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Large Games: Fair and Stable Outcomes*

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We establish that for a broad class of large games with sidepayments, fair outcomes are nearly stable. More precisely, the Shapley value of a large game is in the ε-core and ε is very small if the game is very large. The proof uses two other results of independent interest: for large games the power of improvement is concentrated in small coalitions, and the Shapley value of a small syndicate acting together is nearly the sum of the Shapley values which accrue to the members acting alone. *Journal of Economic Literature* Classification Numbers: 021, 022, 025.

1. INTRODUCTION

Kuhn and Tucker [9] posed, as one of a list of important problems for study, "to establish significant asymptotic properties of n-person games, for large n." This paper takes a step in that direction.

We establish that for a broad class of large games, fair outcomes are nearly stable. Moreover, the larger the game, the more nearly stable is the fair outcome. In more precise terms, the Shapley value of a large game is in the ε-core, and ε is very small if the game is very large. The framework we use is that of games in characteristic function form with sidepayments. This

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framework can acommodate diverse economic models with money, including private goods exchange economies (with divisible and indivisible goods), coalition-production economies, economies with local public goods, and economies with club goods.

At the heart of this paper are two main ideas concerning the role of small coalitions which we think are of interest in themselves. The first of these is that, for large games, the power of improvement is concentrated in small coalitions. Roughly speaking, this means that if an allocation can be improved upon at all, then it can be improved upon by a small coalition. A related idea is familiar in the context of private goods economies (c.f., Mas-Colell [10]) and appears implicitly in other economic models (c.f., Shubik and Wooders [17, 18] on economies with local public goods and on coalition production). In the context of finite games, this idea has been used implicitly by Wooders [19] and Wooders and Zame [20], to show nonemptiness of ε-cores, but to the best of our knowledge, the present paper is the first place where it is given an explicit statement in the strong form herein. The second of these main ideas is that, in large games, the Shapley value cannot be significantly affected by the formation of a small syndicate. That is, the Shapley value which would accrue to the syndicate acting together is very nearly the sum of the Shapley values which accrue to the members acting alone.

The convergence of the Shapley value to the competitive payoff (and hence to the core) for private goods exchange economies was suggested by Shubik, and first demonstrated by Shapley [14] in the context of replication economies with money. It has subsequently been extended by Shapley and Shubik [16], Champsaur [4], Mas-Colell [11], Cheng [5], and others.[1] Some of these extensions treat the case of economies without money, but they are all restricted to the context of private goods exchange economies[2] with divisible goods and concave, monotone utility function. Moreover, they all treat either replicated sequences of economies, or convergent sequences of economies.[3] This kind of private goods framework, however, rules out many natural economic phenomena, such as indivisible goods, nonmonotonic or nonconcave utility functions, general production, local public goods, or club goods. Our framework is broad enough to encompass all these situations (in the presence of money[4]). Moreover, our

[1] This list is not intended to be complete.

[2] Strictly speaking, Champsaur allows for production, but only of a restricted kind (additive production).

[3] The work of Aumann and Shapley [3], Aumann [2], Hart [7], and others on values of economies with a continuum of agents is of a somewhat different nature since it begins with a limit continuum economy.

[4] Without money, such economic settings may be modelled as games without sidepayments. In that framework, the analog of the Shapley value is the NTU value (or λ-transfer value). (See note added in proof.)

framework does not require replicated economies or convergent sequences of economies.

To model large games we introduce the notion of a "technology" (to be understood in a broad sense). A technology encompasses all possible opportunities available to any conceivable group of agents in society. More specifically, a technology consists of a space Ω of possible attributes of agents, and a mapping Λ which specifies the maximum utility obtainable by any group of agents, given the attributes of its members. We make natural assumptions of compactness of the space Ω of attributes, and superadditivity, continuity, and marginal boundedness of the mapping Λ. A game is determined from such a technology by specifying a finite set of players, and the attributes of each player. (In a private goods economy, for example, we would specify, for each agent, an initial endowment and a utility function.)

The stability concept we employ is the individually rational ε-core. A payoff is in this ε-core if it is feasible, Pareto optimal, individually rational, and has the property that no coalition can improve upon it by more than ε for each of its members.

Our main result shows that given an $\varepsilon > 0$, any game, derived from such a technology, which is sufficiently large (in the sense of having many players whose attributes are sufficiently close) has the property that the Shapley value (a fair outcome) is in the individually rational ε-core (the set of nearly stable outcomes).

In the special case of private goods exchange economies with divisible goods and money (but allowing for nonconcave and nonmonotone utility functions), such ε-core payoffs can be "approximately decentralized" by prices (in the sense of Anderson [1]). In this setting, therefore, our result implies that the Shapley value is an approximately competitive payoff (if the economy is sufficiently large). We stress, however, that this is a very special case of our result.

This work is—in part—an outgrowth of an earlier paper (Wooders and Zame [20]), in which the technology framework used herein was established. Although there is an overlap of ideas between the two papers the thrust of the present paper is quite different. In particular, the present paper may be read quite independently.

The remainder of the paper is organized in the following way. We collect general information about game theory in Section 2, and our notion of technologies is described in Section 3. In Section 4 we give a precise statement of our main result and some comments on its meaning, and decribe the two main ideas of its proof. The proof itself is divided into four sections: Section 5 is devoted to the first main idea (the blocking power of small coalitions), Section 6 contains a simple probabilistic estimate, Section 7 is devoted to the second main idea (the power of small syndicates), and the proof is completed in Section 8. Finally, Section 9 relates the ε-cores of a sequence of games to the core of a nonatomic limit game.

2. GAMES

By a *game* (in characteristic function form, with sidepayments) we mean a pair (N, v) where N is a finite set (the set of *players*) and v is a function (the *characteristic function*) from the set 2^N of subsets of N to the set $\mathbb{R}_+$ of nonnegative real numbers, with the property that $v(\emptyset)=0$. We usually refer to subsets S of N as *coalitions*; the number $v(S)$ is the *worth* of S. If the player set N is understood, we frequently refer to v itself as the game. We say that v is *superadditive* if for all disjoint subsets S, S' of N we have

$$v(S \cup S') \geqslant v(S) + v(S').$$

By a *payoff* for (N, v) we mean a vector x in $\mathbb{R}^N$; it is convenient to use functional notation, so for $i \in N$, $x(i)$ is the ith component of x. We say that x is *feasible* if $x(N) \leqslant v(N)$ (where $x(S) = \sum_{i \in S} x(i)$ for each $S \subset N$).

For $\varepsilon \geqslant 0$, a feasible payoff x is in the *ε-core* of (N, v) if

(a) $x(N) = v(N)$ (*Pareto optimality*);

(b) $x(S) \geqslant v(S) - \varepsilon |S|$ for all subsets S of N.

(We use $|S|$ to denote the number of elements of the set S.) We say x is in the *individually rational ε-core* if it is in the ε-core and in addition

(c) $x(i) \geqslant v(\{i\})$ for all $i \in N$ (*individually rationality*).

When $\varepsilon = 0$, the ε-core (which coincides with the individually rational ε-core) is simply the *core.*

Less formally, a feasible, Pareto optimal payoff x belongs to the ε-core of (N, v) if no group of players can guarantee for themselves a payoff which each of them finds better than x by more than ε. As Shapley and Shubik [15] point out, such a payoff can be interpreted as stable if players are nearly optimizing, or satisficing, or if there is an organizational or communicational cost to formation of coalitions (proportional to the size of the coalition).

The *Shapley value* $Sh(v)$ of the game (N, v) is the payoff whose ith component is given by

$$Sh(v, i) = \frac{1}{|N|} \sum_{J=0}^{|N|-1} \frac{1}{\binom{|N|-1}{J}} \sum_{\substack{S \subset N \setminus \{i\} \\ |S| = J}} [v(S \cup \{i\}) - v(S)]$$

In other words, $Sh(v, i)$ is player i's average marginal contribution to coalitions in N. The Shapley value is a feasible, Pareto optimal and individually rational payoff. It is frequently interpreted as representing a "fair" payoff since it yields to each player his expected contribution. It can also be given various other interpretations (as a von Neumann–Morgenstern utility function for example, see Roth [13]).

3. TECHNOLOGIES

We want to formalize the notion of a large game for which the worth of a coalition depends in a continuous fashion on the attributes of its members. To do this, we introduce the notion of a technology.[5]

Let Ω be a compact metric space. By a *profile* on Ω we mean a function f from Ω to the set $\mathbb{Z}_+$ of nonnegative integers for which the *support* of f,

$$\operatorname{support}(f) = \{\omega \in \Omega : f(\omega) \neq 0\},$$

is finite. We denote the set of profiles on Ω by $P(\Omega)$. Note that the sum of profiles (defined pointwise) is a profile, and that the product of a profile with a nonnegative integer is a profile. We write 0 for the profile which is identically zero. We write $f \leqslant g$ if $f(\omega) \leqslant g(\omega)$ for each $\omega \in \Omega$. For ω_0 a point of Ω, we write χ_{ω_0} for the profile given by

$$\begin{aligned} \chi_{\omega_0}(\omega) &= 0 \qquad \text{if} \quad \omega \neq \omega_0; \\ \chi_{\omega_0}(\omega_0) &= 1. \end{aligned}$$

By the *norm* of a profile f we mean

$$\|f\| = \sum_{\omega \in \Omega} f(\omega).$$

(Notice that this is a finite sum, since f has finite support.)

In essence, a profile is simply an (unordered) list of elements of Ω, with each element ω appearing as many times as its multiplicity $f(\omega)$.

By a *technology* we mean a pair (Ω, Λ) where Ω is a compact metric space (the space of *attributes*) and $\Lambda : P(\Omega) \to \mathbb{R}_+$ is a function with the following properties:

(i) $\Lambda(0) = 0$;

(ii) $\Lambda(f + g) \geqslant \Lambda(f) + \Lambda(g)$ (*superadditivity*);

[5] In Wooders and Zame [20] the term "pre-game with attributes" was used; the term "technology" was suggested by Y. Kannai, and seems more descriptive.

(iii) there is a constant M such that $\Lambda(f+\chi_\omega) \leqslant \Lambda(f)+M$ for each $\omega \in \Omega$, $f \in P(\Omega)$ (we say M is an *individual marginal bound*);

(iv) for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\Lambda(f+\chi_{\omega_1})+\Lambda(f+\chi_{\omega_2})| < \varepsilon$$

whenever $f \in P(\Omega)$ and $\omega_1, \omega_2 \in \Omega$ with $\operatorname{dist}(\omega_1, \omega_2) < \delta$ (*continuity*).

