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NEAR-MARKETS AND MARKET GAMES

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by

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## 1. INTRODUCTION

### 1.1. Market Games

The market game, both in its sidepayment and nosidepayment versions, has provided a valuable tool for the utilization of game theoretic analysis for the study of exchange economies. It is known that exchange economies map into totally balanced games\*\* for both sidepayments and nosidepayments. It is known that totally balanced sidepayment games map into exchange economies and it is conjectured, but not yet proved or counterexampled, that every totally balanced nosidepayment game is representable by an exchange economy.

Clearly the mappings are not one to one in both directions, as the market game contains far less information than does the exchange economy.

The intimate relationship between exchange economies and totally balanced games gives no intimation of the relationship of the competitive equilibria to limit points of the core. In order to study this property we need to be able to define a sequence of exchange economies and a related sequence of totally balanced games. The result that we can define a replication sequence of exchange economies and associate with

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\*\*We note that every subgame of a totally balanced game has a non-empty core.

it a sequence of totally balanced games for which the cores converge to the competitive payoffs is an economic result arising from having utilized the economic data to construct the sequence of totally balanced games.

### 1.2. The Purpose of This Paper

A natural question to ask is are there other economic phenomena which give rise to market games or "near-market games" and from which one can construct a sequence of games which may or may not show core convergence. Once we leave the simple structure of the exchange economy a host of modeling difficulties appear. In particular even if core convergence results are obtainable we must ask what is the economic interpretation of such a convergence if no competitive price system exists in the underlying economic model?

In this paper we consider the relationship of market games and near-market games to economies with complexities beyond that of the exchange economy. We argue that a broad class of economies generate near-market games, including private goods economies with non-convexities, coalition production economies, and economies with local public goods. We also suggest that economies with pure public goods do not, without special restrictions, give rise to near-market games.

More specifically, we say that a sequence of replica games is a sequence of near-market games if the games are superadditive and the sequence satisfies a "near - minimum efficient scale for coalitions" property -- all increasing returns to coalition size are eventually exhausted.

We will show that the near minimum efficient scale property ensures that the sequence is asymptotically totally balanced — given any epsilon greater than zero and any subgame of any game in the sequence, when we replicate the set of players in that subgame, for all sufficiently large replications the epsilon-core of the replicated subgame is non-empty.

A model of a sequence of replica economies with coalition production and local public goods, where agents are allowed to be members of possibly more than one jurisdiction is developed and the derived sequence of games is shown to be a sequence of near-market games. Few restrictions are placed on the model; the major ones are that the asymptotic growth of utility functions is no more than linear and the production correspondences are such that positive outputs do not become virtually free in per-capita terms as the economies become large.

When we further assume that all increasing returns to coalition size are realized by some finite economy — there is a minimum efficient scale of coalitions rather than a near minimum efficient scale — we obtain a core convergence result. The  $\epsilon$ -cores of the derived sequence of games converge to the cores of the associated balanced cover games; this result is also applied to show convergence of the  $\epsilon$ -cores to the competitive payoffs for the Shapley-Shubik (1966) model. These results depend on other results showing that: for all sufficiently large replications all payoffs in the  $\epsilon$ -cores have approximately the equal treatment property and; when there is a minimum efficient scale, the limit of the  $\epsilon$ -cores equals the limit of the cores of the balanced

cover games. While an investigation of competitive equilibria (or competitive-like equilibria since we are dealing with a class of economies to which the classic definition of the competitive equilibrium does not apply) and convergence of the core to the equilibrium payoffs\* is beyond the scope of this paper, we conjecture that the minimum efficient scale property will enable stronger forms of convergence than near minimum efficient scale.

Except under some very special conditions (satiation or "asymptotic" satiation, i.e., marginal utilities go to zero as the amount of the public good increases) replica economies with pure public goods do not generate near-market games. Since games derived from given economies with pure public goods may well be market games (i.e., totally balanced),\*\* this suggests that, in line with the incentives literature it is when we consider sequences of economies that game-theoretic properties of private-goods or "market-like" economies and of pure public goods economies differ.

Another question to be addressed is when is an economic situation "adequately" represented by the characteristic function of the derived game? What is "adequate" is, to some extent, a matter of opinion and

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\*Some illustrations of such convergence are demonstrated in Boehm (1974) for economies with coalition production and in Wooders (1980) for ones with local public goods.

\*\*See Rosenthal (1976).

depends on the ultimate purpose of the model. For our purposes, we say an economic situation has the "c-game" property\* (i.e., in our view is adequately represented by its derived game), if what a coalition can achieve once it has formed is independent of the actions of the complementary coalition. Exchange economies obviously have the c-game property; ones with pure public goods might not. Ones with coalition production and local public goods raise modeling and interpretation problems concerning their "c-gameness"; this will be discussed further later.

As indicated above, in this paper we analyze replication sequences of economies -- ones with a fixed number of types of agents and increasing numbers of agents of each type. To extend this analysis to economies with a continuum of agents or with simply a "large" number of agents appears to pose different problems than extensions of this nature for private goods exchange economies. The technical results in this paper are based on results concerning non-emptiness of approximate cores of large replica games\*\* and asymptotic balancedness of sequences of replica games. The extension of the analysis to economies with a large number of agents (but not necessarily replica economies) could, we believe, be carried out if results were available for large (not-necessarily replica) games analogous to those for large replica games. At this point, such results have not been

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\*See Shubik (1982, pp. 130-131, 354).

\*\*See Wooders (1981a) and Shubik and Wooders (1982a).

obtained<sup>\*</sup>. Our purpose in this paper is to introduce concepts of near-market economies and near-market games; it is not to obtain the most general results that might be possible.

The models are also restricted to ones with a freely transferable medium of exchange. There is reason to believe that such a restriction is not necessary.<sup>\*\*</sup>

Table 1 in Section 1.3 provides a sketch of the different classes of economies we consider here. In the remaining parts of this section we provide a preliminary description of the different models noted in Table 1. In Section 2 a mathematical structure and analysis for near-market games is presented. Section 3 contains our model of a sequence of replica economies whose derived games are near-market games. Section 4 deals with the application and interpretation of near-market games together with a closer examination of the modeling problems involved. All proofs are contained in the appendices.

