vcu.edu

***Abstract*—This article discusses the structure of various shift-invariant spaces of cardinal functions generated by a single kernel, and their role in some approximate sampling methods. Particularly, conditions on the generating kernel ϕ are given which imply that the associated L_p shift-invariant space coincides with the shift-invariant space for the cardinal function L_ϕ generated by the kernel. Additionally, we discuss how such spaces are used in summability methods for the classical sampling series.**

I. Introduction

In the beginning was the classical sampling theorem, and while its proper attribution may be disputed, the mathematical and engineering communities took note, and saw that it was good. The theorem states that any L_2 function whose Fourier transform is compactly supported may be represented exactly as

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2\Omega}\right)\operatorname{sinc}(2\Omega x - n), \tag{1}$$

where the *cardinal sine* function is defined by $\operatorname{sinc}(x) := \sin(\pi x)/(\pi x)$, the support of $\widehat{f}$ lies within $[-\Omega,\Omega]$, and the series converges in L_2 and uniformly on $\mathbb{R}$. Thus for a well-behaved class of signals, it suffices to know only their samples at a sufficiently fine lattice to obtain perfect reconstruction.

Nonetheless, the beauty of this representation has its limits. For one thing, the sinc kernel is poorly localized, and approximating the series in (1) may require rather a lot of samples. Oversampling (i.e. taking the samples at a finer lattice) gets around this difficulty, but sometimes at a non-trivial cost. Another method proposed by I. J. Schoenberg and which is used heavily in many areas of approximation theory is that of a *summability method*, i.e. one in which the cardinal sine series on the right-hand side of (1) is suitably modified to converge more rapidly, at the possible cost of replacing equality by approximate equality (in the L_2 norm sense). In [17], Schoenberg discussed summability methods for the interpolating series involving cardinal B–splines. In the intervening time, B–splines have come to enjoy quite a prominent place in sampling theory and signal processing ([8], [18], [19] and the many references therein). Another prominent example of a summability method is the use of the well-known Fejér kernel to make the modified partial sums of the Fourier series of a periodic function converge to the function.

II. Cardinal Functions

One example of a summability method introduced by Schoenberg in [16] is to simply replace the sinc kernel with some other function which behaves similarly on the lattice, but which nonetheless decays more rapidly away from the origin. He considered specifically the following.

Definition II.1. *A function $L:\mathbb{R}\to\mathbb{R}$ is called a* cardinal function *provided $L(k)=\delta_{0,k}$, $k\in\mathbb{Z}$.*

Consequently, the series given by

$$\sum_{n\in\mathbb{Z}} f\left(\frac{n}{2\Omega}\right) L(2\Omega x - n) \tag{2}$$

interpolates f at the lattice $(2\Omega)^{-1}\mathbb{Z}$. The trade-off of using a cardinal function to replace the sinc is that one attempts to design L such that the interpolating series above is close to f in the norm under consideration, while L decays more quickly than sinc. Recalling that $\operatorname{sinc}(x)=O(|x|^{-1})$, this is rather attainable as subsequent examples demonstrate.

From now on, we focus on the canonical case when $\Omega = 1/2$, and the sampling occurs at the integer lattice; the subsequent analysis extends to other band-sizes via dilation. Moreover, the following definition of the Fourier transform is used in the sequel: $\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx$.

Of course, there are many feasible ways to construct such cardinal functions, but we focus on a method which arises from the theory of radial basis functions [5]. Consider a fixed function $\phi:\mathbb{R}\to\mathbb{R}$, and formally define the cardinal function associated with it via its Fourier transform as follows:

$$\widehat{L_\phi}(\xi) := \frac{\widehat{\phi}(\xi)}{\sum_{n\in\mathbb{Z}}\widehat{\phi}(\xi-n)}. \tag{3}$$

If $\widehat{\phi}$ and the periodic symbol $\sigma(\xi) := \sum_{n\in\mathbb{Z}}\widehat{\phi}(\xi-n)$ are sufficiently nice, then the inversion formula holds, and L_ϕ is a cardinal function. Indeed, this follows from the (formal) calculation

$$\begin{aligned} L_\phi(k) &= \int_{\mathbb{R}} \frac{\widehat{\phi}(\xi)}{\sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi-n)} e^{2\pi i k\xi} d\xi \\ &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}+m} \frac{\widehat{\phi}(\xi)}{\sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi-n)} e^{2\pi i k\xi} d\xi \\ &= \int_{\mathbb{T}} \frac{\sum_{m\in\mathbb{Z}} \widehat{\phi}(\xi-m)}{\sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi-n)} e^{2\pi i k\xi} d\xi \\ &= \delta_{0,k}, \end{aligned}$$

where under this normalization of the Fourier transform, we identify the torus $\mathbb{T}$ with the interval $[-1/2, 1/2]$.