The interpretation we have in mind is that a technology encompasses all the economic possibilities for every conceivable group of players. A point of Ω represents a complete description of the relevant attributes of players (endowment, utility function, etc.). A profile f represents a group of players of whom $f(\omega)$ are described by the attribute ω; the total number of players in the group is just $\|f\|$. The number $\Lambda(f)$ represents the maximal possible payoff the members of this groups could achieve (using their own resources) by cooperation. The requirements that $\Lambda(0)=0$ means that the group of no players can achieve nothing. Superadditivity has its usual interpretation: one of the possibilities open to the group represented by $f+g$ is to split into the groups represented by f and by g and share the proceeds. (Notice that we do not require the profiles f, g to have disjoint supports. The groups of players represented by the profiles f, g will have no players in common in any case; to require that the profiles f, g have disjoint supports would be to require that these groups have no types of players in common.) The existence of an individual marginal bound simply means that there are no players whose (potential) contributions to society are arbitrarily large. Continuity of Λ means that players with similar attributes are good substitutes for each other.

If Ω is finite, we frequently refer to its elements as *types*; players of the same type are exact substitutes. (Notice that continuity of Λ is automatic in this case.)

To derive a game from the technology (Ω, Λ), we specify a finite set N and a function $\alpha: N \to \Omega$ (an *attribute function*). We associate with each subset S of N a profile prof $(\alpha \mid S)$ given by

$$\operatorname{prof}(\alpha \mid S)(\omega) = |\alpha^{-1}(\omega) \cap S|;$$

i.e., $\operatorname{prof}(\alpha \mid S)(\omega)$ is the number of players in S possessing the attribute ω. We then define the characteristic function $v_\alpha: 2^N \to \mathbb{R}_+$ by

$$v_\alpha(S) = \Lambda(\operatorname{prof}(\alpha \mid S)).$$

Thus, the worth of a coalition S in a derived game is determined by the technology and depends on the attributes of the players in the coalition. It is easily checked that (N, v_α) is a superadditive game.

4. The Shapley Value and the ε-Core

Having described the framework, we can now state our main result. If the space Ω of attributes is finite (and so consists of a finite number of types), it seems natural to think of a game as large if it has many players of each type; thus each player has many exact substitutes. If Ω is not finite, it thus seems natural to think of a game as large if each player has many near substitutes.

THEOREM 1. *Let (Ω, Λ) be a technology. For each $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ and an integer $n(\varepsilon)$ with the following property*:

> *If (N, v_α) is a game derived from the technology (Ω, Λ), and for each player i in N there exist $n(\varepsilon)$ distinct players $j_1, \ldots, j_{n(\varepsilon)}$ in N such that* $\operatorname{dist}(\alpha(i), \alpha(j_k)) < \delta(\varepsilon)$ *for each $k = 1, \ldots, n(\varepsilon)$, then the Shapley value of (N, v_α) is in the individually rational ε-core of (N, v_α).*

A number of remarks are in order. Most importantly, we stress that *our result is valid for games* and not simply for private goods exchange economies. The case of private goods exchange economies (with money) is covered by our framework, but is a very special case.

Continuity of Λ assures us that players whose attributes are close together are close substitutes. The parameter $\delta(\varepsilon)$ is thus a measure of how close these substitutes must be.

For the remainder of our remarks, we will, for ease of exposition, assume that the attribute space Ω is finite and speak of types.

We stress that our result asserts that the Shapley value is in the ε-core whenever there are enough players of each type; we do not need to know anything at all about the relative proportions of players of each type. By contrast, all the extant value convergence results for private goods exchange economies treat convergent sequences of economies, or sequences of replica economies, so that the relative proportions of players of each type play a crucial role.[6] Sequences of replica games are also of interest to us (we discuss them in Section 9) but our main result is much more general.

It is important that the game have enough players of each type (or at least of each type which is represented at all). It will not suffice merely to have enough players in total; Example 2 of Wooders and Zame [20] shows that this weaker condition does not guarantee that there is any payoff at all in the individually rational ε-core.

[6] Of course, these value convergence theorems also assert the convergence of the value to the competitive equilibrium, which is meaningless in our framework.

We note that the Shapley value may be in the ε-core and yet not be close to any point in the core. The most obvious reason for this is that the core may be empty. However, the Shapley value need not be close to the core even if the core is not empty, as the following simple example illustrates.

EXAMPLE (THE MARKET FOR GLOVES). The attribute space Ω consists of two points, $\Omega = \{L, R\}$. Every profile f on Ω may then be written uniquely in the form $f = n_L \chi_L + n_R \chi_R$; define $\Lambda(n_L \chi_L + n_R \chi_R) = \min(n_L, n_R)$. (This is the market for (indivisible) gloves. Agents of type L own one left glove; agents of type R own one right glove. The value of a pair of gloves is one dollar; odd gloves and fractional combinations are valueless. Money is perfectly divisible and freely transferable.) If we consider a derived game (N, v_α) with k players of type L and $k+1$ players of type R, we see that the core of (N, v_α) consists of the unique payoff for which the players of type L each receive one dollar, while the players of type R receive nothing. The Shapley value, on the other hand, shares the total payoff of k dollars more evenly, with players of type L receiving only a slightly higher payoff then players of type R. (See Shapley and Shubik [16] for a detailed numerical discussion of this example.)

Notice that in this example, if we let k tend to infinity and imagine the limiting continuum market, the imbalance between left and right gloves disappears. In the limit, the Shapley value divides the total payoff equally between players of type L and type R; this payoff is indeed in the core of the limiting continuum market. We will pursue this idea in Section 9.

Finally, we want to make some remarks about the proof. The usual proofs of value convergence theorems for private goods exchange economies depend of course on the special structure: divisible goods, concave utility functions, etc. At the heart of all such proofs is the idea that the marginal contribution of a player to a large coalition depends essentially only on the approximate distribution of types within the coalition. The law of large numbers guarantees that, for most large coalitions, the distribution of types within the coalition is approximately the same as in the economy as a whole. Thus, the Shapley value for each player is approximately the same as his marginal contribution to the entire economy, which is his competitive payoff.[7]

Such an approach cannot work for games in general, since a player's marginal contribution to a coalition may depend very critically on the precise make-up of the coalition. This is the case, for instance, in the glove market, and will commonly be the case whenever we are modelling indivisible goods, or coalition-production, or club goods, etc. Our approach is quite different.

[7] This discussion is over-simplified, but we think it captures the essential idea.

At the heart of our proof are two ideas. The first of these ideas is that, even in large games, the blocking power is concentrated in small coalitions. A little more formally: if the feasible, Pareto optimal payoff x is not in the ε-core of a game derived from the given technology, so that there is a coalition S with $v(S) > x(S) + \varepsilon|S|$, then there is a small coalition S' with $v(S') > x(S') + \frac{1}{2}\varepsilon|S'|$. The second idea is that, in a large game, no small group can profit by forming a syndicate (i.e., an *a priori* coalition). More formally, the Shapley value which would accure to a group if they agreed to act as a single unit is very nearly the sum of their individual Shapley values.

Our results on the blocking power of small coalitions are in Section 5. Section 6 contains a probabilistic estimate which is used in Section 7, where we discuss the power of syndicates. The proof of Theorem 1 is given in Section 8; it follows quite easily from the two basic ideas. We also discuss the question of estimating the parameters $\delta(\varepsilon)$ and $n(\varepsilon)$.

5. The Power of Small Coalitions

Throughout this section, we fix a technology (Ω, Λ). Our goal is to show that for games derived from this technology the blocking power is concentrated in small coalition. The precise statement is as follows:

THEOREM 2. *For each $\varepsilon > 0$, there is an integer $l(\varepsilon)$ with the following property*:

> *If (N, v_α) is any game derived from the technology (Ω, Λ) and $x \in R^N$ is a feasible, Pareto optimal payoff which is not in the ε-core of (N, v_α), then there is a coalition $S \subset N$ such that $|S| \leq l(\varepsilon)$ and $v_\alpha(S) > x(S) + (\varepsilon/2)|S|$.*

Informally: for any derived game, any (feasible, Pareto optimal) allocation which can be ε-improved upon by any coalition whatsoever, can be $(\varepsilon/2)$-improved upon a small coalition.

Before beginning the proof of Theorem 2, it is useful to isolate two lemmas. The first is a combinatorial lemma which will be used several times.

LEMMA A. *Let $\omega_1,..., \omega_K$ be distinct points in Ω and let $\theta_1,..., \theta_K$ be strictly positive real numbers whose sum is one. Let $\{\phi_i\}$ and $\{\psi_j\}$ be sequences of profiles in $P(\Omega)$ such that*:

(a) $\phi_i(\omega) = 0$ *for each i and each $\omega \notin \{\omega_1,..., \omega_K\}$ (no restriction is made for ψ)*;

(b) $\lim_{i\to\infty} \|\phi_i\| = \lim_{j\to\infty} \|\psi_j\| = +\infty$;

(c) $\lim_{i \to \infty} (\phi_i(\omega_k)/\|\phi_i\|) = \lim_{j \to \infty}(\psi_j(\omega_k)/\|\psi_j\|) = \theta_k$ *for each* k, $1 \leqslant k \leqslant K$.

For each i, j *let* r_{ij} *be the largest integer such that* $r_{ij}\phi_i \leqslant \psi_j$. *Then*

$$\lim_{i \to \infty} \limsup_{j \to \infty} (r_{ij}\|\phi_i\|/\|\psi_j\|) = \lim_{i \to \infty} \liminf_{j \to \infty} (r_{ij}\|\phi_i\|/\|\psi_j\|) = 1.$$

Proof. For each i, j the integer r_{ij} is characterized by the inequalities

$$r_{ij}\phi(\omega_k) \leqslant \psi_j(\omega_k) \qquad \text{for each } k;$$

$$(r_{ij} + 1)\, \phi_i(\omega_l) > \psi_j(\omega_l) \qquad \text{for some } l.$$

We now fix an arbitrary real number ρ with $0 < \rho < 1$. By (c) above and the fact that each θ_k is strictly positive, we can find integers i_0 and j_0 such that for $i \geqslant i_0$ and $j \geqslant j_0$,

$$(1-\rho)\,\theta_k \leqslant \phi_i(\omega_k)/\|\phi_i\| \leqslant (1+\rho)\,\theta_k$$

$$(1-\rho)\,\theta_k \leqslant \psi_j(\omega_k)/\|\psi_j\| \leqslant (1+\rho)\,\theta_k$$

for each k. Combining these inequalities with those which characterize r_{ij} and simplifying yields

$$\frac{1-\rho}{1+\rho}\frac{\|\psi_j\|}{\|\phi_i\|} - 1 \leqslant r_{ij} \leqslant \frac{1+\rho}{1-\rho}\frac{\|\psi_j\|}{\|\phi_i\|}$$

or equivalently

$$\frac{1-\rho}{1+\rho} - \frac{\|\phi_i\|}{\|\psi_j\|} \leqslant r_{ij}\frac{\|\phi_i\|}{\|\psi_j\|} \leqslant \frac{1+\rho}{1-\rho}$$

provided that $i \geqslant i_0$ and $j \geqslant j_0$. If we let j tend to infinity (holding i fixed) and recall that $\|\psi_j\|$ tends to infinity, we obtain

$$\begin{aligned} \frac{1-\rho}{1+\rho} &\leqslant \liminf_{j \to \infty} r_{ij}\frac{\|\phi_i\|}{\|\psi_j\|} \\ &\leqslant \limsup_{j \to \infty} r_{ij}\frac{\|\phi_i\|}{\|\psi_j\|} \\ &\leqslant \frac{1+\rho}{1-\rho} \end{aligned}$$

for each $i \geqslant i_0$. Letting our arbitrary ρ tend to zero, so that i_0 tends to infinity, yields the desired result. ∎

The next lemma provides us with some information about payoffs for sequences of profiles with a given limiting distribution. (Note carefully the difference in assumptions between Lemma A and Lemma B.)