### 1.3. The Framework of the Models

Both cooperative game theory and general equilibrium theory are as noninstitutional as is feasible. An attempt is made to strip away references to specific mechanisms. In particular both are resolutely nondynamic. A description of dynamics calls for a specification of the structure of the mechanism which carries the process, but this is

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\* There is reason to believe, however, that they are obtainable.

\*\* In this regard, see Wooders (1982) and Shubik and Wooders (1982a, 1982b). The assumption of a freely transferable medium of exchange simplifies the analysis while allowing quite complex economic situations.

tantamount to adding institutional detail.

In essence, when the core of a market game associated with an exchange economy is studied the technical and institutional assumptions which are made are that individual ownership of all economic goods is recognized and that all goods can be transferred, without costs or technical difficulty, among all individuals.\* In particular, implicit in these assumptions is that the economic reality can be well represented by the characteristic function. Shapley and Shubik (1974) have suggested the term "c-game" for a model that is adequately represented as a game by its characteristic function and we use that term here. If trade and coalition formation are costless it is easy to see that there are essentially no modeling problems with exchange economies; what a coalition  $S$  can achieve via exchange of goods among the members of  $S$  is independent of exchanges carried out among the members of the complementary coalition  $\bar{S}$ .

The introduction of indivisibilities or other nonconvexities does nothing to influence the c-game property. It remains reasonable to assume that a set  $S$  can exchange among its members in any way they desire, even if the items are indivisible.

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\* When we discuss the core here, we refer to the core of the market game as contrasted with the set of undominated distributions of resources in the distribution space of the economy which map into the core of the market game. This distinction may not appear to be of much importance when considering exchange economies, but it is helpful when we consider more general economies.



If we assume that production processes manifest constant returns, then no distinction need be made between individual or joint ownership of the technology. The c-game property is preserved. An group  $S$  can produce and exchange as it pleases regardless of the actions of  $\bar{S}$ .

Production and exchange without constant returns poses new problems. If the technology is commonly owned who gets to use it first now matters. This can be avoided by making the reasonable assumption that it is owned by groups. However if we now wish to consider replication there is a temptation to implicitly or explicitly introduce constant returns to scale when doubling a coalition size.

Table 1 shows ten classes of models, the first of which is the exchange economy. The interpretation of the six column headings is as follows. Column 1, "the sequence generator," refers to the class of economic model being studied. When we wish to study mass behavior, in some situations we can use the economic information present in the initial economic model to generate a sequence of models which in turn are mapped into games. In other situations, as for example, in production economies, we can take as given the production technology sets for all coalitions of agents in the sequence of economies -- and thus avoid restricting ourselves to constant returns when doubling the size of a coalition. We then study solution behavior on the sequence of games.

Column 2 indicates whether there are modeling problems in describing the economic structure as a game in cooperative form.

Column 3 indicates whether the game formulated exhibits the c-game

property; when it does not, this usually means that there are conceptual difficulties in utilizing the characteristic function as an adequate representation of the economic structure.

Column 4 is used to note the type of game generated by the economic model.

Column 5 shows whether, in some appropriate sense, the core "shrinks." There are two problems here; the definition of the meaning of the core becoming small as dimensions change; but also the question of to what do the core and  $\epsilon$ -core converge? In private goods exchange economies, or private goods economies with constant returns, cores converge to competitive equilibria. In ones with pure public goods, we have non-convergence; the core does not shrink (Champsaur, Roberts, and Rosenthal (1975)). For the other economies considered, we have "shrinkage" of the  $\epsilon$ -cores to equal treatment payoffs and, with a minimum efficient scale assumption, convergence of  $\epsilon$ -cores to cores of the associated sequence of balanced cover games; while we suspect that these results will have implications for competitive-like equilibria, at this point the economic significance of these results is (with some exceptions) yet to be explored.

Column 6 notes what is known when instead of considering sequences of games with a finite number of players, we consider games with a continuum of economic agents. Although it is comforting to obtain old results by new methods, if all that game theoretic methods were able to do were to bolster the reasoning behind general equilibrium results this would be a weak justification for the development and application of a

TABLE 1\*

Sequence Generator 1	Modeling Problems? 2	c-Game 3	Game Generated 4	Convergence 5	Continuum Core Equivalence to C.E. or Lindahl 6
1. Exchange Economies	no	yes	market game (Shapley-Shubik, 1969)	yes (Debreu-Scarfi, 1963)	yes (Aumann, 1964)
2a. Two-sided Markets with Indivisibilities and demand constraints	no	yes	balanced	yes (with money, Quinzii, 1982)	not considered yet
2b. Exchange Economies, Nonconvex Preferences or Small Indivisibilities	no	yes	$\epsilon$ -balanced (Shapley-Shubik, 1966)	yes, $\epsilon$ -cores (Hildenbrand, Schmeidler, Zamir, 1973)	yes Mas-Colell (1977)
3. Production and Exchange Economies with Constant Returns	no	yes	market game	yes (Hildenbrand, 1968)	yes (Hildenbrand, 1974)
4. Production and Exchange Economies with Coalition Production without Constant Returns	yes	yes and no	near-market games (Shubik, Wooders, 1982b)	yes, under some conditions	not considered yet (but see Oddou, 1982)
5. Production and Exchange Economies with Setup Costs and Capacities	no	yes	$\epsilon$ -balanced (Shubik, Wooders, 1982b)	yes, under some conditions	not considered yet
6. Local Public Goods	yes	yes and no	$\epsilon$ -balanced (Wooders, 1980, 1982)	yes $\epsilon$ -core (Wooders, 1981b, 1982)	not considered yet
7. Pure Public Goods	in general considerable	no	see below		
7a. Unanimity	no	yes	market game	no	no
7b. Simple Majority	no	yes and no	not balanced	no	no
7c. Minority Rights with Majority Voting	yes	yes	can be made balanced	yes	yes?
7d. Subgroup Use of Own Resources	yes	not addressed	balanced (Foley, 1970)	no (Champsaur Roberts & Rosenthal, 1975)	no (Muench, 1972)

\*The references mentioned are intended, at least in some cases, to be representative of a body of work rather than comprehensive.

new apparatus. Yet it is easy to turn the use of the mathematical methods for studying games with a continuum of agents into a hunt for equivalence theorems establishing a coincidence between the payoffs produced by the competitive equilibria or the Lindahl equilibria. This is done, forgetting that the core and other game theoretic solutions exist in situations where there is no price system.

