A Mini-course on Coalitions and Clubs; Part 1

For presentation at the 2014 Workshop on Game Theory at Sao Paulo

Myrna Wooders

Vanderbilt University

July 2014

Abstract This course primarily treats games with many players. The game-theoretic framework accommodates games derived from a variety of models of economies.

Introduction

- The classic notion of a competitive economy – each individual is motivated solely by the satisfaction of his own wants;
  - given his preferences and his wealth an individual takes prices for all commodities as given and maximizes his own satisfaction.
- This notion has been developed in great depth in general equilibrium theory but has been attacked from a multitude of directions, especially by non-cooperative game theorists and by behavioral economists.
- Noncooperative game theory assumes that individuals behave strategically and that, instead of taking prices as given, they take account of the effects of their purchases of commodities on prices.
- Behavioral economics entertains multiple assumptions contrasting with the self-regarding behavior assumed by Arrow, Debreu and McKenzie's (ADK) classic papers on competitive economies, including the assumption that individuals are affected by those with whom they interact.

- Nevertheless, the notion of a competitive economy is still of great importance; it is used widely in empirical analysis and also is a benchmark against which game-theoretic notions of equilibrium are measured.
- One way in which the ADK models fall short, even within the context of competitive theory, is that group effects are not allowed; individuals are not influenced by others with whom production and consumption are carried out.
- This also holds for economies with public goods; individuals may be influenced by the contributions of others to public good provision but this influence is indirect; an individual does not care about who contributes but is influenced only by how much others contribute to public good provision (cf., Kurz 1994 for a discussion).
- In essence, ADM and much of the subsequent literature on general equilibrium theory assumes that all that is valuable in an economy can be separated from the individuals in the economy. The acts of trade and consumption are *separate* from the individuals engaged in the actions.

- There are numerous important situations in which the individual is inseparable from her acts of consumption or trade; individuals interact in clubs.
- A club is typically taken as a subset of players whose members can collectively produce and/or consume a club good, modelled as a local public good with costless exclusion and with no spillovers of the public goods between clubs.
    - It could also be a firm, a hedonic coalition whose members get together simply to enjoy each other's company (see Dreze & Greenberg 1980), or simply a group of people who meet to trade stamps or other commodities. I
- A coalition – a group of individuals who act collectively – may consist of multiple clubs. If a club economy is modelled directly as a cooperative game, then a coalition is regarded as a club.

- An especially interesting example of clubs is education; the effects of classmates on the academic performance of students have been well studied.
- Another example is the sorting of individuals between firms. Examples that are not commonly studied in the literature but that may well serve to illustrate the sort of considerations that we have in mind include dinner parties, academic departments, and coffee shops.
- The related literature is vast.
- But some essential features of competitive club and coalition economies are already found and most easily made transparent in models in which consumers have quasi-linear utilities and in cooperative games with an idealized money (commonly known as games with transferable utility).
- Thus, I use the frameworks of economies and games with transferable utility to discuss the ideas and concepts at the heart of the analysis of club economies with many participants.

- The competitiveness of a club economy is addressed by whether the set of outcomes that are attainable by cooperation within groups of participants – the core– is nonempty and close to or equivalent to the set of outcomes that result from price-taking competitive equilibrium.
- This kind of approach to the competitiveness of a private-goods economy was initiated by Shubik (1959) for a game-theoretic model with two kinds of firms and by Debreu & Scarf (1963) and Aumann (1964) for exchange economies. I review literature taking this approach to competitive club and coalition economies.
- A competitive club economy must have many participants.
- Even with many participants, the competitiveness of a club economy can depend on the assumptions made to limit “increasing returns” to club size.

- I first provide some basic definitions from cooperative game theory.
- I then introduce a structure that allows us to treat games with many players, each of whom has one of a finite number of types.
- A variety of examples are developed, illustrating the breadth of the framework.
- But ever-increasing returns to larger coalitions must be limited in some way so I spend some time discussing minimal assumptions required for the competitiveness of a club/coalition economy with many participants.

Some Game Theoretic Definitions

- N – a finite set of participants in an economy (i.e., players).
- $S \subset N$ - a *group* or *coalitions*.
- v - a *worth function*, from subsets of N to $\mathbb{R}_+$ with $v(\phi) = 0$.
- $v(S)$ - the maximal total earnings achievable by the group S if the group members cooperate.
- (N, v) - a *TU game* (also called a game with side payments or transferable utility .

An example of a matching game

- Suppose that $N = N_1 \cup N_2$.
- $i \in N_1 \longrightarrow i$ is a worker
- $j \in N_2 \longrightarrow j$ is a machine (owner)
- A worker and a machine can produce one unit of output, worth \$1.
- A player alone can produce 0 so a single player is worth \$0.
- For $S \subset N$

$$v(S) = \min\{|S \cap N_1|, |S \cap N_2|\}.$$

- (N, v) is a game.
- Matching games are special case of much interest. See Gale & Shapley (1962), Shapley & Shubik (1972), Kelso & Crawford (1982), Roth & Sotomayor (1990) and, for extensions of the assignment game property to coalition structure games, Aumann & Dreze (1974) and Kaneko & Wooders (1982).

Back to games

- A *payoff vector* for (N, v) is a vector $\overline{u} = (\overline{u}_1, ..., \overline{u}_{|N|}) \in \mathbb{R}^N$.
- A payoff vector x is *feasible* if

$$\overline{u}(N) \stackrel{\text{def}}{=} \sum_{i \in N} \overline{u}^i \leq \sum v(S^k)$$

  for some partition $\{S^1, ..., S^K\}$ of N.
- We allow players to divide into clubs and for each club to achieve the gains from its collective activities.
- Thus, we assume that the games considered are superadditive and a payoff $\overline{u}$ is feasible if

$$\overline{u}(N) \leq v(N).$$

Epsilon-cores

- We take the fundamental concept governing the distribution of payoff as the ε-core.
- Given $\varepsilon \geq 0$, $\overline{u} \in \mathbb{R}^N$ is in the ε-*core* of (N, v) if it is feasible and if, for all $S \subset N$ it holds that:

$$\overline{u}(S) \stackrel{\text{def}}{=} \sum_{i \in S} \overline{u}^i \geq v(S) - \varepsilon \left[S \right].$$

- A payoff vector is in the *core* if it is in the ε-core for $\varepsilon = 0$.
- A game is totally ε-balanced iff the game (N, v) and every subgame (S, v) has a non-empty ε-core.