LEMMA B. *Let* $\omega_1,\ldots,\omega_T$ *be distinct points in* Ω *and let* $\theta_1,\ldots,\theta_T$ *be non-negative real numbers whose sum is one. Let* $\{f_i\}$ *be a sequence of profiles in* $P(\Omega)$ *such that* $\lim_{i\to\infty}\|f_i\| = +\infty$ *and* $\lim_{i\to\infty}(f_i(\omega_t)/\|f_i\|)=\theta_t$ *for each* t. *Then* $\lim_{i\to\infty}(\Lambda(f_i)/\|f_i\|)$ *exists. Moreover, this limit is independent of the sequence* $\{f_i\}$ *and depends only on the points* $\omega_1,\ldots,\omega_T$ *and the numbers* $\theta_1,\ldots,\theta_T$.

Proof. After renumbering if necessary, we may assume that $\theta_1,\ldots,\theta_K$ are strictly positive and that $\theta_{K+1}=\cdots=\theta_T=0$ for some K, $1\leqslant K\leqslant T$. Define a new sequence of profiles $\{\phi_i\}$ by

$$\phi_i(\omega_k)=f_i(\omega_k) \qquad \text{for} \quad 1\leqslant k\leqslant K-1,$$

$$\phi_i(\omega_K)=\|f_i\|-\sum_{k=1}^{K-1} f_i(\omega_k),$$

$$\phi_i(\omega)=0 \qquad \text{if} \quad \omega\notin\{\omega_1,\ldots,\omega_K\}.$$

It is evident that the points $\omega_1,\ldots,\omega_K$, the numbers $\theta_1,\ldots,\theta_K$ and the sequence $\{\phi_i\}$ satisfy all the conditions of Lemma A (i.e., take $\psi_i=\phi_i$). For each i, j let r_{ij} be the largest integer such that $r_{ij}\phi_i\leqslant\phi_j$. We are first going to show that $\lim(\Lambda(\phi_i)/\|\phi_i\|)$ exists.

Note that if M is an individual marginal bound for the technology (Ω,Λ) then

$$0\leqslant\frac{\Lambda(\phi_i)}{\|\phi_i\|}\leqslant M \qquad \text{for each } i.$$

Hence $\liminf(\Lambda(\phi_i)/\|\phi_i\|)$ and $\limsup(\Lambda(\phi_i)/\|\phi_i\|)$ both exist, and

$$\liminf(\Lambda(\phi_i)/\|\phi_i\|)\leqslant\limsup(\Lambda(\phi_i)/\|\phi_i\|).$$

To see that the reverse inequality holds we note that since $r_{ij}\phi_i\leqslant\phi_j$, the superadditivity of Λ yields

$$\Lambda(\phi_j)\geqslant\Lambda(r_{ij}\phi_i)\geqslant r_{ij}\Lambda(\phi_i)$$

and hence that

$$\frac{\Lambda(\phi_j)}{\|\phi_j\|}\geqslant r_{ij}\frac{\Lambda(\phi_i)}{\|\phi_j\|}=\left(\frac{r_{ij}\|\phi_i\|}{\|\phi_j\|}\right)\left(\frac{\Lambda(\phi_i)}{\|\phi_i\|}\right)$$

for each i, j. Let ε be a small positive number. In view of Lemma A and the definition of lim sup, we can find an index i_0 such that

$$\left(\frac{r_{i_0 j}\|\phi_{i_0}\|}{\|\phi_j\|}\right)\left(\frac{\Lambda(\phi_{i_0})}{\|\phi_{i_0}\|}\right) \geqslant \limsup \frac{\Lambda(\phi_i)}{\|\phi_i\|} - \varepsilon$$

for every index j greater than some j_0 (which depends on i_0). On the other hand, we can; by the definition of lim inf, find an index $j_1 \geqslant j_0$ for which

$$\frac{\Lambda(\phi_{j_1})}{\|\phi_{j_1}\|} \leqslant \liminf \frac{\Lambda(\phi_j)}{\|\phi_j\|} + \varepsilon.$$

Combining these inequalities yields that

$$\liminf \frac{\Lambda(\phi_j)}{\|\phi_j\|} + \varepsilon \geqslant \limsup \frac{\Lambda(\phi_i)}{\|\phi_i\|} - \varepsilon.$$

Since ε was arbitrary, we conclude that $\liminf(\Lambda(\phi_j)/\|\phi_j\|) \geqslant \limsup(\Lambda(\phi_i)/\|\phi_i\|)$ and hence that $L = \lim(\Lambda(\phi_i)/\|\phi_i\|)$ exists.

Returning to our original sequence $\{f_i\}$ of profiles, we want to show that $L = \lim(\Lambda(f_i)/\|f_i\|)$. To see this, note that $\|f_i\| = \|\phi_i\|$ and that our individual marginal bound and the definition of ϕ_i yield:

$$\begin{aligned}\left|\frac{\Lambda(f_i)}{\|f_i\|} - \frac{\Lambda(\phi_i)}{\|\phi_i\|}\right| &= \frac{|\Lambda(f_i) - \Lambda(\phi_i)|}{\|f_i\|} \\ &\leqslant 2M \frac{\sum_{k=1}^{K} f_i(\omega_k) - \|f_i\|}{\|f_i\|}.\end{aligned}$$

Since $(\sum f_i(\omega_k) - \|f_i\|)/\|f_i\|$ tends to zero (as i tends to infinity), we conclude that $\lim_{i\to\infty}(\Lambda(f_i)/\|f_i\|) = \lim_{i\to\infty}(\Lambda(\phi_i)/\|\phi_i\|) = L$.

Finally, to show that this limit L is independent of the sequence $\{f_i\}$, let $\{g_i\}$ be another such sequence. Define a sequence $\{h_i\}$ by $h_{2i} = f_i$ and $h_{2i+1} = g_i$. By the above argument, $\lim(\Lambda(h_i)/\|h_i\|)$ exists. Since all subsequences of a convergent sequence converge to the same limit, we conclude that $L = \lim(\Lambda(f_i)/\|f_i\|) = \lim(\Lambda(g_i)/\|g_i\|)$, as asserted. This completes the proof. ∎

With these two lemmas in hand, we can now give the proof of Theorem 2.

Proof of Theorem 2. Suppose the conclusion were not so. Then we could find, for each integer l, a finite set N_l, an attribute function $\alpha_l\colon N_l \to \Omega$, a vector $x_l \in \mathbb{R}^{N_l}$, and a subset B_l of N_l with the following properties:

(a) $x_l(N_l) = v_{\alpha_l}(N_l)$;

(b) $v_{\alpha_l}(B_l) > x_l(B_l) + \varepsilon|B_l|$,

(c) if $v_{\alpha_l}(S) > x_l(S) + \varepsilon|S|/2$, then $|S| > l$.

In order to obtain a contradiction, we are going to construct small representative subcoalitions S_l of B_l for which $v_{\alpha_l}(S_l)/|S_l|$ is approximately $v_{\alpha_l}(B_l)/|B_l|$ and $x_l(S_l)/|S_l|$ is (almost) less than $x_l(B_l)/|B_l|$.

We begin by choosing a $\delta > 0$ such that if $\omega_1, \omega_2 \in \Omega$ and $\text{dist}(\omega_1, \omega_2) < \delta$ then $|\Lambda(f + \chi_{\omega_1}) - \Lambda(f + \chi_{\omega_2})| < \varepsilon/10$ for each profile f (continuity of Λ). We then use compactness of Ω to write Ω as the disjoint union of a finite number of nonempty subsets $\Omega_1, \ldots, \Omega_T$, each of diameter less than δ. For each t, choose and fix a point $\omega_t \in \Omega_t$.

For each l and t, set

$$\theta_t^l = |\alpha_l^{-1}(\Omega_t) \cap B_l|/B_l$$

so that θ_t^l is the relative proportion of players in B_l with attributes in the set Ω_t. Notice that $0 \leqslant \theta_t^l \leqslant 1$ and that $\sum_{t=1}^T \theta_t^l = 1$ for each l. Passing to a subsequence if necessary, we may assume that $\theta_t = \lim_{l \to \infty} \theta_t^l$ exists for each t. Renumbering if necessary, we may also assume that $\theta_1, \ldots, \theta_K$ are strictly positive and $\theta_{K+1} = \cdots = \theta_T = 0$ for some K, $1 \leqslant K \leqslant T$.

We now define profiles ϕ_l and ψ_l by

$$\begin{aligned}
&\phi_l(\omega_t) = |\alpha_l^{-1}(\Omega_t) \cap B_l| && \text{for } 1 \leqslant t \leqslant K;\\
&\phi_l(\omega) = 0 && \text{for } \omega \notin \{\omega_1, \ldots, \omega_K\};\\
&\psi_l(\omega_t) = |\alpha_l^{-1}(\Omega_t) \cap B_l| && \text{for } 1 \leqslant t \leqslant T;\\
&\psi_l(\omega) = 0 && \text{for } \omega \notin \{\omega_1, \ldots, \omega_T\}.
\end{aligned}$$

(Notice that $\phi_l \leqslant \psi_l$.) It is easily checked that the points $\omega_1, \ldots, \omega_T$, the numbers $\theta_1, \ldots, \theta_T$ and the sequences $\{\phi_l\}$ and $\{\psi_l\}$ satisfy the conditions of Lemmas A and B. Hence,

$$\lim_{l \to \infty} \frac{\Lambda(\phi_l)}{\|\phi_l\|} = \lim_{l \to \infty} \frac{\Lambda(\psi_l)}{\|\psi_l\|}$$

and both these limits exist; call this common limit L. Fix an index l_0 so that $|L - \Lambda(\phi_l)/\|\phi_l\|| < \varepsilon/10$ and $|L - \Lambda(\psi_l)/\|\psi_l\|| < \varepsilon/10$ for every $l \geqslant l_0$.