With the caveat noted above, column 6 indicates where there are core equivalence results.

For those classes of economies where core equivalence results have been obtained, or counterexamples provided, it has been demonstrated that the continuum models approximate models with a large, but finite, number of agents -- with the exception of results for coalition production without constant returns. Models such as Oddou's (1982) have not been investigated as limits of finite replica models such as that of Shubik and Wooders (1982b). After we formally introduce our model of replication economies, in Section 4 we will discuss further the differences between the finite replication models of coalition production and models with a continuum of agents.

We now return to a discussion of the cases covered in the table. By introducing setup costs, indivisibilities, and capacity constraints, items all reasonably related to the functioning of a firm, we lose convexity of the production sets and balancedness of the derived games. However, if we constrain the production correspondences so that positive outputs do not become virtually "free" as the economies grow large, we

obtain non-emptiness of  $\epsilon$ -cores and, if we further assume a minimum efficient scale for productive coalitions we set up the conditions for the non-emptiness and convergence of  $\epsilon$ -cores.

Wooders (1980, 1981a) and Shubik and Wooders (1982a, 1982b) have shown that in essence the same mathematical approach can be applied to local public goods and this investigation is continued in this paper.\* However it must be stressed that the modeling problems involved in describing how the production and distribution of a local public good is to be decided upon become critical. In the models used previously and in this paper, the c-game property is preserved between jurisdictions but not necessarily within given jurisdictions. The feasible consumptions, productions, and exchanges of the members of a collection of jurisdictions, acting as a coalition, is independent of the actions of the complementary collection of jurisdictions; this is because there are no "spill-overs" of the public goods between jurisdictions. Within a jurisdiction, what a subset of agents can consume can be affected by the actions of the other members of the jurisdiction; the public goods are "pure" public goods within the jurisdiction. However, from the viewpoint of the study of the core, it is the c-gameness between jurisdictions that is relevant.

When we consider pure public goods the modeling preliminary to the calculation of the characteristic function is of central importance

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\*In fact the development of the theory has been the other way around. Local public goods economies have potentially the problems of coalition production and also public goods problems (within jurisdictions). Thus, insofar as the study of approximate cores is concerned, coalition production economies are technically relatively easy to investigate.

since the characteristic function must embody the results of some rule specifying the effects of the actions of the complementary coalition on the members of a given coalition. Four methods are suggested here. Under a unanimity rule only the group as a whole can act; such a game is a c-game and is clearly totally balanced. The sequence of larger games with a unanimity rule is such that all games maintain large cores.

A frequently used decision rule is the simple majority vote. But this rule with full taxing authority gives all to any winning coalition. The game is a c-game in the sense that any coalition  $S$  can guarantee everything for itself if it is winning and can guarantee nothing otherwise. This has no core.

In actuality in most societies a considerable body of law and custom exist devoted to protecting minorities. Thus the winner is not always in a position to take all. The specific structure of the minority protection conditions will influence the characteristic function and it may become possible to produce a balanced game even with majority voting by an appropriate selection of the minority protection rule.

One may wish to avoid the specification of "political decision methods" such as a vote in which case some economic usage argument must be made for what a group of individuals can obtain. All of the problems with coalition production are present in addition to the public good distribution problem.

The fourth method is to specify that each subgroup uses its own

resources to produce the public goods for its own members, and any potential production of public goods by the complementary subgroup is ignored in the derivation of the characteristic function. It has been generally assumed that all subgroups have access to the same production possibility set--a closed, convex cone.

It is suggested that market games do provide an overall way of considering a variety of economic structures, but that once we depart from exchange economies or economies with constant return to scale in production a host of modeling questions appear. As is suggested by the headings of Table 1 there are several other properties of economic significance which may or may not be present along with the market game property. The c-game property provides a check on the acceptability of the model and the construction of a sequence of related games permits us to examine the behavior of solutions based upon the characteristic function, for large numbers.

We note that cooperative game analysis based on the market game appears to be of limited value in studying problems involving information, oligopolistic competition, money or virtually any phenomenon where the stress is clearly upon structure and process. For them the extensive or strategic forms are called for. Even in situations noted in Table 1 where the c-game property holds, probably the most satisfactory procedure is to model the economy explicitly as a game in strategic form specifying, for example, a mechanism for price formation, voting control of the corporation and the distribution of corporate profits.

The point that must be stressed is that much of the relevance and realism of the modeling has already been specified in the defining of the characteristic function. If it is unsatisfactory then there is no reason to expect that the core will not reflect the inadequate representation. (This remark holds true for the value as well as any other solution concepts based on the characteristic function).

We conclude this subsection with an example illustrating the differences between the standard Von-Neuman-Morgenstern characteristic function based on min max behavior and the Harsanyi characteristic function, which reflects more of the threat structure between players.

Consider the two person game in strategic form shown below. The Von-Neuman-Morgenstern characteristic function is

		Strategies of Player 2	
		1	2
Strategies of Player 1	1	0, -60	100, 20
	2	0, 0	0, 0

given by\*

$$v(\bar{1}) = v(\bar{2}) = 0$$

$$v(\bar{12}) = 120$$

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\*The notation  $\bar{i}$  represents the set containing player  $i$  only, etc.



Let  $s_i$  stand for a strategy of player  $i$ ,  $i = 1, 2$ , then

$$v(\bar{1}) = \max_{s_1} \min_{s_2} P_1(s_1, s_2)$$

$$v(\bar{2}) = \max_{s_1} \min_{s_2} P_2(s_1, s_2)$$

$$v(\bar{12}) = \max_{s_1} \max_{s_2} \{P_1(s_1, s_2) + P_2(s_1, s_2)\}$$

with  $P_i$  denoting the payoff to player  $i$  from the selection of  $(s_1, s_2)$ .