Assumptions on ϕ which make the previous analysis rigorous will be discussed in later sections, and have been considered in [5], [12]. For a thorough analysis of, and introduction to, the ideas of radial basis functions and their associated interpolation schemes, the reader is invited to consult [6], [20].

Table I below shows the decay rates for cardinal functions associated with some well-known radial basis functions. The proof of decay for the Gaussian is due to the insightful work of Riemenschneider and Sivakumar [13], [14], while that for B–spline is due to Schoenberg [16], whereas the general analysis for the multiquadrics $(|x|^2+c^2)^\alpha$ may be found in [10] (though for half-integer exponents, more precise rates were already considered [2], [5], [7]).

ϕ	Decay Rate of L_ϕ
$e^{-\lambda\|x\|^2}$	$O(e^{-\lambda\|x\|})$
$\underbrace{\mathbb{1}_\mathbb{T} * \dots * \mathbb{1}_\mathbb{T}}_{n}$	$O(e^{-\|x\|})$
$(\|x\|^2+c^2)^\alpha$	$\begin{cases} O(\|x\|^{-\lfloor 2\alpha+1\rfloor}), & \alpha\in(0,\infty)\setminus\mathbb{N} \\ O(\|x\|^{-\lceil 2\alpha-2\rceil}), & \alpha<-1 \\ O(\|x\|^{-k}),\ k\in\mathbb{N}, & \alpha=-1. \end{cases}$

TABLE I
DECAY RATES FOR CERTAIN RBFS.

III. Shift-Invariant Spaces

A *Shift-invariant space* is a space of functions generated by uniform translates of a single window, taking the form

$$V_p(\phi) := \left\{ \sum_{n\in\mathbb{Z}} a_n \phi(\cdot - n) : (a_n) \in \ell_p(\mathbb{Z}) \right\}.$$

Such spaces are often considered in signal processing because they are natural generalizations of the space of bandlimited functions since the sampling theorem and (1) imply that $PW = V_2(\mathrm{sinc})$, where PW is the Paley–Wiener space of L_2 functions satisfying $\mathrm{supp}(\widehat{f}) \subset \mathbb{T}$. (Note that sometimes in the literature these are defined by $\overline{span}^{L_p}\{\phi(\cdot - n) : n\in\mathbb{Z}\}$, but for suitably regular windows, the definitions coincide).

Given the form of (2), it is natural to consider the behavior of the spaces $V_p(L_\phi)$, where L_ϕ is a cardinal function. However, in many cases, the cardinal function itself may be expressed in the form

$$L_\phi(x) = \sum_{n\in\mathbb{Z}} c_n \phi(x-n). \tag{4}$$

Generally, the nature of the convergence of this series and the behavior of the coefficients (c_n) depend on the nature of ϕ itself. Consequently, we ask the following question.

Problem III.1. *Under what conditions on p and ϕ does $V_p(\phi) = V_p(L_\phi)$?*

IV. Structure of Sampling Spaces related to Cardinal Functions

To make some progress on Problem III.1, let us first consider the easier case $p = 2$. Recall the definition of the Wiener amalgam space

$$W(L_\infty, \ell_1) := \left\{ f : \sum_{n\in\mathbb{Z}} \|f(\cdot + n)\|_{L_\infty(\mathbb{T})} < \infty \right\},$$

with the implicit norm there denoted $\|\cdot\|_W$ for simplicity. Consider the following two conditions:

(A) $\widehat{\phi} \in W(L_\infty, \ell_1)$
(B) $\widehat{\phi} \in C(\mathbb{T})$ with $\widehat{\phi}(\xi) \geq \varepsilon > 0$ on $\mathbb{T}$ and $\widehat{\phi} \geq 0$ a.e. on $\mathbb{R}$.

Note that the sinc function satisfies (A) and (B), so they are not overly strict for our purposes.