For each i, j let r_{ij} be the largest integer such that $r_{ij}\phi_i \leqslant \psi_j$. In view of Lemma A, we may choose an index $I \geqslant l_0$ and an index $J_0 \geqslant l_0$ such that (for M the individual marginal bound for the technology):

$$\frac{r_{IJ}\|\phi_I\|}{\|\psi_J\|} > \left(1 + \frac{\varepsilon}{10M}\right)^{-1}$$

whenever $J \geqslant J_0$ (keeping I fixed).

By assumption, $\theta_t > 0$ for each t with $1 \leqslant t \leqslant K$, so in particular $|\alpha_l^{-1}(\Omega_t \cap B_l)| \to \infty$ as $l \to \infty$ for each t with $1 \leqslant t \leqslant K$. We may therefore choose an integer J_1 so that

$$|\alpha_l^{-1}(\Omega_t) \cap B_l| \geqslant \phi_I(\omega_t) = |\alpha_I^{-1}(\Omega_t) \cap B_I|$$

(for each t with $1 \leqslant t \leqslant K$) whenever $l \geqslant J_1$.

For each $l \geqslant \max(I, J_0, J_1)$, we now define a coalition $S_l \subset N_l$ in the following way. For each t, $1 \leqslant t \leqslant K$, we consider all subsets A of $\alpha_l^{-1}(\Omega_t) \cap B_l$ with $|A| = \phi_I(\omega_t) = |\alpha_I^{-1}(\Omega_t) \cap B_I|$; there are of course only finitely many such sets. Among all such sets, we let S_l^t be any one for which the aggregate payoff $x_l(A)$ is as small as possible. Finally, we set $S_l = \bigcup_{t=1}^K S_l^t$. Notice that $|S_l| = \|\phi_I\| \leqslant |B_I|$, so if we can show that

$$v_{\alpha_l}(S_l) > x_l(S_l) + \frac{\varepsilon}{2}|S_l|,$$

or equivalently that

$$\frac{v_{\alpha_l}(S_l)}{|S_l|} > \frac{x_l(S_l)}{|S_l|} + \frac{\varepsilon}{2},$$

we will have obtained a contradiction.

To see this, let us first estimate $v_{\alpha_l}(S_l)/|S_l|$. Our choice of the sets Ω_t and our construction of S_l, together with the facts that $|S_l| = \|\phi_I\|$ and $l \geqslant I \geqslant l_0$ imply that

$$\begin{aligned}
\frac{v_{\alpha_l}(S_l)}{|S_l|} &\geqslant \frac{\Lambda(\phi_I)}{\|\phi_I\|} - \frac{\varepsilon}{10} \\
&> L - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} \\
&> \frac{\Lambda(\psi_I)}{\|\psi_I\|} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} \\
&\geqslant \frac{v_{\alpha_I}(B_I)}{|B_I|} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10}.
\end{aligned}$$

By assumption, $v_{\alpha_I}(B_I) > x_I(B_I) + \varepsilon |B_I|$, so we obtain

$$\frac{v_{\alpha_l}(S_l)}{|S_l|} > \frac{x_I(B_I)}{|B_I|} + \frac{6\varepsilon}{10}.$$

We now estimate $x_l(S_l)/|S_l|$. By construction $r_{lI}\phi_I \leqslant \psi_l$ so that $r_{lI}\phi_I(\omega_t) \leqslant \psi_l(\omega_t)$ for $1 \leqslant t \leqslant K$. But $\phi_I(\omega_t) = |S_l^t| = |S_l \cap \alpha_l^{-1}(\Omega_t)|$, and

$\psi_l(\omega_t) = |\alpha_l^{-1}(\Omega_t) \cap B_l|$, so $r_{ll}|S_l^t| = r_{ll}|S_l \cap \alpha_l^{-1}(\Omega_t)| \leqslant |\alpha_l^{-1}(\Omega_t) \cap B_l|$ for $1 \leqslant t \leqslant K$. Using the fact that S_l^t minimizes $x_l(A)$ over all subsets A of $\alpha_l^{-l}(\Omega_t)$ with $|S_l^t|$ members, we obtain

$$\begin{aligned} x_l(B_l) &\geqslant \sum_{t=1}^{K} x_l(B_l \cap \alpha_l^{-1}(\Omega_t)) \\ &\geqslant \sum_{t=1}^{K} r_{ll} x_l(S_l^t) \\ &= r_{ll} x_l(S_l). \end{aligned}$$

Recalling that $|S_l| = \|\phi_l\|$ and $|B_l| = \|\psi_l\|$, that $l \geqslant J_0$ (which gives an estimate on r_{ll}) and that $x_{\alpha_l}(B_l) < v_{\alpha_l}(B_l) + \varepsilon |B_l|$, and using our individual marginal bound, we obtain:

$$\begin{aligned} \frac{x_l(S_l)}{|S_l|} &\leqslant \frac{x_l(B_l)}{r_{ll}|S_l|} \\ &= \left(\frac{x_l(B_l)}{r_{ll}\|\phi_l\|}\right)\left(\frac{\|\psi_l\|}{|B_l|}\right) \\ &\leqslant \left(\frac{x_l(B_l)}{|B_l|}\right)\left(1 + \frac{\varepsilon}{10M}\right) \\ &< \frac{v_{\alpha_l}(B_l)}{|B_l|} + \frac{\varepsilon}{10}. \end{aligned}$$

Combining this estimate with our estimate on $v_{\alpha_l}(S_l)/|S_l|$ yields

$$\begin{aligned} \frac{x_l(S_l)}{|S_l|} &< \frac{v_{\alpha_l}(S_l)}{|S_l|} - \frac{6\varepsilon}{10} + \frac{\varepsilon}{10} \\ &= \frac{v_{\alpha_l}(S_l)}{|S_l|} - \frac{\varepsilon}{2}, \end{aligned}$$

which is the contradiction we desire. This completes the proof of Theorem 2. ▮

6. A Probabilistic Estimate

As with most calculations involving the Shapley value, ours depends on a probabilistic estimate. The estimate we use is very simple, so we derive it here.

We consider an urn which contains a total of t balls, of which g are green. We draw, at random and without replacement, a sample of size k

($k \leqslant t$), and write h for the number of green balls in the sample. We want to estimate the probability that

$$\left|\frac{h}{g}-\frac{k}{t}\right|<\eta,$$

where η is some predetermined parameter.

To obtain this estimate, we simply follow Feller [6, p. 233]. The number h is a random variable whose expected value is kg/t and whose variance is

$$\operatorname{Var}(h)=\frac{kg(t-g)}{t^2}\left[1-\frac{k-1}{t-1}\right]<\frac{kg(t-g)}{t^2}$$

Hence Chebyshev's inequality yields

$$\begin{aligned}\operatorname{prob}\left(\left|\frac{h}{g}-\frac{k}{t}\right|\geqslant\eta\right)&=\operatorname{prob}\left(\left|h-\frac{gk}{t}\right|\geqslant\eta g\right)\\&<\frac{\operatorname{Var}(h)}{(\eta g)^2}\\&<\frac{kg(t-g)}{t^2\eta^2g^2}\\&<\frac{1}{\eta^2 g}.\end{aligned}$$

Equivalently

$$\operatorname{prob}\left(\left|\frac{h}{g}-\frac{k}{t}\right|<\eta\right)>1-\frac{1}{\eta^2 g}.$$

This is the estimate we shall need.

Note that this estimate is independent of t (the total number of balls) and of k (the sample size). Note too that for η fixed, this probability tends to one as g (the number of green balls) tends to infinity.

7. The Power of Small Syndicates

In this section we prove a general result about the Shapley value of games. This result applies to *all* games, whether or not they are derived from a given technology.

We consider the way in which the Shapley value of a game changes if we allow a group of players to form a syndicate. We will show that, if the syndicate is small (in the sense that there are many near substitutes for members of the syndicate), then the change in the Shapley value is also small.

We consider games (N, v) which are not required to be superadditive. If S is a nonempty subset of N, then by the *syndicate game* (N_S, v_S) we mean the game whose player set is $N_S = (N\backslash S) \cup \{S\}$ and whose characteristic function is given by

$$v_S(W) = v(W) \qquad \text{if} \quad \{S\} \notin W,$$

$$v_S(W) = v([W \cap (N\backslash s)] \cup S) \qquad \text{if} \quad \{S\} \in W.$$

That is, (N_S, v_S) is the game which results if we treat the syndicate S as an indivisible unit. We want to compare $Sh(v_S, \{S\})$ with $\sum_{i \in S} Sh(v, i)$; the difference between these two numbers might be called the *power* of the syndicate, i.e., the gain or loss resulting from formation of the syndicate.

We need some terminology. We will say that the positive number M is an *individual marginal bound* for the game (N, v) if

$$|v(W \cup \{j\}) - v(W)| \leqslant M$$

for each $j \in N$ and each $W \subset N$. For $\gamma > 0$, we will say that players $i, j \in N$ are *γ-substitutes* if

$$|v(W \cup \{i\}) - v(W \cup \{j\}| < \gamma$$

for every $W \subset N\backslash\{i, j\}$. Our result on syndication is the following.

THEOREM 3. *Let M and γ be positive numbers and let s be a positive integer. Then there is an integer $r(M, \gamma, s)$ with the following property*:

If (N, v) is a game with an individual marginal bound of M, S is a coalition of N with $|S| \leqslant s$, and for each player i in S there are at least $r(M, \gamma, s)$ players not in S who are γ-substitutes for i, then

$$\left| Sh(v_S, \{S\}) - \sum_{i \in S} Sh(v, i) \right| \leqslant (\tfrac{3}{2})|S|(|S| + 1)\gamma.$$

We emphasize again that this result is independent of the framework of technologies. It is an assertion about the Shapley value of *every* game. Thus the number $r(M, \gamma, s)$ depends on M, on γ, and on s, but not on any underlying technology—since there is not one. It is possible to give an explicit bound for $r(M, \gamma, s)$, but it would be very messy.

Proof. We are going to obtain an estimate for $|Sh(v_S, \{S\}) - \sum_{i \in S} Sh(v, i)|$ in terms of the number of γ-substitutes for members of S and some other parameters. We will then show that this estimate can be reduced to the one we want by judicious choice of these other parameters provided the number of γ-substitutes is large enough. If $s=1$ there is nothing to prove (since $v_S = v$); we consider first the case $s=2$.