The Harsanyi function however is calculated for a set of players, and its complement (here as there are only two players, there is only one calculation) as follows:

$$h(\bar{1}) + h(\bar{2}) = \max_{s_1} \max_{s_2} \{P_1(s_1, s_2) + P_2(s_1, s_2)\} = 120$$

$$h(\bar{1}) - h(\bar{2}) = \max_{s_1} \min_{s_2} \{P_1(s_1, s_2) - P_2(s_1, s_2)\} = 60$$

which gives in this instance  $h(\bar{1}) = 90$ ,  $h(\bar{2}) = 30$ ,  $h(\bar{12}) = 120$ .

Rosenthal (1971) and others have also noted difficulties with the characteristic function.

#### 1.4. Near Markets

In the remainder of this paper we argue that for large economies, with appropriate conditions essentially limiting the effects of externalities and indivisibilities, near-market games provide a valuable tool for analysis of the first six cases described in Table 1. For nonlocal

public goods we suggest that the physical conditions require a direct modeling of the voting or other socio-political mechanism. This is noted in discussion of cases 7a, b, c and d.

Even though our analysis is carried out in terms of games with sidepayments we conjecture that if there exists a "quasi-transferable" utility (i.e. a good which is divisible, always desirable and a substitute for any other good) our results hold substantially.\*

## 2. NEAR-MARKET GAMES

### 2.1. Games

We first review some game-theoretic concepts.

A game (with sidepayments) is an ordered pair  $(N, v)$  where  $N = \{1, \dots, n\}$  is a finite set, called the set of players and  $v$  is a real-valued function mapping subsets of  $N$  into  $\mathbb{R}_+$  with  $v(\emptyset) = 0$ . Two players  $i$  and  $j$  are substitutes if given any subset  $S$  of  $N$  where  $i \notin S$  and  $j \notin S$ , we have  $v(S \cup \{i\}) = v(S \cup \{j\})$ . A subgame,  $(S, v)$  of  $(N, v)$  is an ordered pair consisting of a non-empty subset  $S$  of  $N$  and the function  $v$  restricted to subsets of  $S$ . The game  $(N, v)$  is superadditive if for all disjoint subsets  $S$  and  $S'$  of  $N$ , we have  $v(S) + v(S') \leq v(S \cup S')$ . A payoff for the game is a

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\*This conjecture is based on the fact that analogous development of approximate core theory has been carried out for games without side payments (see Wooders (1981a) and Shubik and Wooders (1982a)) and some results in the spirit of those herein have been obtained (see Wooders (1980, 1982)).

vector  $B = (B^1, \dots, B^n) \in \mathbb{R}_+^n$ , the non-negative orthant of the  $n$ -fold Cartesian product of the reals. A payoff  $B$  is feasible if  $\sum_{i \in N} B^i \leq v(N)$ .

Given  $\epsilon \geq 0$ , a payoff is in the (weak)  $\epsilon$ -core\* if it is feasible and if, for all non-empty subsets  $S$  of  $N$ ,  $\sum_{i \in S} B^i \geq v(S) - \epsilon|S|$

where  $|S|$  denotes the cardinal number of the set  $S$ . When  $\epsilon = 0$ , the  $\epsilon$ -core is called simply the core.

Given a game  $(N, v)$ , let  $(N, \tilde{v})$  denote the totally balanced cover of  $(N, v)$ ; the function  $\tilde{v}$  is the smallest real-valued function such that, for all non-empty subsets  $S$  of  $N$ , the subgame  $(S, \tilde{v})$  has a non-empty core and  $v(S) \leq \tilde{v}(S)$ .

## 2.2. Sequences of Games

A sequence of games  $(N_r, v_r)_{r=1}^{\infty}$  is superadditive if each game  $(N_r, v_r)$  is superadditive. It is per-capita bounded if there is a constant  $K$ , independent of  $r$ , such that  $v_r(N_r)/|N_r| \leq K$  for all  $r$ .

Let  $(N_r, v_r)_{r=1}^{\infty}$  be a sequence of games where, for some positive integer  $T$ , for each  $r$  the set of players  $N_r$  contains  $rT$  players denoted by  $N_r = \{(t, q) : t = 1, \dots, T, q = 1, \dots, r\}$ . For each  $r$  and each  $t$ , let  $[t]_r = \{(t, q) : q = 1, \dots, r\}$ . The sequence is a sequence of replica games if

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\*This concept was introduced by Shapley and Shubik (1966).

- (1)  $N_r \subset N_{r+1}$  for all  $r$  ;
- (2) for each  $r$  and all subsets  $S$  of  $N_r$  ,  $v_r(S) \leq v_{r'}(S)$  whenever  $r' \geq r$  ;
- (3) for each  $r$  and each  $t$  all players in  $[t]_r$  are substitutes for each other.

Throughout the following, given a sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$  we define  $[t]_r$  as above and call the members of  $[t]_r$  players of type  $t$  . We also assume there are  $T$  types of players and denote the set of players  $N_r$  as above.

Given a sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$  and a subset  $S$  of  $N_r$  for some  $r$  , define the vector  $\rho(S) = (s_1, \dots, s_T)$  by its coordinates  $s_t = |S \cap [t]_r|$  ; the vector  $\rho(S)$  is called the profile of  $S$  and is simply a list of the numbers of players of each type contained in  $S$  . Let  $I$  denote the  $T$ -fold Cartesian product of the non-negative integers. Observe that for any  $r$  and any subset  $S$  of  $N_r$  , we have  $\rho(S) \in I$  . Also, since players of the same type are substitutes, if  $S$  and  $S'$  are two subsets with the same profiles then for any  $r$  such that  $S \subseteq N_r$  and  $S' \subseteq N_r$  , we have  $v_r(S) = v_r(S')$  . Consequently the function  $v_r$  can be completely defined by a mapping from a subset of  $I$  to the reals. In the following, given  $r$  and a profile  $s$  of a subset of  $N_r$  , we define  $v_r(s)$  as  $v_r(S)$  for any  $S \subseteq N_r$  with  $\rho(S) = s$  . \* Given  $s \in I$  we write  $|s| = \sum_{t=1}^T s_t$  since when  $\rho(S) = s$  , we have  $|S| = \sum_{t=1}^T s_t$  .

