A Mini-course on Coalitions and Clubs; Part 2

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- We will now spend some time on games with a continuum of players.
- The following additional definition and result on SGE will be useful.
- In interpretation, the following definition characterizes games with many players as that can be approximated by games with a continuum of players, since in the continuum sets of measure zero have no impact.

Remark: A pregame (Ω, Ψ) satisfies *asymptotic negligibility* if, for any sequence of profiles $\{f^\nu\}$ where

$$\|f^\nu\| \to \infty \text{ as } \nu \to \infty,$$

$$\sigma(f^\nu) = \sigma(f^{\nu'}) \text{ for all } \nu \text{ and } \nu' \text{ and}$$

$$\lim_{\nu\to\infty} \frac{\Psi^*(f^\nu)}{\|f^\nu\|} \text{ exists,}$$

then for any sequence of profiles $\{\ell^\nu\}$ with

$$\lim_{\nu\to\infty} \frac{\|\ell^\nu\|}{\|f^\nu\|} = 0,$$

it holds that

$$\lim_{\nu\to\infty} \frac{\Psi^*\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} \text{ exists, and}$$

$$\lim_{\nu\to\infty} \frac{\Psi^*\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} = \lim_{\nu\to\infty} \frac{\Psi^*(f^\nu)}{\|f^\nu\|} \; .$$

Theorem 4 (Wooders (1992b, 2008b). A pregame (T, Ψ) satisfies SGE if and only if it satisfies PCB and asymptotic negligibility.

- Intuitively, asymptotic negligibility ensures that vanishingly small percentages of players have vanishingly small effects on aggregate per-capita worths.
- It may seem paradoxical that SGE, which highlights the importance of relatively small groups, is equivalent to asymptotic negligibility.
- To gain some intuition, however, think of a marriage model where only two person marriages are allowed. Obviously two-person groups are (strictly) effective, but also, in large player sets, no two persons can have a substantial affect on aggregate per-capita payoffs.

A continuum of players

Since Aumann (1964,1966), who treats economies with a continuum of players, numerous authors have adopted similar models. The idea underlying the continuum model is, as Aumann wrote:

> "a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants. We submit that the most natural model for this purpose contains a continuum of participants, similar to the continuum of points on a line or the continuum of particles in a fluid."

As Aumann notes, with a continuum of participants, an individual player is negligible and so is any countable set of players. Think about that for marriage models!

Kaneko & Wooders (1984,1996) introduce the framework of games with a continuum of players and finite coalitions. Hammond, Kaneko and Wooders (1989) demonstrate core-equilibrium equivalence with finite coalitions. Kaneko and Wooders (1989) show that the model with a continuum of players and finite coalitions is the limit of large, finite economies with relatively small coalitions. Allouch, Conley and Wooders (2009?) show an analogous result with local public goods and clubs. Winter and Wooders ( ) provide an axiomatization of the core with finite coalitions.

A challenge to the study of such models is that partitions of players into finite coalitions may not make sense – they may be inconsistent with relative scarcities. To make this point, we return to an earlier example but with a continuum of players.

Quote from von Neumann and Morgenstern

"It is neither certain nor probable that a mere increase in the number of participants will always lead *in fine* to the conditions of free competition. The classical definitions of free competition all involve further postulates besides the greatness of that number. E.g., it is clear that if great groups of participants will – for any reason whatsoever – act together, then the great number of participants may not become effective ; the decisive decisions will take place directly between large "coalitions," few in number, and not between individuals, many in number acting independently. ... Any satisfactory theory of the "limiting transition" from small numbers of participants to large numbers will have to explain under what circumstances such big coalitions will or will not become effective." (page 16).