Remark: The concept of the core was introduced in Gillies (1959) and ε-core in Shapley & Shubik (1966).

- Some recent work includes Allouch &Wooders (2007) which introduces communication costs, parameterized by ε, into a general equilibrium model of a club economy and demonstrate that, given $\varepsilon > 0 = 0$, for economies with sufficiently many players ε-cores are nonempty.
- Lehrer & Scarsini (2012) provide a dynamic model with discounting that generates outcomes in ε-cores if the discount factor is sufficiently small.

Example: The epsilon-core

A simple majority game.

- $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 \text{ if } |S| = 1, \\ c \text{ otherwise;} \end{cases}$$

  where $c \geq 1$
- For $c < \frac{3}{2}$ the core of the game is empty.
- Given $\varepsilon > 0$, however, for ε sufficiently large, the ε-core is non-empty.
- For $c = 1$, the smallest such ε is $\varepsilon = \frac{1}{6}$; in this case, $\overline{u}' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is in the $\frac{1}{6}$-core.

Substitutes

- i, $j \in N$ are *substitutes* if, for all $S \subset N$ with $i\ j \notin S$ it holds that

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

- Given $\overline{u} \in \mathbb{R}^N$, if $\overline{u}_i = \overline{u}_j$ for all substitutes i and j then $\overline{u}$ has the *equal-treatment property.*
- Note that if there is a partition of N into T subsets, e.g., N_1, ..., N_T, where all players in each subset N_t are substitutes for each other, then we can *represent* $\overline{u}$ by a vector $w \in \mathbb{R}^T$ where, for each t, it holds that $w_t = \overline{u}_i$ for all $i \in N_t$.
- It is easy to verify that if $\overline{u}$ is a feasible payoff for the game then $w \in \mathbb{Z}_+^T$ is also feasible, where

$$w_t \stackrel{\text{def}}{=} \sum_{i \in N_t} w_i$$

for each $t = 1, ..., T$.

Equal treatment and an example

- For the simple majority game a core payoff must have the equal-treatment property.

- For the matching game, for each matched worker-machine pair (i, j) it must hold that $\overline{u}_i + \overline{u}_j = 1$.
- Suppose worker i, matched with machine j gets a higher payoff than worker i', matched with machine j'.
- It follows that $\overline{u}_{i'} + \overline{u}_j < 1$ and (i', j) could each get a higher payoff by being matched to each other.
- Thus, all matched workers much get the same payoff and similarly for all machines.

Remark: Equal treatment property of the core holds for games with many players where there are many close substitutes for each player, cf., Kovalenkov and Wooders (2001). For ε-cores some players may be treated quite unequally.

- For $\varepsilon > 0$, a payoff in the ε-core need not have the equal-treatment property.
- To illustrate, consider again Example 1 with $c = \frac{3}{2}$.
- The payoff vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the unique payoff vector in the core.
- Now let ε be a real number between 0 and $\frac{1}{2}$.
- Create a new payoff vector $\overline{u}'$ by "taxing" all players except one the amount $\frac{\varepsilon}{2}$ and giving all the tax revenue to one player. The payoff vector $\overline{u}'$ is in the ε-core.

Games with many but finite numbers of players

- T – a finite number of player types where all players of the same type are substitutes for each other.
- A *profile* $s = (s_1, \cdots, s_T) \in \mathbb{Z}^T_+$ describes a group of players by the numbers of players of each type in the group.
- Example: Three types, skilled workers, unskilled workers, and managers. The profile $(2, 3, 1)$ describes two skilled workers, three unskilled workers and one manager)
- Given profile s, define the *norm* or *size* of s by

$$\|s\| \stackrel{\text{def}}{=} \sum_t s_t.$$

- A *subprofile of a profile* $n \in \mathbb{Z}^T_+$ is a profile s satisfying $s \leq n$.
- A *partition of a profile* s is a collection of subprofiles $\{s^k\}$ of n, not all necessarily distinct, satisfying

$$\sum_k s^k = s.$$

Pregames

- $\Psi : \mathbb{Z}_+^T \to \mathbb{R}_+$ with $\Psi(0) = 0$.
- $\Psi(s)$ is interpreted as the total payoff a group of players with profile s can achieve from collective activities of the group membership and is called the *worth of the profile* s.
- The pair (T, Ψ) is a *pregame*.
- Given Ψ, define a worth function Ψ^*, called the *superadditive cover* of Ψ, by

$$\Psi^*(s) \stackrel{\text{def}}{=} \max \sum_k \Psi(s^k)$$

  where the maximum is taken over the set of all partitions $\{s^k\}$ of s.
- Ψ is *superadditive* if $\Psi = \Psi^*$.
- We assume that the worth function Ψ is superadditive, which implies that the games generated by a pregame are superadditive.

- A *pregame* is a pair (T, Ψ) where $\Psi : \mathbb{Z}_+^T \to \mathbb{R}_+$.
- To generate a game from a pregame, we specify a total player set N and the number of players of each of T types in the set, say n_t players of type t, $t = 1, ..., T$.
- A *game determined by the pregame* (T, Ψ), called a *game* or a *game with side payments*, is a pair $[n; (T, \Psi)]$ where n is a given profile, the *population*.
- A *subgame* of a game $[n; (T, \Psi)]$ is a pair $[s; (T, \Psi)]$ where s is a subprofile of n.