We fix a game (N, v) and a coalition $S \subset N$ with $|S|=2$. Write $S=\{a, b\}$, $N^* = N \backslash S$ and $n=|N|-1=|N^*|+1=|N_S|$. Let $A \subset N^*$ be a set of γ-substitutes for a, and let $B \subset N^*$ be a set of γ-substitutes for b. We may assume that $|A|=|B|=r$ (simply ignoring some γ-substitutes if necessary). Finally, we fix parameters ρ, τ and ζ with $0<\rho<1/2$, $0<\tau$, and $0<\zeta<1$.

For the remainder of the proof it is convenient to write

$$\Delta(W, i)=v(W \cup \{i\})-v(W)$$

for $W \subset N$ and $i \in N$.

By definition,

$$\begin{aligned}
Sh(v_S, \{S\}) &= \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} [v_S(W \cup \{S\}) - v_S(W)] \\
&= \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} [v(W \cup S) - v(W)] \\
&= \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} [v(W \cup \{a, b\}) - v(W \cup \{a\})] \\
&\quad + \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} [v(W \cup \{a\}) - v(W)] \\
&= \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} \Delta(W \cup \{a\}, b) \\
&\quad + \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n-1}{J}} \sum_{\substack{W \subset N^* \\ |W|=J}} \Delta(W, a).
\end{aligned}$$

Let us write Q for this last double sum and P for the next to last. We are going to estimate $|Sh(v, a)-Q|$ and $|Sh(v, b)-P|$.

As a guide to the intuition, we point out that Q is itself the Shapley value for player a in the game $(\bar{N}, \bar{v})$, where $\bar{N}=N \backslash \{b\}$ and $\bar{v}(W)=v(W)$ for each $W \subset \bar{N} = N \backslash \{b\}$. Since B consists of γ-substitutes for b, it is natural to expect that $Q = Sh(\bar{v}, a)$ should be close to $Sh(v, a)$ if $|B| = r$ is large. Verifying that this is so, however, takes quite a lot of work.

By definition,

$$Sh(v, a) = \frac{1}{n+1} \sum_{J=0}^{n} \frac{1}{\binom{n}{J}} \sum_{\substack{W \subset N \backslash \{a\} \\ |W| = J}} \Delta(W, a).$$

Write

$$E = \frac{1}{n} \sum_{J=0}^{n-1} \frac{1}{\binom{n}{J}} \sum_{\substack{W \subset N \backslash \{a\} \\ |W| = J}} \Delta(W, a)$$

Using the individual marginal bound of M, we see that

$$|E - Sh(v, a)| \leqslant \frac{2M}{n} < \frac{2M}{r}. \tag{1}$$

Now, we want to estimate $|E - Q|$. Note two main differences between the expression for E and the expression for Q: different coefficients, and the presence of coalitions which contain b. We are going to approximate terms involving coalitions which contain b by terms not involving coalitions which contain b; this will also have the effect of "correcting" the discrepancy in the coefficients.

We first introduce some sets of coalitions. For each J and k, and let $\Gamma(J, k)$ be the set of coalitions $W \subset N \backslash \{a\}$ for which $|W| = J$, $|W \cap B| = k$ and $b \notin W$; we let $\Gamma_b(J, k)$ be the set of coalitions $W \subset N \backslash \{a\}$ for which $|W| = J$, $|W \cap B| = k$ and $b \in W$, and let $\Gamma_b(J) = \bigcup_{k=0}^{r} \Gamma_b(J, k)$.

We now use these sets to break up the expressions for E and Q. Set

$$E(J, k) = \sum_{W \in \Gamma(J,k)} \Delta(W, a),$$

$$E_b(J, k) = \sum_{W \in \Gamma_b(J,k)} \Delta(W, a),$$

$$E(J) = \frac{1}{\binom{n}{J}} \sum_{k=0}^{r} [E(J, k) + E_b(J, k)],$$

and

$$Q(J) = \frac{1}{\binom{n-1}{J}} \sum_{k=0}^{r} E(J, k).$$

Notice that

$$E = \frac{1}{n} \sum_{J=0}^{n-1} E(J)$$

and that

$$Q = \frac{1}{n} \sum_{J=0}^{n-1} Q(J).$$

Thus, we want to estimate $|E(J) - Q(J)|$ for each J.

The estimate we use depends on the size of J. If $(J-1) \leqslant \rho(n-1)$ or $(J-1) \geqslant (1-\rho)(n-1)$ we use the obvious estimate given to us by the individual marginal bound:

$$|E(J) - Q(J)| \leqslant 2M. \tag{2}$$

If $\rho(n-1) < (J-1) < (1-\rho)(n-1)$, we need to be more careful. Let us fix such a J, and let I be the set of indices k for which

$$\left| \frac{k+1}{r-k} - \frac{J}{n-J} \right| < \tau$$

(recall that τ was one of our initial fixed parameters); let I' be the complementary set of indices.

Fix an index $k \in I$. For $W \in \Gamma_b(J, k)$ and $c \in B \backslash (W \cap B)$, write $W_c = (W \backslash \{b\}) \cup \{c\}$. The fact that b and c are γ-substitutes yields that

$$|\Delta(W, a) - \Delta(W_c, a)| \leqslant 2\gamma.$$

Hence if we average over the $r-k$ elements of $B \backslash (W \cap B)$, we obtain

$$\left| \Delta(W, a) - \frac{1}{r-k} \sum_{c \in B \backslash (W \cap B)} \Delta(W_c, a) \right| \leqslant 2\gamma.$$

Notice that W was chosen to be in $\Gamma_b(J, k)$ but that each W_c belongs to $\Gamma(J, k+1)$. Now, as W runs over all elements of $\Gamma_b(J, k)$, the sets W_c run over all elements of $\Gamma(J, k+1)$, but each element of $\Gamma(J, k+1)$ occurs exactly $k+1$ times (because there are $k+1$ elements which might have occurred as replacements for b). Hence if we sum over $\Gamma_b(J, k)$, we obtain

$$\left| E_b(J, k) - \frac{k+1}{r-k} E(J, k+1) \right| \leqslant 2\gamma |\Gamma_b(J, k)|.$$

Combining this with the fact that k is in I yields

$$\left|E_b(J,k)-\frac{J}{n-J}E(J,k+1)\right|\leqslant 2\gamma\,|\Gamma_b(J,k)|+\tau\,|E(J,k+1)|$$

$$\leqslant 2\gamma\,|\Gamma_b(J,k)|+2M\tau\,|\Gamma(J,k+1)|. \quad (3)$$

On the other hand, for indices $k\notin I$ we certainly have

$$|E_b(J,k)|\leqslant 2M\,|\Gamma_b(J,k)| \quad (4)$$

and

$$|E(J,k)|\leqslant 2M\,|\Gamma(J,k)|. \quad (5)$$

The next step is to show that most of the coalitions in $\bigcup_{k=0}^{r}\Gamma_b(J,k)$ actually belong to $\bigcup_{k\in I}\Gamma_b(J,k)$. Note first that if $k/r=(J-1)/(n-1)$, then direct calculation gives

$$\frac{k+1}{r-k}=\frac{J-1}{n-J}+\frac{1}{r\left(1-\left(\frac{J-1}{n-1}\right)\right)}. \quad (6)$$

Since we have taken $J-1$ in the range $\rho(n-1)<(J-1)<(1-\rho)(n-1)$ and ρ is fixed, the second term on the right-hand side of Eq. (6) is certainly less than $\tau/3$ if r is sufficiently large. Since $n\geqslant r$, the first term on the right-hand side of (6) differs from $J/(n-J)$ by less than $\tau/3$ if r (and hence n) is large. Hence,

$$\left|\frac{k+1}{r-k}-\frac{J}{n-J}\right|<2\tau/3$$

provided that r is large and $k/r=(J-1)/(n-1)$. Thus,

$$\left|\frac{k+1}{r-k}-\frac{J}{n-J}\right|<\tau \quad (7)$$

provided that r is large and that

$$\left|\frac{k}{r}-\frac{J-1}{n-1}\right|$$

is sufficiently small; if (7) holds, then $k\in I$.

We can now use our probabilistic estimate from Section 6 in the following way. Choosing at random a coalition W in $\Gamma_b(J)$ is the same thing as choosing at random a set of $J-1$ elements from $N\backslash\{a,b\}$, and then adding the element b to this set. If $k = |W \cap B|$, the estimate of Section 7 tells us that with probability at least $1 - \zeta$ (recall that ζ is the third of our fixed parameters), the quantity

$$\left|\frac{k}{r} - \frac{J-1}{n-1}\right|$$

will be very small, provided only that r is sufficiently large. In other words,

$$\left|\bigcup_{k \in I'} \Gamma_b(J, k)\right| \leqslant \zeta \left|\bigcup_{k=0}^{r} \Gamma_b(J, k)\right| \tag{8}$$

provided that r is large.

We now begin to estimate $|E(J) - Q(J)|$ for $\rho(n-1) < (J-1) < (1-\rho)(n-1)$. We write

$$E(J) = \frac{1}{\binom{n}{J}} \sum_{k \in I} [E(J, k) + E_b(J, k)] + \frac{1}{\binom{n}{J}} \sum_{k \in I'} [E(J, k) + E_b(J, k)]. \tag{9}$$

Combining the inequalities (4), (5), and (8), we see that the second of these sums is small; to be precise,

$$\left|\frac{1}{\binom{n}{J}} \sum_{k \in I'} [E(J, k) + E_b(J, k)]\right| \leqslant 4m\zeta. \tag{10}$$

We can use (3) to obtain an approximate expression for the first sum in (9),

$$\begin{aligned} &\frac{1}{\binom{n}{J}} \sum_{k \in I} [E(J, k) + E_b(J, k)] \\ &\qquad \simeq \frac{1}{\binom{n}{J}} \sum_{k \in I} \left[E(J, k) + \frac{J}{n-J} E(J, k+1)\right], \end{aligned} \tag{11}$$

with an error not exceeding $2\gamma + 2M\tau$. Notice that in the right-hand sum of (11), every one of the terms $E(J, k+1)$ appears twice, with the exception of the term corresponding to the smallest value k_0 of k which is in I: the term $E(J, k+1)$ appears the first time with a coefficient of $J/(n-J)$ and a second time with a coefficient of 1. Hence we can rewrite the right-hand side of (11) as

$$\frac{1}{\binom{n}{J}} \sum_{k \in I} \left[E(J, k) + \frac{J}{n-J} E(J, k+1) \right]$$

$$= \frac{1}{\binom{n}{J}} \sum_{\substack{k \in I \\ k \neq k_0}} \left(\frac{J}{n-J} + 1 \right) E(J, k+1) + \frac{1}{\binom{n}{J}} E(J, k_0). \tag{12}$$

Combining (5) with the same probabilistic estimate we used before, we can see that the last of these terms can be made small:

$$\frac{1}{\binom{n}{J}} E(J, k_0) \leqslant 2M\zeta, \tag{13}$$

if r is large.