Admissible club structures

- A feasible state of the economy must specify a partition of the set of players into clubs/coalitions that is "consistent" with the measure on the total consumer set.
- When the player set is an atomless continuum, some care must be taken in defining a partition.
  - For example, if there are more (a larger measure) of females than of males then consistency dictates that not all females can have a (male) dance partner.
- To retain the notion of relative scarcities, a partition of a continuum player set should be consistent with the measures (proportions) given by the measure on the total player set.
  - Thus, an admissible club structure is required to be a *measurement-consistent partition* in the sense of Kaneko and Wooders (1986).

Examples of consistency and inconsistency

Since Kaneko and Wooders (1986), variants of the following example have appeared in a number of papers.

Example Let $N = [0, 3)$. The players in $[0, 1)$ are girls and those in $[1, 3)$ are boys.

- A non-measurment consistent partition:

$$\pi = \{(i, j) : i \in [0, 1),\ j = 1 + 2i\};$$

  that is, girl i is partnered with boy $1 + 2i$.

- Measurement-consistency is not satisfied by the partition π.
- A measurement-consistent partition:

$$\pi' = \{(i, 1 + i) : i \in [0, 1)\} \cup \{\{i\} : i \in [2, 3)\}.$$

- Note: there are measurement-consistent partitions with "first members" of both genders.

Canadian census example

- In a Canadian census, for each positive integer k each k-person household was required to list the people in the household as first person, second person, ..., kth person.
- Consider any subset of the population consisting of 1^{st}-persons in 4-person households, for example, the set of 1first persons in 4-person households who reside in the cities of Toronto and Edmonton.
- A *number-consistency property* would be that the numbers of 2^{nd} persons, 3^{rd} persons and 4^{th} persons in these households would equal the number of 1^{st} persons. It is obvious.
- Measurement-consistency requires that, for each $k \in \mathbb{Z}_+$ the players in k-member households can be labelled in an analogous fashion (using measure-preserving isomorphisms) from the set of first members to the set of ℓth members for $\ell = 1, ..., k)$ and the measure of any measurable set of 1^{st} members equals the measure of members of ℓ^{th} members of the same clubs.

Games with a continuum of players and finite coalitions/groups.

The TU case

- Let (T, Ψ) be a pregame. Let N be a continuum of players with Lesbegue measure,
- $N = \cup N_t$ where each N_t is a measurable set.
- Let S be a finite subset of N – that is, S contains only a finite number of points and let s be the profile of S. Take the payoff to S as $\Psi(s)$.
- We now have a game with a continuum of players.

NTU games

- An *NTU game* (in coalitional function form) is a pair (N, V) where N is a finite set (the set of *players*) and V is a set-valued function that assigns to each nonempty subset S of N (a *group* or *coalition*) a nonempty subset $V(S)$ of $\mathbb{R}^S$, called a *payoff possibilities set* or simply a *payoff set*, with the following properties:
  - $V(S)$ is a closed subset of R^S, comprehensively generated by $V(S) \cap \mathrm{R}_+^S$;
  - $0 \in V(S)$;
  - $V(S) \cap \mathbb{R}_+^S$ is bounded.

NTU games

- A *payoff vector for a game* (N, V) is a vector x in $\mathbb{R}^N$. A payoff vector x is *feasible* for N if there exists a partition $\{S^k\}$ of N with the property that $x_{S^k} \in V(S^k)$ for each k. With this definition of feasibility, we say that the game (N, V) is *essentially superadditive*.
- Given $\varepsilon \geq 0$, a payoff vector x is in the *weak ε-core* of (N, V) if it is feasible and if there is a subset $N^0 \subset N$ such that $\frac{|N^0|}{|N|} \leq \varepsilon$ and, for every subset S of $N \setminus N^0$, $x_S + \varepsilon 1_S \notin int\ V(S)$.
- A payoff vector x *uniform ε-core* of a game (N, V) if it is feasible for N and if, for every subset S of N, $x_S + \varepsilon 1_S \notin int\ V(S)$.