Proposition IV.1. *If ϕ satisfies (A) and (B), then $L_\phi(x) := \int_{\mathbb{R}} \widehat{L_\phi}(\xi) e^{2\pi i \xi x} d\xi$ with $\widehat{L_\phi}$ defined as in* (3) *is a cardinal function. Moreover, L_ϕ satisfies (A) and (B).*

Proof. First notice that (A) and (B) imply that $\widehat{L_\phi} \in L_1 \cap L_2(\mathbb{R})$. Indeed, (B) implies that

$$\widehat{L_\phi}(\xi) \leq \varepsilon^{-1}\widehat{\phi}(\xi) \tag{5}$$

almost everywhere, whereby the straightforward fact that $W(L_\infty, \ell_1) \subset L_1 \cap L_2$ yields the conclusion. Consequently, L_ϕ defined by the Fourier inversion formula is continuous and square-integrable on $\mathbb{R}$. Moreover, the formal calculation in Section II showing that $L_\phi(k) = \delta_{0,k}$, $k\in\mathbb{Z}$ is justified by the Monotone Convergence Theorem on account of (B).

To verify the moreover statement, first notice that (5) implies that $\|\widehat{L_\phi}\|_W \leq \varepsilon^{-1}\|\widehat{\phi}\|_W < \infty$, which is (A). Secondly, on $\mathbb{T}$, $\widehat{L_\phi}(\xi) \geq \frac{\varepsilon}{\|\widehat{\phi}\|_W} > 0$, whilst $\widehat{L\phi} \geq \frac{\widehat{\phi}(\xi)}{\|\widehat{\phi}\|_W} \geq 0$ a.e. on $\mathbb{R}$, which is (B). ☐

Theorem IV.2. *If ϕ satisfies (A) and (B), then $V_2(\phi) = V_2(L_\phi)$.*

Proof. By Plancherel theory, it suffices to show that the Fourier transforms of the spaces in question are equal. To wit, notice that $\widehat{V_2}(\phi) = \{\widehat{\phi}\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\cdot} : (a_n)\in\ell_2(\mathbb{Z})\}$, and similarly for $\widehat{V_2}(L_\phi)$. However, since (a_n) runs over all of ℓ_2, we equivalently have

$$\widehat{V_2}(\phi) = \{\widehat{\phi}Q : Q \text{ is periodic, and } Q|_\mathbb{T} \in L_2(\mathbb{T})\}.$$

Since the symbol $\sigma(\xi) := \sum_{n\in\mathbb{Z}} \widehat{\phi}(\xi-n)$ is continuous, periodic, bounded, and bounded away from 0, any $\widehat{g} = \widehat{\phi}Q \in \widehat{V_2}(\phi)$ may be expressed as

$$\widehat{g} = \frac{\widehat{\phi}}{\sigma}(\sigma Q) \in \widehat{V_2}(L_\phi)$$

since σQ is evidently periodic and in $L_2(\mathbb{T})$. Thus $\widehat{V}_2(\phi) \subset \widehat{V}_2(L_\phi)$, and the reverse inclusion follows similarly. $\square$

Note that this, in particular, implies that $L_\phi \in V_2(\phi)$, i.e. (4) holds with ℓ_2 coefficients, with the series converging at least in L_2. However, it is not necessary that ϕ satisfy (A) and (B) for the cardinal function L_ϕ to do so. Indeed, for the multiquadrics with positive exponent α, the space $V_2(\phi)$ is not even well-defined, but nonetheless the decay of the Fourier transform of L_ϕ is such that $V_2(L_\phi)$ is well-defined and L_ϕ satisfies (A) and (B) (see Table I and [10]).

Since it will lead to some additional structural conclusions in the case of general p, we record the following fact.

Proposition IV.3. *If ϕ satisfies (A) and (B), then $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is a Riesz basis for $V_2(\phi)$.*

Proof. It suffices to verify that the system $\{\widehat{\phi} e^{-2\pi i n \cdot} : n \in \mathbb{Z}\}$ is a Riesz basis for its span. Recall a system $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis for a Hilbert space $\mathcal{H}$ if $c_1\|a\|_{\ell_2}^2 \leq \|\sum_{n\in\mathbb{Z}} a_n f_n\|_{\mathcal{H}} \leq c_2\|a\|_{\ell_2}^2$ for every $a \in \ell_2$ for some positive constants c_1, c_2. To verify this for the system in question, notice that