Induced games

- With any game $[n; (T, \Psi)]$ we can associate a game (N, ν).
- Let
$$N = \{(t, q) : t = 1, \cdots, T \text{ and } q = 1, \cdots, n_t\}$$
be a *player set* for the game.
- For each $S \subset N$, define the *profile of* S, denoted by *prof(S)*$\in \mathbb{Z}_+^T$, by its components
$$prof(S)_t \overset{\text{def}}{=} \left|\{S \cap \{(t', q) : t' = t \text{ and } q = 1, \cdots, n_t\}\right|$$
and define
$$v(S) \overset{\text{def}}{=} \Psi(prof(S)).$$
- Then the pair (N, ν) satisfies the usual definition of a game with side payments.

- Varying assumptions limiting returns to club size and composition have appeared in the literature. We describe three, going from the weakest to the strongest.

An apparently very mild assumption on a pregame (T, Ψ) is that per-capita payoffs are uniformly bounded.

- **Per-capita boundedness, PCB:** A pregame (T, Ψ) satisfies PCB if there is a constant K such that for all profiles $n \in \mathbb{Z}_+^T$ it holds that

$$\frac{\Psi(n)}{\|n\|} \leq K.$$

Remark: Wooders (1979) identifies PCB as implying nonemptiness of approximate cores. The more general NTU case (in which payoff can not necessarily be transferred at a one-to-one rate) of Wooders (1983) uses boundedness of the set of equal-treatment payoffs to show nonemptiness of approximate cores. Both papers require thickness.

- SGE, introduced in Wooders (1992,1994a), dictates that, given $\varepsilon > 0$ there is a bound $\eta_0(\varepsilon)$ on group size so that almost all gains (within ε per capita) to collective activities can be realized by cooperation only within groups bounded in size by $\eta_0(\varepsilon)$.

Small group effectiveness, SGE: A pregame (T, Ψ) satisfies SGE if, given $\varepsilon > 0$, there is an integer $\eta_0(\varepsilon)$ such that for every profile n there is a partition $\{n^k\}$ of n satisfying:

$$\|n^k\| \leq \eta_0(\varepsilon) \text{ for each subprofile } n^k, \text{ and}$$

$$\Psi(n) - \Sigma_k \Psi(n^k) \leq \varepsilon\|n\|.$$

- Consider a pregame with two types $n = (n_1 > 0,\ n_2 > 0)$.
- $\Psi(s) = \|s\|$ if $s_1 > 0,\ s_2 > 0$.
- Otherwise $\Psi(n) = 0$.
- If we rule out games with arbitrarily small, positive percentages of players of some types (that is, if we rule out scarce types), then PCB and SGE are equivalent! (Wooders 1994a, Theorem 4).
- This is a result of much interest as it relates to determining when groups of players might have market power over outcomes.

Relating PCB and SGE.

- Given a real number $\rho \in (0,1)$, the *ρ-thick restriction of* (Ω, Ψ) is the pregame (Ω, Ψ_ρ) with admissible profiles f required to satisfy the condition that for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$.
- Note that a sequence of profiles derived from the ρ-thick restriction of (Ω, Ψ) does not allow vanishingly small but positive percentages of players of any type.

MW 1994, Econometrica, Theorem 4. *With 'thickness,' SGE=PCB.*

1. Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.
2. Let (T, Ψ) be a pregame satisfying PCB. Then given any $\rho > 0$ construct a new pregame (T, Ψ_ρ) with the domain of Ψ_ρ is restricted to profiles f where, for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$. Then (T, Ψ_ρ) satisfies SGE on its domain.

- Many studies of clubs, going back to Buchanan (1965), require that group sizes are strictly bounded.

Strict small group effectiveness, SSGE: (T, Ψ) satisfies *SSGE* if there is an integer η_1 such that for every profile n there is a partition $\{n^k\}$ of n satisfying:

$$\|n^k\| \leq \eta_1 \text{ for each subprofile } n^k, \text{ and}$$

$$\Psi(n) - \Sigma_k \Psi(n^k) = 0.$$

- Note that this definition of SSGE does not rule out the possibility of large clubs.
- SSGE is satisfied by our first two examples but it may not be satisfied by exchange economies or economies with hedonic coalitions with ever-increasing returns to group size, as in the next example.
- Note also that SSGE implies SGE. It is easy to develop examples that satisfy SGE but not SSGE. For example, take $T = 1$ and define $\Psi(n) = n - \frac{1.}{n}$.

Preview

- The main ideas rest on three properties:
  - (a) SGE,
  - (b) substitution (In games with many players most players have many substitutes, and
  - (c) superadditivity **(**An option open to a group of players is to divide into smaller groups and realize the sum of worths achievable with collective activities only within the smaller groups).
- Many kinds of economies generate games satisfying these conditions. The following example illustrates that economies with quasi-linear utilities generate cooperative games. The example differs from others I provide in that the optimal club consists of all consumers in the economy.

Limiting gains to coalition formation

Example: A pure public goods economy

- All consumers are identical with utility function:

$$u(x, y) = x - e^{-y}$$

  where x is private good and y is public good.
- Each consumer has an endowment $\omega > 0$ of private good.
- The production function:

$$y = bz$$

  where z is the input of private good and b is a positive constant.
- $N = \{1, ..., n\}$ denote a finite number of consumers.
- An *allocation* is a vector $(x^1, ..., x^n; y)$.
- The allocation is Pareto optimal if and only if

$$y = \ln(bn)$$

  and

$$\sum_{i=1}^{n} x^i = n\omega - \frac{\ln bn}{b}.$$

- We describe the economy as a cooperative game.
- Given any positive integer $s \leq n$ define

$$\Gamma(s) = \sum_{i=1}^{s}(x^i - e^{-y}) \text{ where}$$
$$y = \ln(bs)$$
$$\sum_{i=1}^{s} x^i = sw - \frac{\ln(bs)}{b}$$
$$x^i \geq 0 \text{ for all } i = 1, ..., s.$$

- $$\Gamma(s) = \omega s - \frac{\ln(bs)}{b} - \frac{1}{b}.$$
- Define $\Gamma(0) = 0$.
- Then (N, v_n) is a cooperative game where, for any $S \subset N$ with $|S| = s$, $v(S) = \Gamma(s)$.
- The game is *anonymous.*
- We can generate an entire family of games by varying n.
- Note: As n grows large, the average Pareto optimal utility, given by $\frac{\Gamma(n)}{n} = \omega - \frac{\ln(bn)}{nb} - \frac{1}{bn}$ converges to ω .