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\* This abuse of notation should create no confusion. We note that we typically denote subsets by upper case letters and profiles by lower case ones.

We say a sequence of games satisfies the property of "near minimum efficient scale" (for coalitions), NMES, if it is per-capita bounded. If the sequence is also superadditive, we say the sequence is a sequence of near market games.

We remark that when a sequence of replica games is per-capita bounded and superadditive, both  $\tilde{v}(N_r)/|N_r|$  and  $v(N_r)/|N_r|$  converge and to the same limit. In this case, for  $r$  sufficiently large, the per-capita gains to forming a coalition larger than  $N_r$  are small. This motivates our term "near-minimum efficient scale."

A sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$  is asymptotically totally balanced if, given any  $r$ , any subset  $S$  of  $N_r$ , and any  $\epsilon > 0$ , there is an  $n^*$  such that for all  $n \geq n^*$  we have

$$\frac{\tilde{v}_{nr}(S_n)}{|S_n|} - \frac{v_{nr}(S_n)}{|S_n|} < \epsilon,$$

where  $v_{nr}$  denotes the function  $v_r$ , with  $r' = nr$  and  $S_n$  is any subset of  $N_r$ , with  $\rho(S_n) = n\rho(S)$ . It can easily be verified (and follows from well-known results, cf. Shapley (1967)), that given any game  $(N, v)$ , we have  $\tilde{v}(N)/|N| - v(N)/|N| < \epsilon$  if and only if the  $\epsilon$ -core of the game is non-empty. Consequently, given any subset  $S$  of  $N_r$  for some  $r$  and any sequence of subsets  $(S_n)$  satisfying the properties required above, for all  $n$  sufficiently large the subgames  $(S_n, v_{nr})$  have non-empty  $\epsilon$ -cores.

The following theorem provides sufficient conditions for asymptotic total balancedness of sequences of replica games.

Theorem 1. Let  $(N_r, v_r)_{r=1}^{\infty}$  be a sequence of near market games. Then the sequence is asymptotically totally balanced.

Proof. All theorems in this section are proven in Appendix I.

### 2.3. Convergence of Cores

In the remainder of this section, we consider convergence properties of  $\epsilon$ -cores of sequences of replica games.

Our next theorem, informally, shows that as  $r$  goes to infinity and  $\epsilon$  goes to zero, in "the limit," only equal-treatment payoffs are in the  $\epsilon$ -cores where an equal treatment payoff  $x$  for  $N_r$  has the property that, for each  $t$ ,  $x^{tq} = x^{tq'}$  for all  $q$  and  $q'$ .

Theorem 2. Let  $(N_r, v_r)_{r=1}^{\infty}$  be a sequence of near-market games. Then, given any  $\delta > 0$  and any  $\lambda > 0$ , there is an  $\epsilon^*$  and an  $r^*$  such that for all  $\epsilon \in [0, \epsilon^*]$  and for all  $r \geq r^*$ , if  $B$  is in the  $\epsilon$ -core of  $(N_r, v_r)$  then

$$|\{(t,q) \in N_r : \|B^{tq} - \bar{B}_t\| > \delta\}|/r < \lambda$$

where  $\bar{B}_t = \sum_{q=1}^r B^{tq}/r$  and  $\|\cdot\|$  denotes the absolute value.

Less formally, Theorem 2 states that for small  $\epsilon$  and large  $r$ , any payoff in the  $\epsilon$ -core has the property that the percentage of players whose payoffs differ significantly (by more than  $\delta$ )

from the average payoff for their type can be made arbitrarily small (less than  $\lambda$ ). We remark that since the theorem holds for  $\epsilon \geq 0$ , it follows that for all sufficiently large  $r$  any payoff in the core of the  $r^{\text{th}}$  balanced cover game has "nearly" the equal-treatment property.

Given a sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$ , let  $B$  be an equal-treatment payoff in the  $\epsilon$ -core of  $(N_r, v_r)$  for some  $r$  and some  $\epsilon \geq 0$ . Since, for each  $t$ ,  $B^{tq} = B^{tq'}$  for all  $q$  and  $q'$ ,  $B$  can be completely described by a vector  $\bar{B} \in \mathbb{R}^T$  where, for each  $t$ ,  $\bar{B}_t = B^{tq}$  for (any)  $q$ . We say that  $\bar{B}$  represents an equal-treatment payoff in the  $\epsilon$ -core of  $(N_r, v_r)$ . Given  $\epsilon > 0$ , for each  $r$  let  $B_r(\epsilon) = \{\bar{B} \in \mathbb{R}^T : \bar{B} \text{ represents an equal-treatment payoff in the } \epsilon\text{-core of } (N_r, v_r)\}$ .

Given  $\epsilon > 0$ , let  $L(B(\epsilon))$  denote the closed limit\* of the sequence of sets  $(B_r(\epsilon))_{r=1}^{\infty}$ . In the appendix it is shown this limit exists.\*\* Define  $A^* = \bigcap_{\epsilon > 0} L(B(\epsilon))$ ; then  $A^*$  is the asymptotic core of  $(N_r, v_r)_{r=1}^{\infty}$ . Informally,  $A^*$  is the limit of the  $\epsilon$ -cores and  $B^* \in \mathbb{R}^T$  is in the asymptotic core if, given any  $\epsilon > 0$ , there is a sequence  $(\bar{B}_r^{\epsilon})_{r=1}^{\infty}$  where  $\bar{B}_r^{\epsilon} \in B_r(\epsilon)$  for each  $r$  and  $\bar{B}_r^{\epsilon} \rightarrow B^*$  as  $r$  goes to infinity.

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\* See Hildenbrand (1974), pp. 15-17 for a definition and some properties of the closed limit of a sequence of sets.

\*\*This is Theorem 6 in Appendix 1.

Given a sequence of replica games,  $(N_r, v_r)_{r=1}^{\infty}$ , let  $C(r)$  denote the subset of  $\mathbb{R}^T$  where each member  $C$  of  $C(r)$  represents an equal-treatment payoff in the core of the balanced cover game  $(N_r, \tilde{v}_r)$ . Let  $L(C)$  denote the closed limit of the sequence of sets  $(C(r))_{r=1}^{\infty}$ .