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n \cdot}\widehat{\phi}\right\|_{L_2}^2 &= \sum_{\ell\in\mathbb{Z}}\int_{\mathbb{T}+\ell}\left|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\right|^2 |\widehat{\phi}(\xi)|^2 d\xi \\ &= \int_{\mathbb{T}}\left|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\right|^2 \sum_{\ell\in\mathbb{Z}}|\widehat{\phi}(\xi+\ell)|^2 d\xi.\end{aligned}$$

By (A) and the fact that the ℓ_2 norm is subordinate to the ℓ_1 norm, the quantity on the right-hand side above is majorized by $\|\widehat{\phi}\|_W^2\|a\|_{\ell_2}^2$. On the other hand, it is bounded below by $\varepsilon^2\|a\|_{\ell_2}^2$ by (B), which verifies the Riesz basis inequality. $\square$

Having made some conclusions in the $p = 2$ case, let us now turn to the case of other p. First, note that for any $1 \leq p \leq \infty$, Proposition IV.1 implies that $V_p(\phi)$ and $V_p(L_\phi)$ are closed subspaces of $L_p(\mathbb{R})$ and that for $1 \leq p < \infty$, $\{\phi(\cdot - n) : n \in \mathbb{Z}\}$ is an unconditional basis for $V_p(\phi)$, and likewise for the analogous system in $V_p(L_\phi)$ (see [1, Theorem 2.4]).

Next, recalling that the Fourier transform may be extended to functions in $L_p(\mathbb{R})$ for $1 \leq p \leq 2$ via the Riesz–Thorin Theorem, we begin there. To proceed, we make another assumption on the window:

(C) $(\phi(k))_{k\in\mathbb{Z}} \in \ell_1$.

The primary reason for this assumption is that it implies that the periodic symbol $\sigma(\xi) := \sum_{n\in\mathbb{Z}}\widehat{\phi}(\xi - n)$ is in the Wiener algebra $A(\mathbb{T})$ of functions with absolutely summable Fourier coefficients. Indeed, one need only notice that $\hat{\sigma}(k) = \phi(k)$ via the same periodization argument used in the calculation in Section II.

It follows then that $\sigma(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-2\pi i n\xi}$ for some $c \in \ell_1$, and moreover by Wiener's Tauberian Theorem that $1/\sigma(\xi) = \sum_{n\in\mathbb{Z}} d_n e^{-2\pi i n\xi}$ for some $d \in \ell_1$. With this observation in hand, we can make the following conclusion.

Theorem IV.4. *If ϕ satisfies (A)–(C), then $V_p(\phi) = V_p(L_\phi)$ for every $1 \leq p \leq 2$.*

Proof. Let (c_n) and (d_n) be the Fourier coefficients of σ and $1/\sigma$ as discussed above. Then consider a function $\sum_{n\in\mathbb{Z}} a_n\phi(x-n)$ such that $(a_n) \in \ell_p$; the following formal calculation follows by noticing that $\widehat{\phi} = \sigma\widehat{L_\phi}$ and writing $\phi(x-n)$ via the Fourier inversion formula:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} a_n\phi(x-n) &= \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\widehat{\phi}(\xi)e^{2\pi i x\xi}d\xi \\ &= \int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}(a*c)_n e^{-2\pi i n\xi}\widehat{L_\phi}(\xi)e^{2\pi i x\xi}d\xi \\ &= \sum_{n\in\mathbb{Z}}(a*c)_n L_\phi(x-n).\end{aligned}$$

The interchange of the sum and the integral follows from Fubini's theorem, which is justified below. Subsequently, the fact that $\ell_p * \ell_1 \subset \ell_p$ yields the conclusion that $V_p(\phi) \subset V_p(L_\phi)$. To justify the use of Fubini's Theorem above, consider that, by periodization and positivity of $\widehat{\phi}$,

$$\begin{aligned}&\int_{\mathbb{R}}\left|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\widehat{\phi}(\xi)e^{2\pi i x\xi}\right| d\xi \\ &\leq \int_{\mathbb{T}}\left|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\right|\sum_{m\in\mathbb{Z}}\widehat{\phi}(\xi - m)d\xi. \\ &= \int_{\mathbb{T}}\left|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\xi}\right|\sigma(\xi)d\xi.\end{aligned}$$

By Hölder's inequality, the final term above is majorized by $\|\sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n\cdot}\|_{L_p(\mathbb{T})}\|\sigma\|_{L_q(\mathbb{T})}$, where $1/p + 1/q = 1$. This is finite since $\sigma \in A(\mathbb{T})$ is continuous, and the trigonometric series is bounded in L_p by utilizing the facts that $\|\cdot\|_{L_p(\mathbb{T})} \leq \|\cdot\|_{L_2(\mathbb{T})}$, and $\|\cdot\|_{\ell_2} \leq \|\cdot\|_{\ell_p}$ for $1 \leq p \leq 2$.