We now point out a very convenient identity:

$$\frac{1}{\binom{n}{J}} \left(\frac{J}{n-J} + 1 \right) = \frac{1}{\binom{n-1}{J}}. \tag{14}$$

If we plug this identity into (12), and combine (9) with the estimates (10), (11), and (13), we conclude that

$$E(J) \simeq \frac{1}{\binom{n-1}{J}} \sum_{\substack{k \in I \\ k \neq k_0}} E(J, k+1), \tag{15}$$

with an error not exceeding $4M\zeta + 2\gamma + 2M\tau + 2M\zeta$. But the right-hand side of (15) is just part of the expression for $Q(J)$; the terms which are missing are just the ones for which $k \in I'$, and just as before we can see that the sum of the missing terms does not exceed $4M\zeta$, for large r.

To summarize, we have shown that

$$|E(J) - Q(J)| \leqslant 10M\zeta + 2M\tau + 2\gamma, \tag{16}$$

provided that $\rho(n-1) < (J-1) < (1-\rho)(n-1)$ and r is large. Recall from (2) that

$$|E(J) - Q(J)| \leqslant 2M$$

if $(J-1) \leqslant \rho(n-1)$ or $(J-1) \geqslant (1-\rho)(n-1)$.

In order to estimate $|E-Q|$, we need only add all these estimates and divide by n. This yields

$$|E-Q| \leqslant \frac{1}{n}[2\rho n M + n(10M\zeta + 2M\tau + 2\gamma)]$$
$$= M(2\rho + 10\zeta + 2\tau) + 2\gamma.$$

Combining this with (1) gives us the estimate we are after:

$$|Sh(v,\ a) - Q| \leqslant M(2\rho + 10\zeta + 2\tau) + 2\gamma + \frac{2M}{r},$$

for r sufficiently large. It is now clear that we need not be very careful in our choice of the parameters ρ, τ, and ζ. If we simply choose each of them very small, we obtain

$$|Sh(v, a) - Q| \leqslant 3\gamma,$$

provided that r is sufficiently large. This is the desired estimate for Q.

The intuition underlying our estimate for P is similar to the intuition underlying our estimate for Q. We want to see that P is nearly the Shapley value for player b in the game $(\bar{\bar{N}}\,\bar{\bar{v}})$, where $\bar{\bar{N}} = N \backslash \{a\}$ and $\bar{\bar{v}}(W) = v(W)$ for each $W \subset N \backslash \{a\}$. Since the preceding argument, with the roles of a and b reversed, shows that $|Sh(\bar{\bar{v}}, b) - Sh(v, b)| \leqslant 3\gamma$ if r is large, this will give us the estimate we need.

Our previous argument was based on the idea of systematically replacing every term which involved a coalition containing b by terms which do not involve coalitions containing b. This time, we want to systematically replace every term involving a coalition containing a by terms which do not involve coalitions containing a. The twist is that all these terms occur in the expression for P and not in the expression for $Sh(\bar{\bar{v}}, b)$. The terms we seek to replace are of the form $\Delta(W \cup \{a\}, b)$, and following the same procedure as before, we obtain the approximation

$$\Delta(W \cup \{a\}, b) \simeq \Delta(W \cup \{d\}, b),$$

for $d \in A \backslash (W \cap A)$. Notice that $W \cap \{d\}$ is a coalition in $\bar{\bar{N}} \backslash \{b\} = N \backslash \{a, b\}$, and $|W \cup \{d\}| = |W| + 1$. Thus if we follow the same averaging procedure as before, the terms in the expression for P corresponding to coalitions W of size J become terms in the expression for $Sh(\bar{\bar{v}}, b)$ corresponding to coalitions $W \cup \{d\}$ of size $J+1$. The averaging procedure also gives us an extra factor which (for most of the terms) is nearly equal to $(J+1)/(n-1-J)$. This extra factor is just what we need, since we can then use, instead of the identity (14), the identity

$$\frac{1}{\binom{n-1}{J}}\left(\frac{J+1}{n-1-J}\right)=\frac{1}{\binom{n-1}{J+1}}.$$

After carrying out the same sort of approximations as before, we obtain that

$$|P-Sh(\bar{\bar{v}}, b)| \leqslant 3\gamma,$$

provided that r is sufficiently large. Since, as we have already noted, the first argument (with the roles of a and b reversed) $|Sh(\bar{\bar{v}}, b)-Sh(v, b)| \leqslant 3\gamma$, we conclude that

$$|P-Sh(v, b)| \leqslant 6\gamma,$$

if r sufficiently large. Combining our estimates for P and Q gives

$$\begin{aligned}|Sh(v_S, \{S\})-Sh(v, a)-Sh(v, b)| &= |P+Q-Sh(v, a)-Sh(v, b)| \\ &\leqslant 9\gamma,\end{aligned}$$

provided that r is sufficiently large. This is the estimate we want, so the proof in the case $|S|=s=2$ is complete.

Finally, we come to the general case. Let $|S| \leqslant s$ and write $S=\{a_1,\ldots, a_{|S|}\}$. We write, as before,

$$Sh(v_S, \{S\})=P_1+\cdots+P_{|S|}.$$

Just as above, we find that $|P_{|S|}-Sh(v, a_S)| \leqslant 3\gamma$, that $|P_{|S|-1}-Sh(v, a_{S-1})| \leqslant 6\gamma$, and so forth. Summing yields

$$\left|Sh(v_S, \{S\})-\sum_{i=1}^{|S|} Sh(v, a_i)\right| \leqslant \frac{3|S|(|S|+1)\gamma}{2},$$

provided that r (the number of γ-substitutes for members of S) is sufficiently large. This is the estimate we needed, so the proof is complete. ∎

Although we shall have no need of it, we point out that a similar argument can be used to show that for $i \notin S$ we also have

$$|Sh(v_S, i)-Sh(v, i)| < \frac{3|S|(|S|+1)\gamma}{2},$$

provided that the number of γ-substitutes for members of S is sufficiently large.

7. Proof of Theorem 1

It is now very easy to combine Theorem 2 of Section 5 and Theorem 3 of Section 7 to give a proof of Theorem 1.

We are given a technology (Ω, Λ) and a positive number ε. Let $l(\varepsilon)$ be the integer by Theorem 2. Set

$$\gamma = \frac{\varepsilon}{3(l(\varepsilon)+1)},$$

and choose a positive number $\delta(\varepsilon) > 0$ such that for each $f \in P(\Omega)$,

$$|\Lambda(f+\chi_{\omega_1}) - \Lambda(f+\chi_{\omega_2})| < \gamma$$

whenever ω_1 and ω_2 belong to Ω and $\operatorname{dist}(\omega_1, \omega_2) < \delta(\varepsilon)$. Let $r(M, \gamma, l(\varepsilon))$ be the integer given by Theorem 3. Finally set $n(\varepsilon) = r(M, \gamma, l(\varepsilon)) + l(\varepsilon)$.

Now suppose we are given a game (N, v_α) derived from (Ω, Λ) and that for each player $i \in N$ there are at least $n(\varepsilon)$ distinct players $j_1, ..., j_{n(\varepsilon)}$ such that $\operatorname{dist}(\alpha(i), \alpha(j_k)) < \delta(\varepsilon)$ for $1 \leqslant k \leqslant n(\varepsilon)$. We must show that $Sh(v_\alpha)$ belongs to the individually rational ε-core of (N, v_α). Certainly, $Sh(v_\alpha)$ is feasible, Pareto optimal and individually rational. If $Sh(v_\alpha)$ were not in the ε-core, we could, by Theorem 2, find a coalition $S \subset N$ such that $|S| \leqslant l(\varepsilon)$ and

$$\sum_{i \in S} Sh(v_\alpha, i) < v_\alpha(S) - \frac{\varepsilon}{2}|S|.$$

On the other hand, our choice of $\delta(\varepsilon)$ guarantees that each member of S has at least $n(\varepsilon)$ γ-substitutes in N; our choice of $n(\varepsilon)$ guarantees that at least $r(M, \gamma, l(\varepsilon))$ of these γ-substitutes do not belong to S. Hence Theorem 3 tells us that

$$\sum_{i \in S} Sh(v_\alpha, i) \geqslant Sh((v_\alpha)_S, \{S\}) - \frac{3|S|(|S|+1)\gamma}{2}.$$

On the other hand, individual rationality of the Shapley value for the game $(v_\alpha)_S$ guarantees that $Sh((v_\alpha)_S, \{S\}) \geqslant (v_\alpha)_S(\{S\}) = v_\alpha(S)$. Combining this fact with the previous inequality and the definition of γ yields

$$\sum_{i \in S} Sh(v_\alpha, i) \geqslant v_\alpha(S) - \frac{\varepsilon}{2}|S|.$$

which is a contradiction. We conclude that $Sh(v_\alpha)$ indeed belongs to the individually rational ε-core of (N, v_α), as asserted. This completes the proof of Theorem 1. ∎

It is of interest to obtain good estimates on $\delta(\varepsilon)$ and $n(\varepsilon)$, in terms of certain natural parameters of the technology (Ω, Λ). The chief obstacle to doing so is that the integer $l(\varepsilon)$ given by Theorem 2 depends on the technology in a way that seems difficult to quantify. If we think of $l(\varepsilon)$ as known, however,[8] it is not hard to obtain an estimate for $n(\varepsilon)$. Simply keeping track of the various parameters in the proofs of Theorem 3 and Theorem 1 gives an estimate for $n(\varepsilon)$ on the order of a constant times $(M/\varepsilon)^3(l(\varepsilon))^3$.[9] The number $\delta(\varepsilon)$ of course, depends only on the modulus of continuity of Λ.

9. Limiting Behavior

The example presented in Section 4 shows that the Shapley value of a large game, while in the individually rational ε-core, need not be close to any point in the core (even if the core is not empty). The same example also suggests that the Shapley value should be close, not to the core of the large finite game, but to the core of a limiting nonatomic game. The purpose of this section is to show that this is indeed the case. In fact, we show quite a lot more: the individually rational ε-cores of a sequence of games converge to the core of the limit game. To make this precise we require some preliminary discussion.