NTU games

- (Wooders 1983) Two players $i, j \in N$ are *substitutes* if

1. For any $S \subset N$ such that $i, j \notin S$ if $x \in V(S \cup \{i\})$ then $x' \in V(S \cup \{j\})$ where x' is defined by $x'_j = x_i$ and $x'_\ell = x_\ell$ for all $\ell \in S$.
2. For any $S \subset N$ such that $i, j \in S$ if $x \in V(S)$ then $x' \in V(S)$ where x' is defined by $x'_j = x_i$, $x'_i = x_j$ and $x'_\ell = x_\ell$ for all $\ell \in S$, $\ell \neq i, j$.

- Let (N, V) be a game and let $x \in V(N)$. Then x has the *equal treatment property* if and only if, for all players i and j who are substitutes, it holds that $x_i = x_j$.

NTU pregames

1. Ω denotes a compact metric space (the space of attributes) with the distance function d.
2. $F(\Omega)$ denotes the set of all pairs (S, α), called populations, where S is a finite non-empty set (of players) and $\alpha : S \to \Omega$ is a function (an attribute function).
3. For ω in Ω, the set of players in S with attribute ω is $\alpha^{-1}(\omega)$ and $\left|\alpha^{-1}(\omega)\right|$ is the number of players in S with that attribute.

NTU pregames

- A (*NTU*) *pregame* is an ordered pair (Ω, ϕ) where Ω is a space of attributes and ϕ is a function (the worth function or payoff possibilities function) that associates to each population (S, α) in $F(\Omega)$ a subset $\phi(S, \alpha)$ of R^S, called a *payoff possibilities set* or simply a *payoff set*, that is
  - closed,
  - comprehensively generated by $\phi(S, \alpha) \cap \mathrm{R}^S_+$,
  - contains the origin, and
  - has bounded intersection with the positive orthant $\mathbb{R}^S_+$.
- In interpretation, $\phi(S, \alpha)$ represents the set of payoff vectors corresponding to all possible payoff vectors that the group of players S can achieve for its members, given that the attributes of the members of S are as described by the attribute function α.

Example: To illustrate a NTU pregame, we will convert the glove game (a matching game) into a NTU pregame. Let $\Omega = \{\omega_1, \omega_2\}$ denote a set of attributes, where ω_1 denotes the attribute "is endowed with a RH glove" and ω_2 denotes the attribute "is endowed with a LH glove". As previously, given a population (S, α), the attribute function α assigns one glove to each player. For a population (S, α) consisting of only one player we define

$$\phi(S, \alpha) = \{x \in \mathbb{R} : \ x \leq 0\}.$$

For a population (S, α) consisting of a pair of players with attributes ω_1, ω_2 we define

$$\phi(S, \alpha) = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \leq \frac{1}{2}, x_2 \leq \frac{1}{2}\}.$$

Using this data, we can construct the payoff possibilities for any population (S, α) by taking $\phi(S, \alpha)$ as the set of payoff possibilities achievable by the population (S, α) when it is partitioned into groups consisting of pairs of players where one player is endowed with a RH glove and the other player is endowed with a LH glove and one-player groups. We could also modify the example to assume that payoff sets for the pregame are given by the convex hulls of the payoff sets $\phi(S, \alpha)$. In this

We will require that a pregame satisfies superadditivity. Superadditivity for the NTU case is in interpretation the same as for the TU case and the discussion of essential superadditivity of TU games applies also to NTU games. The function ϕ is *superadditive* if for all (S, α), (T, β) in $F(\Omega)$ we have

$$\phi(S, \alpha) \times \phi(T, \beta) \subset \phi(S \vee T, \alpha \vee \beta).$$

This is superadditivity in the usual sense: the union of two disjoint groups can always obtain for itself anything that the groups could obtain separately.

Let (Ω, ϕ) be a pregame, let N be a finite set, and let α be an attribute function mapping N into Ω. The *derived game* (N, V_α) is the game with V_α defined by

$$V_\alpha(S) \stackrel{\text{def}}{=} \phi(S, \alpha_S)$$

for every nonempty subset S of N, where α_S denotes the restriction of α to S. From the definition of a pregame it follows that a pregame (Ω, ϕ) satisfies *substitution*, the property that, for all populations (S, α), whenever $\alpha(i) = \alpha(j)$ for two players i, j in S then i and j are substitutes.