A market game example

- Take as given two continuous, concave utility functions u_1 and u_2 and two endowment vectors e^1 and e^2, both in $\mathbb{R}^L_+$.
- Type 1 players have utility function u_1 and endowment e^1 and similarly for type 2 players.
- Given a profile $f \in \mathbb{Z}^2_+$ define

$$\Psi(f) = \max_{x,y} \sum (f_1 u_1(x) + f_2 u_1(y))$$
$$\text{subject to the conditions that}$$
$$f_1 x + f_2 y = f_1 e^1 + f_2 e^2$$
$$x, y \geq 0.$$

- Thus, a pregame is derived from the pre-economy.

Discussion

- In the context of club economies modelled as cooperative games, if there are many players of each type that appears in the population then PCB suffices to obtain the nonemptiness of approximate cores and convergence of approximate cores to equal-treatment outcomes. The distinction between PCB and SGE has consequences in models with scarce types, as I will illustrated further.
- In the literature on clubs, having only one private good and club structures satisfying SSGE greatly simplifies the analysis. See Wooders (1978,1980) and Scotchmer & Wooders (1987).
- In some general equilibrium models of club economies with multiple private goods, it is assumed that there are only a finite number of kinds of clubs and club sizes are uniformly bounded, Cole & Prescott (1997), Wooders (1997), Ellickson et al. (1999,2001). This enables the authors to apply techniques from the analysis of general equilibrium models of exchange economies – the list of commodities is expanded from only private goods to include club memberships. If prices for private goods were taken as given or if there were only one

- But there may be ever-increasing returns to group size. Think of the membership of Facebook or religions that seek to embrace all people. If we allow ever-increasing returns to group size then the situation becomes even more complex and other techniques need to be employed.
- A first approach to general equilibrium in such settings was Wooders (1989), which required only PCB. Allouch & Wooders (2007) allow ever-increasing returns to club size and demonstrate limiting core-equilibrium equivalence for large finite economies with overlapping clubs and multiple private goods while Allouch et al.(2009), in a setting with a continuum of players, allow unbounded finite club sizes but require that clubs uniformly bounded in size can realize all gains to club formation.
- Which assumption to make limiting returns to group size depends on the sort of phenomena being investigated. For example, to study whether scarce types can have significant impacts, then whether SGE is satisfied appears to be at the heart of the issues.
- **End of Lecture 1.**

Direct markets and market-game equivalence

- Shapley and Shubik (1969) introduced the notion of a direct market derived from a totally balanced game.
- In the direct market, each player is endowed with one unit of a commodity and all players in the economy have the same utility function.
  - In interpretation, a labor market or as a market for productive factors, (as in Owen 1975, for example) where each player owns one unit of a commodity.
- For pregames, we take the player types of the game as the commodity types of a market and assign all players in the market the same utility function, derived from the worth function of the game.

- Let (T, Ψ) be a pregame and let $[n; (T, \Psi)]$ be a derived game.
- Let $N = \{(t, q) : t = 1, ..., T \text{ and } q = 1, ..., n_t \text{ for each } t\}$ denote the player set in the induced game where all participants $\{(t', q) : q = 1, ..., n_{t'}\}$ are of type t' for each $t' = 1, ..., T$.
- Take the commodity space as $\mathbb{R}^T_+$.
- Suppose that each participant in the market of type t is endowed with one unit of the t^{th} commodity, and thus has endowment $\mathbf{1}_t = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^T_+$ where "1" is in the t^{th} position.
- The total endowment of the economy is then given by $\sum n_t \mathbf{1}_t = n$.

- For any vector $y \in \mathbb{R}^T_+$ define

$$u(y) \stackrel{\text{def}}{=} \max \sum_{s \le n} \gamma_s \Psi(s),$$

the maximum running over all $\{\gamma_s \ge 0 : s \in \mathbf{Z}^T_+,\ s \le n\}$ satisfying

$$\sum_{s \le n} \gamma_s s = y.$$

- It can be verified that the function u is concave and one-homogeneous. This does not depend on the balancedness of the game $[n; (T, \Psi)]$. Indeed, one may think of u as the "balanced cover of $[n; (T, \Psi)]$ extended to $\mathbb{R}^T_+$.
- If Ψ were not necessarily superadditive and Ψ^* is the superadditive cover of Ψ then it holds that

$$\max \sum_{s \le n} \gamma_s \Psi(s) = \max \sum_{s \le n} \gamma_s \Psi^*(s).$$

- Taking the utility function u as the utility function of each player $(t, q) \in N$ where N is now interpreted as the set of participants in a market, we have generated a market, called the *direct market*, denoted by $[n, u; (T, \Psi)]$, from the game $[n; (T, \Psi)]$.

Shapley-Shubik market games = totally balanced games

- Again, the following extends a result of Shapley and Shubik (1969) to pregames.

Theorem. Let $[n, u; (T, \Psi)]$ denote the direct market generated by a game $[n; (T, \Psi)]$ and let $[n; (T, u)]$ denote the game derived from the direct market. Then, if $[n; (T, \Psi)]$ is a totally balanced game, it holds that $[n; (T, u)]$ and $[n; (T, \Psi)]$ are identical.