To see the reverse inclusion, one need only perform the same calculation using the fact that $\widehat{L_\phi} = \widehat{\phi}/\sigma$, and notice that by (3), $\sum_{m\in\mathbb{Z}}\widehat{L_\phi}(\xi - m) = 1$. Consequently, $V_p(L_\phi) \subset V_p(\phi)$, which yields the conclusion for the given range of p. $\square$

It is natural to ask if Theorem IV.4 remains true for $p > 2$, but thus far we have been unable to prove this. However, the assumptions on the decay of the generating function ϕ are such that equality of the spaces for large p seems quite reasonable.

Remark IV.5. Note that for the Gaussian and the Poisson kernel $(1+x^2)^{-1}$, their Fourier transforms satisfy conditions (A)–(C), and consequently their associated shift-invariant spaces coincide with those associated with their cardinal functions. This was already known under slightly different conditions [6], but our conditions and method of proof are somewhat different here.

V. Relation to Approximate Sampling Schemes

Here we briefly discuss the appearance of these shift-invariant spaces in some approximate sampling schemes which arise naturally from the radial basis function literature. To begin, let us note that for bandlimited functions, we may find unique interpolants in the shift-invariant spaces discussed above:

Theorem V.1. *If ϕ satisfies (A) and (B), then for every $f \in PW$, there is a unique function $I_\phi f \in V_2(\phi)$ such that $I_\phi f(n) = f(n)$, $n \in \mathbb{Z}$.*

Proof. Conditions (A) and (B) imply the conditions (A1)–(A3) in [11], whereby the conclusion follows from Proposition 1 therein. □

Naturally, Theorem IV.2 implies that this interpolant is in $V_2(L_\phi)$, and by uniqueness, it must have the form

$$I_\phi f(x) = \sum_{n\in\mathbb{Z}} f(n) L_\phi(x-n),$$

which matches that of (2).

With this in mind, it is pertinent to consider the behavior of the spaces $V_2(\phi)$ in relation to PW. One way of doing this is to estimate $\mathrm{dist}_2(PW, V_2(\phi)) := \inf_{f\in PW, g\in V_2(\phi)} \|f-g\|_{L_2}$. While for a fixed ϕ this may be difficult in general, some asymptotic considerations have been done.

To wit, consider a family of kernels $(\phi_\alpha)_{\alpha\in A}$ which are indexed by some parameter (for example, $g_\alpha(x) := e^{-|x|^2/\alpha}$, $\alpha > 0$). Then under suitable regularity conditions on the family of kernels (see [11], [12]), we have that

$$\lim_{\alpha\to\infty} \mathrm{dist}_2(PW, V_2(L_{\phi_\alpha})) = 0,$$

and moreover that for any given $f \in PW$, its interpolants given by Theorem V.1 converge (this is the content of [12, Theorem 2] for general families, whereas for the Hardy multiquadric $\sqrt{x^2+1}$ see [2]). For similar results for the Gaussian kernel but nonuniform sampling sites, see [15], and for other families of kernels, [9]. In another vein, interpolation and approximation methods from so-called ladders of shift-invariant spaces of the form $\{V_2(\phi_h, h\mathbb{Z})\}_{h>0}$, with $\phi_h(x) := \phi(x/h)$ have been thoroughly explored ([3], [4] and many subsequent papers).

VI. Conclusion

We have studied briefly the structure of some L_p principal shift-invariant spaces generated by so-called cardinal functions, and their relationship to the (typically) radial basis functions associated with them. Some basic conditions involving only the Fourier transforms of the generating kernel have been given under which the spaces coincide for a restricted range of p, and we have given a brief review of their use in summability methods related to the classical sampling theorem as well as some of the considerations of approximation orders made in the classical approximation theory literature. In the end, it is worth mentioning the heuristic idea that any aspect of sampling theory in which B–splines enjoy success should be amenable to the use of radial basis functions, which sometimes due to their nice structure, may lead to more computational feasibility. That is not to say that B–splines should be abandoned altogether, but nonetheless there are other options available which may allow for some additional flexibility.

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