Let (N, V_α) be a game derived from a pregame (Ω, ϕ). A payoff vector $x \in V_\alpha(N)$ has the *equal-treatment property* if

$$x_i = x_j \text{ whenever } \alpha(i) = \alpha(j).$$

In this section we introduce two concepts of small group effectiveness and show that they both imply nonemptiness of approximate cores.

Weak approximate core property: A NTU pregame (Ω, ϕ) has the *weak approximate core property* if: Given any real number $\varepsilon > 0$ there is an integer $n_1(\varepsilon)$ such that for all populations (S, α) with $|S| \geq n_1(\varepsilon)$, the weak ε-core of (S, V_α) is nonempty.

We actually will demonstrate that there exists equal-treatment payoff vectors in weak ε-cores of derived games with sufficiently many players.

An economic model with clubs

- Assume one private good and K public goods.
- Players are endowed only with the private good.
- An *endowment* is a function ω^0 from N to $\mathbb{R}_{++}$ such that for all i and j with the same taste type, $\omega^0(i) = \omega^0(j)$,
  - that is, players of the same taste type have the same endowment.
- Typically, we denote an amount of private good by $x \in \mathbb{R}_+$ and a bundle of public goods by $y \in \mathbb{R}_+^K$.

- Each player of type $(c, t) \in \mathcal{C} \times \mathcal{T}$ has consumption set

$$\mathcal{X} = \mathbb{R}^1_+ \times \mathbb{R}^K_+ \times \mathbb{Z}^{\mathcal{C}},$$

  where $\mathbb{Z}^{\mathcal{C}}$ is the set of crowding profiles $\mathcal{C}pro(S)$.
- For ease in notation and presentation (only), we do not restrict the consumption set of a player of type (c, t) to bundles with crowding profiles containing players of his crowding type. We will place restrictions on the model, however, so that in an equilibrium or core state of the economy, each player is assigned to some club containing players of his crowding type.

Example 6 (a continuum matching model).. Consider a matching model but the set of workers given by the points in the interval $[0, 1)$, and the machine owners are the points in the interval $[1, 3)$, endowed with Lebesgue measure μ .

- Take the worth of a player in a singleton coalition as equal to zero and the worth of a worker-machine pair (I, j) as $i + j$. Thus, we might think of the index number of a player as a measure of her productivity when she cooperates with another player of the opposite type.
- Because the set of players is now a continuum, the core is described by functions from the set of players to payoffs (real numbers).
- For this example, a measurement-consistent partition of the total player set into coalitions which yields a core payoff places the set of machines in $[1, 2)$ in singleton coalitions and matches worker $i \in [0, 1)$ with machine $j = 1 + i$.
- Because workers are relatively scarce, an payoff function in the core assigns worker i a payoff of $1 + 2$, a machine j in the interval $[2, 3)$ a payoff of $j - 2$. and machine indexed j in the interval $[1, 2)$ a payoff of 0.

- Kaneko & Wooders (1986) demonstrate that the f-core of a (general) game with finite types of players and a continuum of players of each type is nonempty.
- They require only a weak form of PCB (that equal-treatment payoffs are bounded in a neighborhood of the population proportions given by the measures on the sets of players of each type). The result builds on the lemmata in Wooders (1983).
- They also show that for an exchange economy, the f-core coincides with the set of competitive equilibrium payoffs and thus with the Aumann-core. Kaneko & Wooders (1994) shows that, with a compact metric space of player types, SSGE ensures that the f-core is nonempty.