Remark. If the game $[n; (T, \Psi)]$ and every subgame $[s, (T, \Psi)]$ has a nonempty core – that is, if the game is 'totally balanced'– then the game $[n; (T, u)]$ generated by the direct market is the initially given game $[n; (T, \Psi)]$. If however the game $[n; (T, \Psi)]$ is not totally balanced then $u(s) \geq \Psi(s)$ for all profiles $s \leq n$. But, whether or not $[n; (T, \Psi)]$ is totally balanced, the game $[n; (T, u)]$ is totally balanced and coincides with the totally balanced cover of $[n; (T, \Psi)]$.

Remark. Another approach to the equivalence of markets and games is taken by Garratt and Qin (1997), who define a class of direct lottery

Equivalence of markets and games with many players

- The requirement of Shapley and Shubik (1969) that utility functions be concave is restrictive. It rules out, for example situations such as economies with indivisible commodities.
- It also rules out club economies; for a given club structure of the set of players – in the simplest case, a partition of the total player set into groups where collective activities only occur within these groups – it may be that utility functions are concave over the set of alternatives available within each club, but utility functions need not be concave over all possible club structures.
- For example, take utility functions

$$u(x, y, s) = x - e^{-y} - c \|s\| .$$

where c is a constant and $\|s\|$ denotes the number of players in the same club in a partition of the set of players into clubs.

Generating a limiting market utility function from a pregame

Let (T, Ψ) satisfy SGE. For each x in $\mathbb{R}_+^T$ define

$$U(x) \stackrel{\text{def}}{=} \|x\| \lim_{\nu\to\infty} \frac{\Psi^*(s^\nu)}{\|s^\nu\|}$$

where the sequence $\{s^\nu\}$ satisfies

$$\lim_{\nu\to\infty} \frac{s^\nu}{\|s^\nu\|} = \frac{x}{\|x\|}$$
and
$$\|s^\nu\| \to \infty.$$

Proposition (MW 1988, 1994, Lemma 2). Assume SGE holds. Then for any $x \in \mathbb{R}_+^T$ the limit above exists and $U(\cdot)$ is a well-defined concave, 1-homogeneous function.

Example of a limiting utility function

- **The pregame:** There are two types - cooks and helpers. Suppose a banquet is worth 10 dollars and unemployment insurance is worth 1 dollar.

(a) 1 cook and 2 helpers can make a banquet; $\Psi(1, 2) = 10$
(b) 4 cooks alone can make a banquet (cooks are not very efficient as helpers); $\Psi(4, 0) = 10$
(c) A helper can find his way to the unemployment insurance office and collect unemployment benefits. $\Psi(0, 1) = 10$
(d) All other groups in the kitchen are useless: $\Psi(m_1, m_2) = 0$ otherwise.

The utility function: $u(x, y) = \begin{bmatrix} \frac{5x}{2} + \frac{15y}{4} & \text{if } 2x \geqslant y \\ 8x + y & \text{if } 2x < y \end{bmatrix}$

Defining a utility function with PCB?

Example. (Wooders 2008a) Let $T = 2$ and let (T, Ψ) be the pregame given by

$$\Psi(s_1, s_2) = \begin{cases} s_1 + s_2 & \text{when } s_1 > 0 \\ 0 & \text{otherwise} \end{cases} .$$

- Ψ obviously satisfies PCB.
- There is a problem in defining $\lim \frac{\Psi(s_1, s_2)}{s_1 + s_2}$ as $s_1 + s_2$ tends to infinity.
- Consider the sequence (s_1^ν, s_2^ν) where $(s_1^\nu, s_2^\nu) = (0, \nu)$; then $\lim \frac{\Psi(s_1^\nu, s_2^\nu)}{s_1^\nu + s_2^\nu} = 0$.
- But consider $(s_1^\nu, s_2^\nu) = (1, \nu)$; then $\lim \frac{\Psi(s_1^\nu, s_2^\nu)}{s_1^\nu + s_2^\nu} = 1$.
- This illustrates why, to obtain the result that games with many players are market games either it must be required that there are no scarce types or some assumption limiting the effects of scarce types must be made.

With only PCB uniform approximate cores of games with many players may be empty

Example.. (Wooders 2008a) Let $T = \{1, 2\}$ and let Ψ is the superadditive cover of the function Ψ' defined by:

$$\Psi'(s) \stackrel{\text{def}}{=} \begin{cases} |s| & \text{if } s_1 = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if a profile $s = (s_1, s_2)$ has $s_1 = 2$ then the worth of the profile according to Ψ' is equal to the total number of players it represents, $s_1 + s_2$, while all other profiles s have worth of zero. In the superadditive cover game the worth of a profile s is 0 if $s_1 < 2$ and otherwise is equal to s_2 plus the largest even number less than or equal to s_1.

- Consider a sequence of profiles $(s^\nu)_\nu$ where $s_1^\nu = 3$ and $s_2^\nu = \nu$ for all ν. Given $\varepsilon > 0$, for all sufficiently large player sets the (uniform) ε-core is empty. Take, for example, $\varepsilon = 1/4$.
- To obtain a contradiction, suppose that $u^\nu = (u_1^\nu, u_2^\nu)$ represents an equal treatment payoff vector in the uniform ε-core of $[s^\nu; (T, \Psi)]$. The following inequalities must hold:

$$\begin{gathered} 3u_1^\nu + \nu u_2^\nu \leq \nu + 3, \\ 2u_1^\nu + \nu u_2^\nu \geq \nu + 3, \text{ and} \\ u_1^\nu \geq \tfrac{3}{4}. \end{gathered}$$

  which is impossible.
- A payoff vector which assigns each player zero is, however, in the weak ε-core for any $\varepsilon > \frac{3}{\nu+3}$. But it is not very appealing, in situations such as this, to ignore a relatively small group of players (in this case, the players of type 1) who can have a large effect on per capita payoffs. ■

Derivation of markets from pregames satisfying SGE

Theorem 5. (Wooders 1988; 1994). Assume the pregame (T, Ψ) satisfies SGE. Then for any $x \in \mathbb{R}_+^T$ the limit exists. Moreover, $U(\cdot)$ is well-defined, concave and 1-homogeneous and the convergence is uniform in the sense that, given $\varepsilon > 0$ there is an integer η such that for *all profiles* s with $\|s\| \leq \eta$ it holds that

$$\left| U\left(\frac{s}{\|s\|}\right) - \frac{\Psi^*(s)}{\|s\|} \right| \leq \varepsilon.$$

- From Wooders (1994, Theorem 4), if arbitrarily small percentages of players of any type that appears in games generated by the pregame are ruled out, then the above result holds under per capita boundedness (Wooders 1994, Theorem 6).