We now state conditions under which  $L(C)$  exists and equals  $A^*$ ,

We say a sequence of superadditive replica games  $(N_r, v_r)_{r=1}^{\infty}$  has a minimum efficient scale for coalitions, MES, if there is an  $r^*$  such that for all  $r \geq r^*$  and  $r' \geq r^*$  we have  $\tilde{v}_r(N_r)/r = \tilde{v}_{r'}(N_{r'})/r$ .<sup>\*</sup> We call the replication number  $r^*$  an MES bound (a minimum efficient scale bound). We remark that if  $r^*$  is an MES bound for the sequence, given  $r > r^*$  if  $B$  is a payoff in the core of  $(N_r, v_r)$  then  $B$  has the equal-treatment property; this is easily verified.

Theorem 3. Let  $(N_r, v_r)_{r=1}^{\infty}$  be a sequence of near-market games. If the sequence has the MES property, then  $L(C)$  exists and equals  $A^*$ .

To illustrate an application of this result recall that in Shapley and Shubik (1966), although it is conjectured that for a class of private goods exchange economies, the weak  $\epsilon$ -cores converge to the set of competitive payoffs of the concavified economy,<sup>\*\*</sup>

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\* Alternative formulations of MES are contained in Shubik and Wooders (1982a) and Wooders (1981a).

\*\* An economy derived from the original economy when the utility functions have been concavified.



this is not proven. The concavified game (i.e., the one derived from the concavified economy), is exactly the balanced cover game of the game derived from the (non-concavified)economy. For the concavified economy the competitive equilibrium exists and the cores of the concavified games converge to the competitive payoffs. Since the sequence of derived games satisfies the MES property, Theorem 3 applies and the limit of the  $\epsilon$ -cores equals the limit of the cores of the balanced cover games. This limit equals the set of competitive payoffs. Thus, the proof of Shapley and Shubik's conjecture is a corollary to Theorem 3.

### 3. NEAR-MARKET ECONOMIES

#### 3.1. Introduction to Near-Market Economies

In this section we develop a model of a sequence of replica economies with private goods, local public goods, and coalition production. Minimal restrictions are imposed on the model yet we are able to show that the sequence of derived games is superadditive and per-capita bounded and thus is asymptotically totally balanced. Therefore, the class of replication economies we consider are near-market economies--the derived games are near-market games for large replications.

#### 3.2. The Model

A sequence of replica economies  $(E_r)_{r=1}^{\infty}$  is defined as a sequence of septuples

$$E_r = (N_r, R^l, R^m, U_r, W_r, (J_r, Z_r), (F_r, Y_r))$$

where  $N_r = \{(t,q) : t = 1, \dots, T, q = 1, \dots, r\}$  is the set of agents.

$R^l$  is the private commodity space;

$R_+^m$  is the public commodity space;

$U_r = \{u^{tq} : (t,q) \in N_r\}$  is an indexed collection of sublinear utility functions mapping  $R_+^{m+l}$  into  $R_+^1$  with the property that for

some linear function  $L$  and some real number  $c$  we have

$u^{tq}(x,y) \leq L(x,y) + c$  for all  $(x,y)$  in  $R_+^{m+l}$  and for all  $(t,q)$

in  $N_r$ .

$W_r = \{w^{tq} \in R^l : (t,q) \in N_r\}$  is an indexed collection of initial endowment vectors, each in  $R_+^l$  (no public goods are initially endowed);

$(J_r, Z_r)$  is a pair of correspondences, where  $J_r$ , called the allowable jurisdiction structure correspondence, maps non-empty subsets  $S$  of  $N_r$  into non-empty subsets of  $S$  and  $Z_r$ , called the public goods production correspondence, maps subsets  $S$  of  $N_r$  into subsets of  $R_+^m \times R^l$ ;

and  $(F_r, Y_r)$  is another pair of correspondences where  $F_r$ , called the allowable firm structures correspondence maps non-empty subsets  $S$  of  $N_r$  into subsets of  $S$  and  $Y_r$ , the private goods production correspondence, maps non-empty subsets  $S$  of  $N_r$  into subsets of  $R^l$ .

A number of further specifications are made on the components of a sequence of replica economies:

- (1)  $N_r \subset N_{r+1}$  for each  $r$ ;
- (2) for each  $t$ , each  $q$  and  $q'$  in  $\{q'' : q'' = 1, \dots, r\}$ , and all  $r$ ,  $u^{tq} = u^{tq'}$  and  $w^{tq} = w^{tq'}$ ; i.e., all agents of the same type have the same utility functions and the same initial endowments. Also,  $u^{tq}(w^{tq}) > 0$  for all  $(t,q) \in N_r$  and for all  $r$  (this assumption is for technical convenience);

(3) Given  $r$ ,  $S \subseteq N_r$ , and  $J(S) \in J_r(S)$ ,  $J(S)$  is an allowable jurisdiction structure of  $S$ . Allowable jurisdiction structures  $J(S)$  are required to satisfy the properties that

- (a)  $S \subseteq \bigcup_{S' \in J(S)} S'$  (allowable jurisdiction structures of  $S$  cover  $S$ );
- (b) if  $J(S) \in J_r(S)$  and  $J(S') \in J_r(S')$ , where  $S$  and  $S'$  are non-empty, disjoint subsets of agents, then  $\{S'' \subseteq N_r : S'' \in J(S) \cup J(S')\} \in J_r(S \cup S')$ ;
- (c) given  $S \subseteq N_r$  and  $r' \geq r$ , if  $J(S)$  is in  $J_r(S)$ ; then  $J(S)$  is in  $J_{r'}(S)$ ;
- (d) if  $S$  and  $S'$  are non-empty subsets of  $N_r$  with the same profiles, then there is a one-to-one mapping, say  $\psi$ , of  $J_r(S)$  onto  $J_r(S')$  such that if  $\psi(J(S)) = J(S')$ , then the collection of profiles of members of  $J(S)$  (not all necessarily distinct), equals those of  $J(S')$ .