Hammond et al. (1989) treat the f-core of an exchange economy with a continuum of participants and demonstrate that the equivalence of the f-core and the set of price-taking equilibrium outcomes continues to hold when there are widespread externalities and, in this case, the equivalence of the Aumann-core and the price-taking equilibrium outcomes *may fail* to hold. Thus, the f-core is more closely connected to the set of competitive outcomes than the Aumann core. This is not surprising as the continuum

CLUBS, FORMS OF CROWDING AND CROWDING TYPES

First, it is important to note that we use the term crowding in a different way than the term congestion as crowding effects may be positive – players may want to have many people in the same club. This section discusses models with more underlying structure. The models are all variants of situations in which individuals have utility functions $u_i(x, y, s)$ where x is a vector of private goods (often only one), y is a vector of public good(s) and s represents crowding in the individuals club or jurisdiction of membership.

The simplest versions of models studying the core in economies with clubs have anonymous crowding, that is, players are affected only by the numbers of players of in their chosen clubs. Pauly (1970) considered such a model in which players of two types are indifferent to the size of their club of membership as a game with side payments. (This is actually a result that follows from Pauly's assumption that the marginal contribution of a player to a coalition was independent of the type of the player.)

Wooders (1978, 1980) extends the anonymous crowding model to situations in which players of different types could have different preferences for both crowding and local public goods and preferences are described by utility functions $u_t(x, y, s)$ and in which all players of the same type have the same utility function and endowment. The variable s is simply the number of members belonging to the same club.

The next generation of models (beginning with Wooders 1989, studying economies with local public goods or clubs and specifying details of the economic situation underlying games generated by the economies), introduced differentiated crowding – individuals of different types could have different impacts on others (also known as nonanonymous crowding). For example, players may care about whether they are matched with a skilled worker or an unskilled worker. With anonymous crowding, players only care only about the number of other players in their club; thus, anonymous crowding cannot handle matching models, for example. With differentiated crowding the variable s in the utility function $u_t(x, y, s)$ is a vector listing the numbers of players of each type in the same club (see Allouch et al. 2009 for the most recent model treating the equivalence of the core and competitive outcomes and for further references to papers

Another version of the club economy model, introduced in Conley & Wooders (1997, 2001), regards a player as having two distinct attributes – his taste type and his crowding type. (See also Ellickson et al. 1999,2001, Allouch & Wooders 2008). The crowding type of a player determines his direct effects on others whereas his taste type is regarded as private information, which does not directly affect others. For example, a player may be directly affected by whether the other players in his club are skilled, and this may be observable. Whether the other players enjoy their work may not be observable. Or a player may be a great singer but prefer to listen to music. However, s in the utility function $u_t(x, y, s)$ is a list of the numbers of players of each *crowding type* in the same club.
Let us turn to another example.

Example 7. Consider a pregame (T, Ψ). Suppose that associated with each $t = 1, ..., T$ there is a pair (κ_ℓ, τ_m) where $\ell \in \{1, ..., L\}$ and $m \in \{1, ...M\}$. (Thus, $T \leq LM$.). Suppose that κ_t denotes crowding type and τ_m denotes taste type. Assume, for simplicity, that we are in a situation such as that described in Example 1 but workers with different taste types simply may get different amounts of (dis)utility from using a machine so that the total worth of a worker-machine pair is $1 - \frac{1}{m}$; thus, in interpretation, the higher the index number of the worker is, the less disutility the worker experiences from working. The utility of a singleton remains zero.

Consider a total player set with the number of machines smaller than the total number of workers. Assume all machines are identical, that there are fewer machines than workers and all types of workers appear in the population. Thus, for each $m \in \{1, ...M\}$ there is some worker with taste type m. Let $\overline{u}$ be in the core. Since all machines are identical, each machine must be assigned the same core payoff. Only those workers with highest payoff from working will be assigned a machine. The core may not be a singleton set, but let us suppose that $\overline{u}$ is the worst for workers (and there is such an element). Thus, the worker who likes working the