Theorem 5 follows from the facts that the function U is superadditive and 1-homogeneous on its domain. Since U is concave, it is continuous on the interior of its domain; this follows from PCB. Small group effectiveness ensures that the function U is continuous on its entire domain (Wooders 1994, Lemma 2).

Theorem 6. (Wooders 1994) Let (T, Ψ) be a pregame satisfying small group effectiveness and let (T, U) denote the derived direct market pregame. Then (T, U) is a totally balanced market game. Moreover, U is one-homogeneous, that is, $U(\lambda x) = \lambda U(x)$ for any non-negative real number λ.

In interpretation, T denotes a number of types of players/commodities and U denotes a utility function on $\mathbb{R}_+^T$. Observe that when U is restricted to profiles (in $\mathbf{Z}_+^T$), the pair (T, U) is a pregame with the property that every game $[n; (T, U)]$ has a nonempty core; thus, we will call (T, U) the *premarket generated by the pregame* (T, Ψ). That every game derived from (T, U) has a nonempty core is a consequence of the Shapley and Shubik (1969) result that market games derived from markets with concave utility functions are totally balanced.

It is interesting to note that, as discussed in Wooders (1994, Section 6), if we restrict the number of commodities to equal the number of player types, then the utility function U is *uniquely* determined. (If one allowed more commodities then one would effectively have 'redundant assets'.) In contrast, for games and markets of fixed, finite size, as demonstrated in Shapley and Shubik (1975), even if we restrict the number of commodities to equal the number of player types, given any nonempty, compact, convex subset of payoff vectors in the core, it is possible to construct utility functions so that this subset coincides with the set of competitive payoffs. Thus, in the Shapley and Shubik approach, equivalence of the core and the set of price-taking competitive outcomes for the direct market is only an artifact of the method used there of constructing utility functions from the data of a game and is quite distinct from the equivalence of the core and the set of competitive payoff vectors as it is usually understood (that is, in the sense of Debreu and Scarf 1963 and Aumann 1964). See also Kalai and Zemel (1982a,b) which characterize the core in multi-commodity flow games.

Cores and approximate cores

- As illustrated above, ε-cores of games with many players may be empty.
- Under SGE, however, given any $\varepsilon > 0$ all games with sufficiently many players have nonempty ε-cores.
- The intuition for the simplest case is clear.

- Suppose any two players can earn 1 and all other sized clubs can earn 0.
- Take the superadditive cover, so that the worth of n players is $\frac{[n]}{2}$ where $[n]$ is the largest even integer less than or equal to n
- If n is an even integer then the core is nonempty.
- If n is odd, one payoff vector in the ε-core is given by $\overline{u}_i = \frac{1}{2} - \frac{\varepsilon}{2}$ for each player except $i = n$ and the n^{th} player is assigned the payoff $(n-1)\frac{\varepsilon}{2}$.
- For n sufficiently large, the ε-core is nonempty.

Theorem 1 (nonemptiness of approximate cores of many-player games). Let (T, Ψ) be a pregame satisfying SGE. Then:

> Given any positive real number $\varepsilon > 0$ there is a positive real number $\nu(\varepsilon)$ such that, for any induced game (N, v), if $|N| > \nu(\varepsilon)$ then the game has a nonempty ε-core.

Remark: This result was first proven under PCB for sequences of games with a fixed distribution of player types in early working papers due to this author and then for such games without side payments in Wooders (1983) but still with a fixed distribution of player types. Wooders & Zame (1984) relaxes the restriction to fixed distributions of players types but requires the assumption of *boundedness of marginal contributions to coalitions*; this condition is more restrictive than SGE. The Wooders-Zame condition bounds marginals whereas SGE bounds average contributions. Wooders (1992) demonstrates the result above for games with a compact metric space of player types and obtains an analogous result for nontransferable utility games in Wooders (2008). Kovalenkov & Wooders (2003), by extending lemmata from Wooders (1983), obtains an analogous result for situations satisfying a form of SGE for arbitrary games. Kaneko & Wooders (1986,1996) provide versions with a continuum of players. Many of the ideas exposited can be applied to modified notions of the core, c.f., Furth (1998), Askoura (2011).

- We first present the a variation, from Wooders (1983), of the well-known (c.f., Shapley 1967, Owen, 1975) linear programming characterization of the core but for a game derived from a pregame. Let (T, Ψ) be a pregame and let $[n, (T, \Psi)]$ be a game derived from the pregame.
- A game with side payments has a nonempty core if and only if it has an equal-treatment payoff in its core so we will consider only equal-treatment payoffs $z \in \mathbb{R}^T$ where z_t is the payoff to each player of type t.

- The game $[n, (T, \Psi)]$ has a nonempty core if and only if the linear program problem

$$\text{minimize} \sum_{t=1}^{T} n_t z_t = c$$
$$\text{subject to} \sum s_t z_t \geq \Psi(s) \text{ for all profiles } s \leq n$$

has a minimum $c^* \leq \Psi(n)$.

- If the core is nonempty, any minimizing z^* that satisfies $\sum_{t=1}^{T} n_t z_t^* = \Psi(n)$ represents an equal-treatment payoff in the core of the game.