We remark that in Wooders (1980) the allowable jurisdiction structures of a set of agents was the set of all partitions of agents. The present formulation of allowable jurisdiction structures is sufficiently general to include the possibility that  $J_r(S)$  is the set of all partitions of  $S$  and also to include the possibility of "overlapping" jurisdictions (an agent could possibly belong to several jurisdictions simultaneously).

The public goods production correspondence is required to satisfy the properties that

(e) given  $S \subseteq N_r$  and  $r' > r$ ,  $Z_r(S) \subseteq Z_{r'}(S)$  ;

(f) if  $S$  and  $S'$  are non-empty subsets of  $N_r$  for

any  $r$ , with the same profiles then

$Z_r(S) = Z_r(S')$ , i.e. the production possibility

set for a subset of  $N_r$  depends only on the profile

of that coalition;

(g)  $0 \in Z_r(S)$  for all non-empty subsets  $S$  of  $N_r$  and for

all  $r$ .

(4) Given a subset  $S$  of  $N_r$ , allowable firm structures  $F(S) \in F_r(S)$

are required to satisfy the same properties as allowable jurisdiction structures, i.e. (a), (b), (c), and (d) of (3) above.

Also, the private goods production correspondence is assumed to satisfy the same properties as the public goods production correspondence, (e), (f), and (g) above.

For private goods, we also define the aggregate production correspondences. Given  $r$  and  $S \subseteq N_r$ , define

$$\bar{Y}_r(S) = \bigcup_{F(S) \in F_r(S)} \sum_{S' \in F(S)} Y_r(S') :$$

then  $\bar{Y}_r(\cdot)$  is the  $r^{\text{th}}$  aggregate production correspondence. Note that  $\bar{Y}_r(\cdot)$  is superadditive; given any two disjoint, non-empty subsets  $S$  and  $S'$  of  $N_r$ , we have  $\bar{Y}_r(S) + \bar{Y}_r(S') \subseteq \bar{Y}_r(S \cup S')$ .

The above consists of a description of the components of a sequence of replica economies and members of the sequence. In the following, we introduce additional definitions which enable us to relate production decisions to consumption decisions and to define feasible states of an economy.

An  $N_r$ -allocation is a vector  $(x, y) = (x^{11}, \dots, x^{Tr}, y^{11}, \dots, y^{Tr}) \in \mathbb{R}_+^{Tr(m+l)}$  where  $(x^{tq}, y^{tq})$  is a commodity bundle for the  $(t, q)^{th}$  agent. Given any  $r$  and any non-empty subset  $S$  of  $N_r$ , an  $S$ -allocation is an  $N_r$ -allocation where  $x^{tq} = 0$  and  $y^{tq} = 0$  if  $(t, q) \notin S$ .

Given  $r$  and a non-empty subset  $S$  of  $N_r$ , an  $S$ -private goods production plan is a vector  $y \in \bar{Y}_r(S)$ .

It is possible, and obvious, that we could define an aggregate public goods production correspondence analogously to the definition of the aggregate private goods production correspondence. However, because of the "non-transferability" of public goods produced in one jurisdiction to the members of another jurisdiction, we proceed differently in defining public goods production plans. In particular, we keep track of the jurisdiction structure associated with a public goods production plan. Given a non-empty subset  $S$  of  $N_r$ , an  $S$ -public goods production plan is an ordered pair,  $\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$  where  $J(S) \in J_r(S)$ .

Given  $r$  and a non-empty subset  $S$  of  $N_r$ , an  $S$ -state of the economy,  $e(S)$ , is an ordered triple,  $e(S) = (\bar{y}, \varphi(S), (x, y))$  where  $\bar{y}$  is a private goods production plan for  $S$ ,

$\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$  is an S-public goods production plan, and  $(x, y)$  is an S-allocation where  $x^{tq} = \sum_{\substack{tq \in S' \\ S' \in J(S)}} x_{S'}$ ,

for all  $(t, q) \in S$  (the  $(t, q)^{\text{th}}$  agent consumes the total outputs of public goods produced in all jurisdictions of which he is a member).

The state is S-feasible if  $\sum_{(t, q) \in S} (y^{tq} - w^{tq}) + \sum_{S' \in J(S)} z_{S'} = \bar{y}$ . An

$N_r$ -state of the economy is called simply a state of the  $r^{\text{th}}$  economy, or, when no confusion is likely to arise, simply a state of the economy.

Given  $S \subseteq N_r$ , let  $A_r(S) = \{(x, y) : \text{there is an S-feasible state of the economy with the associated S-allocation } (x, y)\}$ . The set  $A_r(S)$  is called the set of S-attainable allocations. We note that if  $r' \geq r$ , then

$$\text{Proj}_S A_{r'}(S) \subseteq \text{Proj}_S A_r(S)$$

where  $\text{Proj}_S A_{r''}(S)$  denotes the projection of the set  $A_{r''}(S)$  onto the subset of  $\mathbb{R}^{|S|(m+l)}$  associated with the members of  $S$  for any replication number  $r''$ .

### 3.3. The Derived Games

We now define the sequence of games derived from the sequence of economies.

Given  $r$  and  $S \subseteq N_r$ , define  $v_r(S) = \sup_{(x, y) \in A_r(S)} \sum_{(t, q) \in S} u^{tq}(x^{tq}, y^{tq})$

when  $S \neq \emptyset$ . Define  $v_r(S) = 0$  when  $S = \emptyset$ .

Observe that the pair  $(N_r, v_r)$  is a game with sidepayments. The generation of a game with sidepayments presumes, as usual, the existence of a freely

transferable medium of exchange. There is no need to introduce this explicitly. It is straightforward to verify that the sequence of derived games is a sequence of replica games.

### 3.4. Near-Market Economies

Without further restrictions on the economies, in particular on production, there is no assurance that the derived sequence of games is a sequence of near-market games. The restrictions required are, informally, that positive production does not become virtually free as the economies become large. Formally, we assume

A1. There is a closed convex cone  $Y^* \subset \mathbb{R}^l$ , with  $-\mathbb{R}_+^l \subseteq Y^*$  and  $Y^* \cap \mathbb{R}_+^l = \{0\}$ , such that  $Y_r(S) \subseteq Y^*$  for all subsets  $S$  of  $N_r$  and for all  $r$ .