We will restrict our attention to the case where Ω is a finite set, say $\Omega = \{\omega_1, ..., \omega_T\}$ so that we will speak of types. (Similar results could undoubtedly be established in more general contexts, but would probably be much more complicated, and lose much of their intuitive flavor.)

To define a nonatomic limiting game, we fix strictly positive[10] real numbers $\theta_1, ..., \theta_T$ with $\sum \theta_t = 1$, and disjoint intervals $I_1, ..., I_T$ on the real line for which length $(I_t) = \theta_t$. Set $I = \bigcup I_t$, let $\mathscr{B}$ be the family of Borel subsets of I, and let μ be the restriction to I of Lebesgue measure. (The interpretation we have in mind is that I represents a continuum of players of which the fraction $\theta_t = \mu(I_t)$ are of type t. The family $\mathscr{B}$ of all Borel subsets if I is the family of admissible coalitions.)

To define a nonatomic game on I, in the sense of Aumann and Shapley [13], we must define a set function λ on $\mathscr{B}$ which is of bounded variation. To this end, let B be an element of $\mathscr{B}$. If $\mu(B) = 0$, we define $\lambda(B) = 0$. Otherwise, we write $\beta_t = \mu(B \cap I_t)/\mu(B)$ for each t; note that

[8] That is, if we regard $l(\varepsilon)$ as a known parameter of the technology.

[9] This may be compared with the estimate obtained by Mas-Colell, which is of the same order.

[10] The case in which some of the θ_t's are zero can be treated by restricting attention to a subset of Ω.

$\beta_t \geqslant 0$ and that $\sum \beta_t = 1$. Choose a sequence $\{f_k\}$ of profiles on Ω such that $\|f_k\| \to \infty$ and $f_k(\omega_t/\|f_k\| \to \beta_t$ for each t. We then define

$$\lambda(B) = \left(\lim_{k \to \infty} \frac{\Lambda(f_k)}{\|f_k\|}\right) \mu(B).$$

(In view of Lemma 2, this limit exists and is independent of the particular sequence $\{f_k\}$ of divisors we choose.) To see that λ is of bounded variation, we note that $0 \leqslant \Lambda(f)/\|f\| \leqslant M$ for any divisor f (where M is the individual marginal bound for the technology (Ω, Λ)) so that $0 \leqslant \lambda(B) \leqslant M\mu(B)$. (The construction we have given corresponds to the usual "fractionating process" for constructing nonatomic economies. The limit $\lim_{k \to \infty} \Lambda(f_k)/\|f_k\|$ is to be interpreted as the limiting per-capita payoff to a coalition with a given distribution of types, so $\lambda(B)$ is the limiting payoff, normalized relative to the number of players.)

Recall that the *core* of λ consists of all nonnegative, finitely-additive set functions $\sigma: \mathscr{B} \to \mathbb{R}$ such that $\sigma(I) = \lambda(I)$ and $\sigma(B) \geqslant \lambda(B)$ for each $B \in \mathscr{B}$. (If σ is in the core, we may interpret $\sigma(I_t)$ as the total (normalized) payoff to the set of players of type t.) It is a useful fact that every elements of the core treats players of the same type equally.

LEMMA C. *Let σ belong to the core of λ. Then for each t and each Borel subset A of I_t.*

$$\sigma(A) = \sigma(I_t)\, \mu(A)/\mu(I_t).$$

Proof. Let us first perform a preliminary calculation. Suppose that B is a Borel subset of I such that $\mu(B \cap I_t)/\mu(B) = \mu(I_t) = \theta_t$ for each t (so that B has the same relative distribution as I); write $B' = I \backslash B$. The definition of λ implies that

$$\lambda(B) = \lambda(I)\, \mu(B),$$
$$\lambda(B') = \lambda(I)\, \mu(B').$$

Since σ is in the core of λ, we obtain

$$\sigma(B) \geqslant \lambda(B) = \lambda(I)\, \mu(B),$$
$$\sigma(B') \geqslant \lambda(B') = \lambda(I)\, \mu(B').$$

Additivity of σ and μ imply that

$$\sigma(I) = \sigma(B) + \sigma(B') \geqslant \lambda(I).$$

Since $\sigma(I) = \lambda(I)$, we conclude that $\sigma(B) = \lambda(B)$ and $\sigma(B') = \lambda(B')$.

Now suppose that A_1, A_2 are subsets of I_t with $\mu(A_1)=\mu(A_2)$; we claim that $\sigma(A_1)=\sigma(A_2)$. To see this, we write $\rho=\mu(A_1)/\theta_t=\mu(A_2)/\theta_t$, and choose a subset C of $I \backslash I_t$ such that $\mu(C \cap I_s)/\theta_s=\rho$ for each $s \neq t$. Our preliminary calculation implies that

$$\sigma(A_1 \cup C)=\lambda(A_1 \cup C)=\lambda(I)\,\mu(A_1 \cup C),$$
$$\sigma(A_2 \cup C)=\lambda(A_2 \cup C)=\lambda(I)\,\mu(A_2 \cup C).$$

Since $\mu(A_1 \cup C)=\mu(A_2 \cup C)$, we conclude that $\sigma(A_1 \cup C)=\sigma(A_2 \cup C)$, and additivity of σ implies that $\sigma(A_1)=\sigma(A_2)$.

Finally, let A be a Borel subset of I_t, and let m be a positive integer. There is a unique integer r such that

$$\frac{r}{m} \leqslant \frac{\mu(A)}{\theta_t} < \frac{r+1}{m}.$$

We may then choose disjoint Borel subsets $E_1,\ldots, E_m$ of I_t such that $\bigcup E_i=I_t$, $\mu(E_i)=\theta_t/m$ for each i, $\bigcup_{i=1}^{r} E_i \subset A \subset \bigcup_{i=1}^{r+1} E_i$. Our previous calculations, together with additivitiy of σ yield that $\sigma(E_i)=\sigma(I_t)/m$ for each i, and hence that

$$r\sigma(I_t)/m \leqslant \sigma(A) \leqslant (r+1)\,\sigma(I_t)/m.$$

Letting m tend to infinity now yields the desired result. ∎

For each σ in the core of λ, define $\bar{\sigma} \in \mathbb{R}^T$ by $\bar{\sigma}(t)=\sigma(I_t)/\mu(I_t)$. In view of Lemma C, we can unambiguously interpret $\bar{\sigma}(t)$ as the per-capita payoff to players of type t. Let

$$\bar{C}(\lambda)=\{\bar{\sigma}\colon \sigma \text{ is in the core of } \lambda\},$$

so that $\bar{C}(\lambda)$ is a subset of $\mathbb{R}^T$.

We now fix a sequence $\{(N_k, v_{\alpha_k})\}$ of games derived from the technology (Ω, A). We assume that $|N_k| \to \infty$ and that $\alpha_k^{-1}(\omega_t)/|N_k| \to \theta_t$ for each t; there is also no loss of generality in assuming that $\alpha_k^{-1}(\omega_t)>0$ for each k and t. We view the nonatomic game λ as a (normalized) limit of the games (N_k, v_{α_k}). For $\varepsilon>0$ we will say that a payoff x in the individually rational ε-core of (N_k, v_k) is an *equal-treatment payoff* if $x(i)=x(j)$ whenever $\alpha_k(i)=\alpha_k(j)$ (so that players of the same type receive the same payoff). For such an x, we define $\bar{x} \in \mathbb{R}^T$ by $\bar{x}(t)=x(i)$ for any $i \in N_k$ with $\alpha(i)=t$; we write $\bar{C}_\varepsilon(N_k, v_{\alpha_k})$ for the set of such vectors, so that $\bar{C}_\varepsilon(N_k, v_k)$ is a subset of $\mathbb{R}^T$. Of course, for $\bar{x} \in \bar{C}_\varepsilon(N_k, v_k)$, we may interpret $\bar{x}(t)$ as the per-capita payoff to players of type t. Evidently, then, to show that the sets $\bar{C}_\varepsilon(N_k, v_{\alpha_k})$ and $\bar{C}(\lambda)$ are close is to show that, in a natural sense, the individually rational ε-core of $(N_k, v_{\alpha k})$ (and the Shapley value in particular) is close to the core of λ; this is what we are going to do. We use the Hausdorff distance between sets as the measure of closeness; for information about the Hausdorff distance and the lim sup and lim inf of a sequence of sets, we refer to Hildenbrand [8].

THEOREM 4. *Given $\delta_0 > 0$ and $\varepsilon_0 > 0$ there is an ε_1 with $0 < \varepsilon_1 < \varepsilon_0$ and an integer k_0 such that*

$$\operatorname{dist}(\bar{C}_{\varepsilon_1}(N_k, v_{\alpha k}), \bar{C}(\lambda)) < \delta_0$$

for every $k \geqslant k_0$. Equivalently,

$$\bar{C}(\lambda) = \bigcap_{\varepsilon > 0} \limsup_{k \to \infty} \bar{C}_\varepsilon(N_k, v_{\alpha k})$$

$$= \bigcap_{\varepsilon > 0} \liminf_{k \to \infty} \bar{C}_\varepsilon(N_k, v_{\alpha k}).$$

In particular, if $\zeta_k = Sh(N_k, v_{\alpha k})$, then

$$\lim_{k \to \infty} \operatorname{dist}(\bar{\zeta}_k, \bar{C}(\lambda)) = 0.$$

Proof. We write $C^k_\varepsilon = \bar{C}_\varepsilon(N_k, v_{\alpha k})$. Note that each C^k_ε is a compact set. Moreover, if $\bar{x} \in C^k_\varepsilon$, then for each t,

$$\bar{x}(t) \leqslant v_{\alpha k}(N_k)/|\alpha_k^{-1}(\omega_t)| \leqslant \frac{M|N_k|}{|\alpha_k^{-1}(\omega_t)|}.$$

Since $|N_k|/|\alpha_k^{-1}(\omega_t)| \to \theta_t$, we conclude that the sets C^k_ε are in fact uniformly bounded. We have already noted that, given $\varepsilon > 0$, the sets C^k_ε are nonempty for sufficiently large k. It follows then that $\limsup_{k \to \infty} C^k_\varepsilon \neq \varnothing$ for each $\varepsilon > 0$. Since $C^k_{\varepsilon'} \subset C^k_\varepsilon$ whenever $\varepsilon' < \varepsilon$, we also have that $\limsup C^k_{\varepsilon'} \subset \limsup C^k_\varepsilon$ whenever $\varepsilon' < \varepsilon$. Thus $\{\limsup C^k_\varepsilon\}$ is a nested family of compact sets, and in particular, $\bigcap_{\varepsilon > 0} \limsup C^k_\varepsilon \neq \varnothing$.