Competition between clubs for players determines competitive prices just as in private goods economies, competition between individual players for commodities determines prices. In a club model, if two players have the same crowding type, then they have the same admission price – they are equally valued by *other* members of a club and their tastes are irrelevant. It may be that individuals can belong to multiple clubs. Shubik & Wooders (1982,1999) introduces a model of economies with transferable utility but otherwise very broad and demonstrates that large replica economies generate games satisfying the conditions of Wooders (1979); thus results for games with many players can be applied. Ellickson et al. (1999,2001) also allow multiple memberships but, since they use techniques from finite-dimensional private goods exchange economies, they must restrict the number of kinds of clubs to be bounded and also restrict the club goods feasibly provided to be finite and uniformly bounded. Allouch & Wooders also allow multiple memberships but allow clubs to be as small as singletons or as large as the entire set of economic participants. To obtain the nonemptiness of approximate cores and their convergence to price-taking economic equilibrium in a model comparable to that of Shubik and Wooders appears to be an extremely challenging open problem.

Pricing and club admission rules

It has long been recognized that, in general, to ensure optimality of a price-taking equilibrium there must be universality of markets – that is, there must be a market for all components of utility functions and production functions. Moreover, budget constraints are determined by market prices. In a competitive equilibrium, these prices are taken as given.

In club economies, a variety of models have lead to a variety of price systems for club membership. To illustrate, in Conley & Wooders (1997,2001) models, there is only a finite set of types of individuals, and prices for admission to clubs are given for each possible club profile (a list of the number of players of each crowding type in the club) and level of public good. Because there is only one private good there is no need for private good prices. With an assumption bounding group sizes, Conley & Wooders demonstrate the equivalence of the core and the equilibrium outcomes. Cole & Prescott (1997) treat valuation equilibrium and allow lotteries over clubs to obtain existence. Ellickson et al. (1999,2001) also use a crowding-types model but allow multiple private goods. They limit their models to a finite set of kinds of clubs and thus can use techniques from private-goods exchange economies (Kakutani's Theorem). An example is be a marriage model where, even though they may have different endowments of private goods and different tastes, as partners all males are identical and all females are identical so club sizes are bounded by two. Allouch et al. (2009) also allow multiple private goods but require an assumption that is more similar to the SSGE. That is, all gains to group formation can be realized by groups bounded in size but larger

If clubs can offer a continuum of different levels of the club goods then, without further restrictions, a complete price system will have a continuum of admission prices, one for each level of club good for each crowding type; Allouch et.al. (2009) or Scotchmer & Wooders (1986) provide examples. As demonstrated in Conley & Wooders (1998), with further restrictions on the price system, such as Lindahl pricing within clubs, existence of equilibrium and core-equilibrium equivalence may be lost. If the set of possibilities open to the members of a club is restricted (as in Ellickson et al, 1999, 2001 and Allouch & Wooders, 2007, so, for example, so that a club provides only a given, specific amount of some club commodity or is purely hedonic, for example) then a finite dimensional price system may suffice.

THE CLASSICS: TIEBOUT AND BUCHANAN

With the above in place, we now return to two seminal papers, Tiebout (1956) and Buchanan (1965).

In his seminal paper, Tiebout (1956) observes that if public goods are subject to exclusion and congestion, the benefits of sharing costs of the provision of public goods over a large number of individuals will eventually be offset by the negative effects of crowding. When it is optimal or near optimal for there to be many jurisdictions providing local public goods, Tiebout conjectured that the movement of consumers to their preferred jurisdictions will lead to a market-type, near-optimal outcome and the free-rider problem of economies with pure public goods will not arise: the Tiebout Hypothesis. Although the general ideas of his paper were quite informally expressed (there were no precise definitions or conjectures) he did describe a severe comparison model that could easily be formalized. His severe model, Tiebout (1956, p. 441), supposes that there exists an infinite number of communities, each offering a different public-goods package (implicitly, so that all possible levels of public goods are provided). There is no congestion nor any increasing returns to scale within jurisdictions and the per capita costs of providing the public goods