- Consider the dual program:

$$\text{maximize} \sum_{\substack{s \in \mathbb{Z}_+^T \\ s \leq n}} \omega_s \Psi(s) = q,$$

$$\text{subject to } \sum_{\substack{s \in \mathbb{Z}_+^T \\ s \leq n}} \omega_s s = n \text{ and}$$

$$\omega_s \geq 0 \text{ for all profiles } s \leq n..$$

- Denote the maximal value of q by q^* and let

$$\{\omega_s^* \geq 0 : s \in \mathbb{Z}_+^T,\ s \leq n\}$$

denote a set of weights satisfying

$$\sum_{s \in \mathbb{Z}_+^T,\ s \leq n} \omega_s^* \Psi(s) = q^*$$

From the Fundamental Theorem of Linear Programming, it follows that the equal-treatment core, and hence the core, of the game is nonempty if and only if $c^* = q^*$. Thus, the core is nonempty if and only if $q^* \leq \Psi(n)$.

- Recall the definition of SSGE:

Strict small group effectiveness, SSGE: (T, Ψ) satisfies *SSGE* if there is an integer η_1 such that for every profile n there is a partition $\{n^k\}$ of n satisfying:

$$\|n^k\| \leq \eta_1 \text{ for each subprofile } n^k, \text{ and}$$

$$\Psi(n) - \Sigma_k \Psi(n^k) \leq \varepsilon\|n\|.$$

- Start with a given player profile n with, for all t, either $n_t = 0$ or $n_t > \eta_1$, given in the definition of SSGE.
- Since the set of weights" $\{\omega_s \geq 0\colon\ s \in \mathbb{Z}^T_+,\ s \leq n\}$ satisfying the constraint $\sum \omega_s s = n$ is a finitely generated compact polyhedron,

  generated by constraints all in terms of integers, the set components of the finite set of of points generating the polyhedron are all rational.
- Thus, there is an integer m_0 such that for every generating point ω_s of the polyhedron, $m_0\omega_s$ is an integer. (Lemma 5 of Wooders 1983, showing that when we replicate a game and its player set an appropriate number of times, then the core of the replicated game is nonempty.)
- This implies that, for the weights w^*_s we can partition the population $m_0 n$ into profiles so that there are $m_0\omega^*_s$ profiles s in the the partition.
- From superadditivity, the definition of SSGE and the Fundamental Theorem of Linear Programming, it holds that for any positive integer r,

$$\sum_{i=1}^{T} rm_0\omega^*_s\Psi(s) \quad =$$

- Thus, if we replicate the player set rm_0 times the value of the maximum is multiplied by rm_0.
- We can create a partition of the profile $rm_0 n$ into subprofiles where, if $\omega_s^* > 0$ there are $r(m_0\omega_s^*)$ profiles s in the collection.
- Now let r be any large integer. Observe that we can write $rn = r'm_0 n + \ell$ where $\frac{\|\ell\|}{\|rn\|}$ is small. Then, given a payoff vector in the $\frac{\varepsilon}{2}$-core of the game $[r'm_0 n, (T, \Psi)]$ we can tax the players in the subgame by some small amount, distribute the taxes to the left-over player (with profile ℓ) and create a payoff vector in the ε-core of the game $[rn, (T, \Psi)]$. Also observe that every large profile is approximately a replication of small profile plus some leftovers.

Remark: Note that m_0 in the above is independent of the worth function. Kaneko &Wooders (1982) apply this insight to replications of a given game and nonemptiness of cores of rm_0 replications of partitioning games, also known as coalition structure games. See Kovalenkov & Wooders, 2003, for the strongest form of this result.

- Assume that a pregame (T, Ψ) satisfies SGE.
- Then, given $\frac{\varepsilon}{2} > 0$ there is a bound $B(\varepsilon)$ such that for any s there is a partition of s, $\{s^1, ..., s^K\}$ of s, with $\left\| s^k \right\| \leq \varepsilon$ for each k and

$$\Psi(s) - \sum_k \Psi(s^k) \leq \frac{\varepsilon}{2} \|s\| .$$

- Let (T, Ψ') be a another pregame where

$$\Psi'(s) = \max \sum_k [\Psi(s^k) - \frac{\varepsilon}{2} \|s\|]$$

where the maximum is taken over all partitions of s with $\left\| s^k \right\| \leq B(\varepsilon)$ for each k.

- Suppose that n is sufficiently large so that $[n, (T, \Psi')]$ has a nonempty $\frac{\varepsilon}{2}$-core.
- Let $z \in \mathbb{R}_+^T$ represent an equal-treatment payoff vector in its $\frac{\varepsilon}{2}$-core.
- Observe that $\Psi'(s) \leq z \cdot s - \frac{\varepsilon}{2} \|s\|$for each $s \leq n$ implies that $\Psi(s) \leq z \cdot s - \varepsilon \|s\|$ for each $s \leq n$ so z is in the ε-core of $[n, (T, \Psi)]$.
- End of proof.

Remark: The basic intuition has been extended to a broader classes of games with side payments (such as Wooders & Zame 1984), to games without side payments (see Wooders, 2008, and references therein), to games with a continuum of players (see Kaneko & Wooders, 1986,1996) and has been used in economies with clubs (Wooders,1980,1989,1997) and Allouch & Wooders, 2007), and other papers.

Equality and inequality

- In games with many players derived from a pregame satisfying SGE payoffs in approximate (ε) cores have the property that for player types that are abundant in the population most players of the same type are treated nearly equally.
- More formally, given a sufficiently small non-negative real number ε, for any game with a finite set of player attributes (or types) any payoff vector x in the ε-core of the game has the property that, for each type that appears in sufficient abundance in the population, most players of that type are treated approximately equally. Note that in interpretation of the Theorem the numbers γ and λ are to be thought of as 'small'.
- The following result is from Wooders (2010), which allows a larger set of types (a compact metric space). The first version of the following Theorem appeared in Wooders (1979).

Equality and inequality

- In games with many players derived from a pregame satisfying SGE payoffs in ε-cores have the property that for player types that are abundant in the population most players of the same type are treated nearly equally.
- That is, given a sufficiently small non-negative real number ε, any payoff vector x in the ε-core of the game has the property that, for each type that appears in sufficient abundance in the population, most players of that type are treated approximately equally.
- Note that in interpretation of the following Theorem the numbers γ and λ are to be thought of as 'small'. The following result is from Wooders (2010), which allows a larger set of types (a compact metric space). The first version of the Theorem appeared in Wooders (1979).