The next step is to show that $\bigcap_{\varepsilon > 0} \limsup C^k_\varepsilon \subset \bar{C}(\lambda)$. Let $\bar{x} \in \bigcap_{\varepsilon > 0} \limsup C^k_\varepsilon$; then $\bar{x} \in \limsup C^k_\varepsilon$ for each $\varepsilon > 0$, so there is an increasing sequence $\{k_n\}$ of positive integers, a decreasing sequence $\{\varepsilon_n\}$ of positive numbers tending to zero, and a sequence $\{\bar{x}_n\}$ converging to $\bar{x}$ with $\bar{x}_n \in C^{k_n}_\varepsilon$ for each n. We claim that $\bar{x}$ is in $\bar{C}(\lambda)$.

To see this, set $\sigma = \sum_{t=1}^T \bar{x}(t)\, \mu_t$; we need to show that σ belongs to the core of λ. First of all, note that

$$
\begin{aligned}
\sigma(I) &= \sum_{t=1}^{T} \bar{x}(t)\,\mu_t(I) \\
&= \sum_{t=1}^{T} \bar{x}(t)\,\theta_t \\
&= \sum_{t=1}^{T} \left(\lim_{n\to\infty} \bar{x}_n(t)\right)\left(\lim_{n\to\infty} \frac{|\alpha_{k_n}^{-1}(\omega_t)|}{|N_{k_n}|}\right) \\
&= \sum_{t=1}^{t} \lim_{n\to\infty} \left(\frac{\bar{x}_n(t)\,|\alpha_{k_n}^{-1}(\omega_t)|}{|N_{k_n}|}\right) \\
&= \lim_{n\to\infty} \frac{\sum_{t=1}^{T} \bar{x}_n(t)\,|\alpha_{k_n}^{-1}(\omega_t)|}{|N_{k_n}|} \\
&= \lim_{n\to\infty} \frac{v_{\alpha_{k_n}}(N_{k_n})}{|N_{k_n}|} \\
&= \lambda(I).
\end{aligned}
$$

Thus σ is feasible. If σ were not in the core we could find a Borel set B for which $\lambda(B) > \sigma(B)$; hence there would be a positive number ρ such that

$$\frac{\lambda(B)}{\mu(B)} > \frac{\sigma(B)}{\mu(B)} + \rho.$$

Choose coalitions $S_n \subset N_{k_n}$ such that $|S_n| \to \infty$ and

$$\frac{|\alpha_{k_n}^{-1}(\omega_t) \cap S_n|}{|S_n|} \to \frac{\mu(B \cap I_t)}{\mu(B)}$$

for each t. The definition of λ yields that

$$\frac{\lambda(B)}{\mu(B)} = \lim_{n\to\infty} \frac{v_{\alpha_{k_n}}(S_n)}{|S_n|}$$

so we have

$$\frac{v_{\alpha_{k_n}}(S_n)}{|S_n|} > \sum_{t=1}^{T} \frac{\bar{x}_n(t)\,|\alpha_{k_n}^{-1}(\omega_t) \cap S_n|}{|S_n|} + \frac{\rho}{4}$$

for n large. If we write x_n for the equal-treatment payoff in the ε_n-core of $(N_{k_n}, v_{\alpha_{k_n}})$ which gives rise to $\bar{x}_n$, then our last inequality implies that

$$v_{\alpha_{k_n}}(S_n) > x_n(S_n) + \frac{\rho}{4}|S_n|$$

for large n. However, since $\varepsilon_n \to 0$, this contradicts the fact that x_n is in the ε_n-core of $(N_{k_n}, v_{\alpha_{k_n}})$ for large n. We conclude that σ is in the core of λ and that $\bar{x}$ belongs to $\bar{C}(\lambda)$, as asserted. Thus $\bigcap_{\varepsilon>0} \limsup C_\varepsilon^k \subset \bar{C}(\lambda)$; note that in particular, $\bar{C}(\lambda) \neq \varnothing$.

The next step is to show that $\bar{C}(\lambda) \subset \bigcap_{\varepsilon>0} \liminf C_\varepsilon^k$. Given $\bar{\sigma} \in \bar{C}(\lambda)$ and an $\varepsilon > 0$, we must therefore construct, for each large k, a vector $\bar{x}_k \in C_\varepsilon^k$ such that $\bar{x}_k \to \bar{\sigma}$. To this end, define, for each k, a vector $y_k \in \mathbb{R}^{N_k}$ by $y_k(i) = \bar{\sigma}(\alpha_k(i))$. We will show that a small perturbation of y_k is in the individually rational ε-core of (N_k, v_{α_k}).

It is convenient and involves no loss of generality to assume that our games are zero-normalized, so that $v_{\alpha_k}(\{i\}) = 0$ for each k and i.

Write $\sigma = \sum \bar{\sigma}(t)\, \mu_t$, so that σ is in the core of λ. By definition,

$$\sigma(I) = \lim \frac{v_{\alpha_k}(N_k)}{|N_k|}.$$

Since

$$y_k(N_k) = \sum y_k(i) = \sum_{t=1}^{T} \bar{\sigma}(t)\, |\alpha_k^{-1}(\omega_t)|$$

and

$$|\alpha_k^{-1}(\omega_t)|/|N_k| \to \theta_t = \mu(I_t),$$

we conclude that

$$\frac{y_k(N_k)}{|N_k|} \to \sigma(I),$$

and

$$\frac{v_{\alpha_k}(N_k)}{y_k(N_k)} \to 1.$$

Thus if we set

$$z_k = \frac{v_{\alpha_k}(N_k)}{y_{\alpha_k}(N_k)}\, y_k,$$

we obtain a feasible, Pareto optimal, individually rational equal-treatment payoff for the game (N_k, v_{α_k}), and clearly $\bar{z}_k \to \bar{\sigma}$. We need to see that z_k is in the ε-core of (N_k, v_{α_k}) for k large. If this were not so then for each k we would find a coalition $S_k \subset N_k$ such that

$$\frac{v_{\alpha_k}(S_k)}{|S_k|} > \frac{z_k(S_k)}{|S_k|} + \varepsilon.$$

Passing to a subsequence if necessary, we may assume that $\beta_t = \lim_{k\to\infty} |\alpha_k^{-1}(\omega_t) \cap S_k|/|S_k|$ exists for each t. Let B be any Borel subset of I such that $\mu(B\cap I_t)/\mu(B) = \beta_t$ for each t. Then

$$\frac{\lambda(B)}{\mu(B)} = \lim_{k\to\infty} \frac{v_{\alpha_k}(S_k)}{|S_k|}.$$

On the other hand, the equal-treatment nature of z_k and the fact that $\bar{z}_k \to \bar{\sigma}$ imply that

$$\frac{\sigma(B)}{\mu(B)} = \lim_{k\to\infty} \frac{z_k(S_k)}{|S_k|}.$$

Hence

$$\lambda(B) > \sigma(B) + \varepsilon\mu(B)$$

which contradicts the fact that σ is in the core of λ. We conclude that z_k is in the ε-core of (N_k, v_k) for k sufficiently large. Hence $\bar{C}(\lambda) \subset \bigcap_{\varepsilon>0} \liminf C_\varepsilon^k$.

We have now shown that

$$\bigcap_{\varepsilon>0} \limsup_{k\to\infty} C_\varepsilon^k \subset \bar{C}(\lambda) \subset \bigcap_{\varepsilon>0} \liminf_{k\to\infty} C_\varepsilon^k.$$

Since $\limsup C_\varepsilon^k \supset \liminf C_\varepsilon^k$ for every ε, it follows that

$$\bigcap \limsup C_\varepsilon^k = \bar{C}(\lambda) = \bigcap \liminf C_\varepsilon^k.$$

The first statement of the Theorem is an easy consequence of this fact. To see this fix $\delta_0 > 0$, $\varepsilon_0 > 0$, and let U be the δ_0-neighborhood of $\bar{C}(\lambda)$; i.e.,

$$U = \{\bar{y} \in \mathbb{R}^T : \operatorname{dist}(\bar{y}, \bar{C}(\lambda)) < \delta_0\}.$$

Since $\bar{C}(\lambda) = \bigcap \limsup C_\varepsilon^k$, we can find an ε_1, $0 < \varepsilon_1 < \varepsilon_0$, such that $U \supset \limsup C_{\varepsilon_1}^k$. Hence $U \supset C_{\varepsilon_1}^k$ for all sufficiently large k. Writing U_k for the δ_0-neighborhood of $C_{\varepsilon_1}^k$, we need to show that $\bar{C}(\lambda) \subset U_k$ for all sufficiently large k. If this were not so we could find a sequence of integers $\{k_n\}$ tending to infinity and a sequence $\{\bar{\sigma}_n\}$ in $\bar{C}(\lambda)$ such that $\operatorname{dist}(\bar{\sigma}_n, C_{\varepsilon_1}^{k_n}) \geqslant \delta_0$ for each n. Passing to a subsequence if necessary, we may assume that $\bar{\sigma}_n \to \bar{\sigma}$. But then $\operatorname{dist}(\bar{\sigma}, C_{\varepsilon_1}^{k_n}) \geqslant \frac{1}{2}\delta_0$ for k_n large, so that $\bar{\sigma}$ would be a point of $\bar{C}(\lambda)$ which was not in $\liminf C_{\varepsilon_1}^k$. This is a contradiction, so we must have $\bar{C}(\lambda) \subset U_k$ for all sufficiently large k. In other words

$$\operatorname{dist}(\bar{C}_{\varepsilon_1}^k, \bar{C}(\lambda)) < \delta_0$$

for all sufficiently large k, as desired.

The final assertion, about the Shapley value, is an immediate consequence of this distance estimate and Theorem 1, so the proof is complete. ∎

We remark that a similar argument may be used to show that, for every $\varepsilon > 0$, $\liminf C_\varepsilon^k = \limsup C_\varepsilon^k$; this set may be interpreted as representing the set of equal-treatment payoffs in the ε-core of λ. (Of course, this is not true if $\varepsilon = 0$, as the example in Section 4 shows.) We shall not go into the details here.

It is tempting to suppose that the (per-capita normalized) Shapley values of the games (N_k, v_{α_k}) converge to the asymptotic Shapley value of the non-atomic game λ. Unfortunately, λ need not have an asymptotic Shapley value (see the "three-handed glove" market in Aumann and Shapley [3], for instance). Perhaps such a result could be proved using the more general value of Mertens [12].

Note added in proof. Since this paper was accepted, we have obtained analogous results for NTU games, Wooders and Zame, "NTU values of large games," University of Toronto, Department of Economics Working Paper (forthcoming).

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