Example (Tiebout's severe model). Suppose all levels of public goods are possible. We identify a level of the public good y with a community, so community $y \in \mathbb{R}_+$ offers quantity (or quality) y of the public good. Suppose that the costs of providing y to s consumers is \$$ys$ and that consumer who chooses community y must pay the cost (or tax) y. Let N denote a set of consumers, either a continuum or a finite set. Each consumer $i \in N$ has a quasi-linear utility function

$$u^i(y, \xi) = f^i(y) + \xi.$$

Also assume $f^i(y)$ is concave. Then consumer i faces the problem:

$$\text{maximize}_x \; f^i(y) + \xi - y.$$

Since $f^i(y) - y$ is concave, a maximum exists and, with free mobility, the utility maximizing consumer may move to exactly that community which maximizes his preferences (subject to his budget constraint).
Note that in this simple example the size of the population is of no real relevance. It could be finite or (with the technical measure-theoretic conditions) a continuum (with or without atoms – no real problem either way).

In the theory of competitive exchange economies, the most well-accepted test of the competitiveness of an economy may be the convergence of the core to the price-taking (or Walrasian) equilibria. Thus, one test of the Tiebout Hypothesis is suggested; the test of core convergence to price-taking equilibrium outcomes. The convergence of the core to price-taking equilibrium outcomes in economies with congestable public goods is now well-established (see Wooders 1978,1980, 1989,1997; Scotchmer & Wooders 1986), Ellickson et al. 2001, Allouch &Wooders 2008 and also, for the equivalence of the core and equilibrium outcomes with a continuum of players, Ellickson et al. 2001 and Allouch et al. 2009). The view of the Tiebout Hypothesis of this author, outlined in this paper, is broader than sometimes expressed by other authors. I believe that Tiebout's severe model, in which all actors in a jurisdiction have the same demands for the public goods and thus there is homogeneity of demands, is meant only to be illustrative and not to indicate that his theory predicts homogeneity of types of agents goods within jurisdictions (although it does predict homogeneity of demands for public goods by members of the same jurisdiction or club in some circumstances, Wooders 1978).

Buchanan (1965) takes another approach. He assumes that utility functions and production functions depend on both the amounts of public goods consumed and on the numbers of consumers of the goods. (This was also allowed by Tiebout but was not formalized.) Suppose that preferences are quasi-linear and can be written as

$$u^i((x_1, s_1), ...(x_K, s_K)) = h^i((x_1, s_1), ...(x_K, s_K), \zeta)$$

where ζ can be thought of as money, x_k is an amount of the kth club good, and s_k is the number of consumers of the kth good, $k = 1, ..., K$. In interpretation, x_k is the amount of the kth club good to be shared among the s_k consumers of the good. It is assumed that there is equal sharing so that each consumer of the kth good will consume x_k / s_k units of the good. Similarly, the cost function facing the individual, denoted by

$$F = F^i((x_1^i, s_1), ...(x_K, s_K))$$

has the same variables. Buchanan characterized optimal choices of individuals (if they were free to choose values for all variables x_k, s_k and ζ, in terms of derivatives.

A number of authors have noted that there is a problem with both Tiebout's and Buchanan's works of existence of an price-taking equilibrium. In fact, Bewley (1981) provides a formulation of Tiebout's *severe* model but with the condition of *free entry* replacing the assumption of an infinite number of communities all providing different public-goods packages. The free-entry condition is that, given prices for private goods, no subset of consumers could provide a preferred level of public goods for themselves. Bewley collects a number of examples from the literature demonstrating problems in defining an appropriate notion of equilibrium yielding both existence and optimality. Bewley reaches the same conclusion as Tiebout does for his severe model. Unless there are as many communities as types of consumers, an equilibrium may not exist. Moreover, as Tiebout does, Bewley concludes that this severe model does not make much sense. Unlike Tiebout and this article, Bewley did not consider large numbers of jurisdictions and/or a large population.