Theorem 2 (equal treatment). Let (T, Ψ) be a pregame satisfying SGE. Given $\gamma > 0$ and $\lambda > 0$ there is an $\varepsilon^* > 0$ and a $\rho > 0$ such that for each $\varepsilon \in [0,\varepsilon^*]$ and for every $n \in \mathbb{Z}_+^T$ with $\|n\|_1 > \rho$, if $x \in \mathbb{R}^N$ is in the ε-core of the game $[n, \Psi]$ with

$$N = \{(t, q) : t = 1, ..., T \text{ and, for each } t, \ q = 1, ..., n_t\}$$

then, for each $t \in \{1, ..., T\}$ with $\frac{n_t}{\|n\|_1} \geq \frac{\lambda}{2}$ it holds that

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda n_t \ \},$$

where, for each $t = 1, ..., T$,

$$z_t = \frac{1}{n_t} \sum_{q=1}^{n_t} x^{tq},$$

the average payoff received by players of type t.

- Define:
  - The rich: those assigned a payoff greater than the average for their type plus γ.
  - The poor: those assigned a payoff less than the average for their type minus γ.
  - The middle class: everyone else.

- The intuition of the Theorem is most apparent under the stronger assumption of SSGE.
- Since SSGE bounds effective group sizes, given any core or ε-core payoff vector, the "poor", in total, can be discriminated against by only some bounded amount say B.
- The "middle class" can each only be discriminated by at most ε each.
- The "rich" can only receive, in aggregate no more than the total payoff $\varepsilon \left| \{\text{middle class}\} \right| + B$ plus ε times the number of members of the middle class.
- Thus, the rich must be bounded in number.
- The proof with SGE proceeds by approximation.

- With scarce types, equal treatment may no longer holds. The following two examples are from Wooders (RoED, 2010).

Example 3. (*Unequal treatment of 'scarce types'*) Let (Ω, Ψ) be a pregame where $T = 2$ and the payoff $\Psi(n)$ to any profile $n = (n_1, n_2)$ is given by:

$$\Psi(n) = \begin{cases} n_1 + n_2 \text{ if } n_1 \geq 2 \\ n_2 \qquad\quad \text{if } n_1 = 0 \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Observe that the pregame satisfies SGE. Now consider a sequence of games (N^ν, v^ν) where the profile of N^ν is $(2, \nu)$. Then for any ν, the payoff vector assigning 0 to one player of type 1, 2 to the other player of type 1, and 1 to each of the n_2 players of type 2 is in the core of the game (N^ν, v^ν).

Example 4. (*Without thickness, PCB does not imply equal treatment of players of abundant types.*) Let (Ω, Ψ) be a pregame where $T = 2$ and the payoff $\Psi(n)$ to any profile $n = (n_1, n_2)$ is given by:

$$\Psi(n) = \begin{cases} n_1 + n_2 \text{ if } n_1 > 0,\ \ n_2 > 0 \\ \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Now consider a sequence of games (N^ν, v^ν) where the profile of N^ν is $(1, \nu)$. Then given ν, consider a payoff vector $x^\nu \in \mathbb{R}^{N^\nu}$ assigning x^ν_{2q} to the q^{th} player of type 2, $q = 1, ..., \nu$, where the points $\{x_{2q}\}$ are uniformly distributed on the interval $[0, 1]$, and assigning $n_1 + n_2 - \sum_q x^\nu_{2q}$ to the one player of type 1. Then, for any ν, x^ν is in the core of the game (N^ν, v^ν). With some additional work, the same conclusion can be obtained for approximate cores.

- The following Proposition illustrates the importance of outside options for equal treatment. Related results for games/economies with exact substitutes have a long history and in the game-theoretic literature, one such result appears in Owen (1975).

Proposition 2: Let (N, v) be a game and let $x \in \mathbb{R}^N$ be in the core of the game. Suppose that there are two players i and j who are δ-substitutes for each other, for some $\delta > 0$ and also suppose that there are two disjoint groups S, $S' \subset N$ satisfying $i \in S$, $j \in S'$ and $x(S) = v(S)$, $x(S') = v(S')$. Then it follows that $\left|x^i - x^j\right| \leq \delta$.

Equivalence of SGE for feasibility and for improvement

- The following result was motivated by a result due to Mas-Colell another due to and Kaneko and Wooders.
- Both show convergence of cores to equilibrium payoffs in the context of private goods exchange economies.
- Mas-Colell uses a form of SGE for improvement while Kaneko-Wooders use SGE for feasibility.

Theorem 5: (**Equivalence of small group effectiveness for feasibility and for improvement)** A pregame (T, Ψ) satisfies small group effectiveness for improvement if and only if it satisfies SGE.

With a compact metric space of attributes

- Let Ω be a compact metric space of attributes.
- Let f be a function with finite support from Ω to the non-negative integers.
  - $f(\omega) \neq 0$ for only a finite set of points $\omega \in \Omega$.
- Let $\mathcal{F}$ be the space of all such functions.
- Let Ψ: $\mathcal{F} \to \mathbb{R}_+$
- The pair (Ω, Ψ) is a pregame.
- This sort of construct was introduced in Wooders and Zame (1987).

- The main assumption used were continuity (so that players with close attributes are approximately substitutes) and boundedness of individual marginal contributions.
- Let $\chi_\omega \in \mathcal{F}$ be the function which is identically 0 except $\chi_\omega(\omega) = 1$.

Boundedness of individual marginal contributions: (Ω, Ψ) satisfies BIMC if there is a constant K such that for any $f \in \mathcal{F}$ and any χ_ω it holds that

$$\Psi(f + \chi_\omega) - \Psi(f) < K.$$

- BIMC implies SGE.