A2. There is a closed convex cone  $Z^* \subseteq \mathbb{R}_+^m \times \mathbb{R}^l$ , with  $\{0\} \times -\mathbb{R}^l \subseteq Z^*$  and  $Z^* \cap \mathbb{R}_+^m \times \mathbb{R}_+^l = \{0\}$ , such that for any  $r$ , any non-empty subset  $S \subseteq N_r$ , and any allowable jurisdiction structure  $J(S) \in J_r(S)$ , we have  $\sum_{S' \in J(S)} (x_{S'}, z_{S'}) \in \{(x, z) \in \mathbb{R}^{m+l} : (|S|x, z) \in Z^*\}$ .

The first assumption is clear. The second is that there is some set  $Z^*$  satisfying the properties of a standard private goods production set and public goods are never cheaper per capita than private goods would be if they were produced with the public set  $Z^*$ .

An example of production correspondences which satisfy A2 in the one-private-good, one-public-good case is given by the production functions

$x + z/|S| = 0$  for each subset  $S$  of  $N_r$  and for all  $r$  with  $J_r(S)$  equal to the set of partitions of  $S$ . Here the per capita costs of the public good, in terms of the inputs, is constant and independent of the size and composition of the jurisdiction structure.

To see what A2 rules out, suppose all coalitions have the production set  $Z$  determined by the production function  $x+z = 0$ , there is only one private good and one public good, and again  $J_r(S)$  is the set of all partitions of  $S$ . To show that A2 is not satisfied, let  $x_r = 2/r$  and  $z_r = -(2/r)$  for each positive integer  $r$ . Observe that  $(x_r, z_r) \in Z(S)$  for all  $S \subseteq N_r$  and for all  $r$ . Choose a sequence of subsets  $S_r \subseteq N_r$  for each  $r$  such that  $|S_r| = r$ . We then have  $\lim_{r \rightarrow \infty} |S_r| x_r = 2$  and  $\lim_{r \rightarrow \infty} z_r = 0$ . This contradicts A2 since  $Z^*$  is closed and  $Z^* \cap \mathbb{R}_+^2 = \{0\}$ .

The consequences of A1 are described in Shubik and Wooders (1982b).

Together, the assumptions on  $Z_r$  and  $Y_r$  and the sublinearly of the utility functions imply that the sequence of derived games is per-capita bounded. Since the sequence of derived games is also superadditive, it is a sequence of near-market games and asymptotically totally balanced.

Theorem 4. Let  $(E_r)_{r=1}^{\infty}$  be a sequence of replica economies satisfying A1 and A2. Then the derived sequence of replica games is a sequence of near-market games and is asymptotically totally balanced.

In Appendix 2, we show that the derived sequence of games is per capita bounded and superadditive. Theorem 4 then follows as a Corollary to Theorem 1. (All theorems in this section are proven in Appendix 2.)



### 3.5. Convergence of $\epsilon$ -Cores

We now impose a further restriction on production which will have the result that the derived sequence of games has the MES property. This assumption ensures that there is a "minimum efficient scale for production coalitions."

We require the following definition: given  $r$  and  $(x,y) \in A_{N_r}(N_r)$  a permutation by types of  $(x,y)$  is an  $N_r$ -allocation, say  $(x',y')$  where for each type  $t$  there is a one to one mapping of  $\{(x^{tq}, y^{tq}) : q=1, \dots, r\}$  onto itself. Note that since  $(x,y)$  is an  $N_r$ -attainable allocation,  $(x',y')$  is an  $N_r$ -attainable allocation.

A3. There is an  $r^*$  such that for all positive integers  $n$ , given  $r' = nr^*$  for some  $n$  and  $(x,y) \in A_{N_{r'}}(N_{r'})$ , there is a permutation by types of  $(x,y)$ , say  $(x',y')$ , and an  $(x^*,y^*) \in A_{N_{r^*}}(N_{r^*})$  where, for each  $(t,q) \in N_{r^*}$ , we have  $u^{tq}(x^{*tq}, y^{*tq}) \geq u^{t'q'}(x'^{t'q'}, y'^{t'q'})$  for all  $(t',q') \in N_{r'}$ , with  $t' = t$  and  $q' = q, 2q, \dots, nq$ .

Informally, A3 is the assumption that agents can do as well for themselves in coalitions with profiles less than or equal to  $\rho(N_{r^*})$  as they can in coalitions with profiles less than or equal to  $n\rho(N_{r^*})$  --increasing returns to coalition size are exhausted by economies with profiles less than or equal to that of  $N_{r^*}$ .

Theorem 5. Let  $(E_r)_{r=1}^{\infty}$  be a sequence of replica economies satisfying assumptions A1 to A3. Then the derived sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$  has the MES property.

It is an immediate consequence of Theorems 2, 3, and 5, that the  $\epsilon$ -cores of the sequence of derived games  $(N_r, v_r)_{r=1}^{\infty}$  converge to the cores of the balanced cover games as  $r$  becomes large, i.e. for the derived sequence of games  $A^* = L(C)$ . (Since we have made an MES assumption, only equal-treatment payoffs are in the cores of the balanced cover games for sufficiently large replications. Theorem 2 ensures the convergence of the  $\epsilon$ -cores to equal-treatment payoffs.)

#### 4. The Framework Revisited

We begin this section by returning to the table in the introduction.

In terms of the questions addressed in columns 1 to 6 of the table, private goods exchange economies have desirable properties; the generation of sequences of replica economies is straightforward; the derived games are totally balanced  $c$ -games (see Shapley and Shubik (1969) for games with sidepayments and Billera and Bixby (1974) for ones without); convergence of the core payoffs to the competitive allocations is well-known (see Shubik (1959), Debreu and Scarf (1963), and, more recently, Anderson (1978) and others). The continuum case was first studied by Aumann (1964); equivalence of the core and competitive equilibrium was attained.

Some of the above referenced results apply to exchange economies with non-convex preferences and ones with indivisibilities; see also for example, MasCollé (1977). Again, no serious problems arise.

Production and exchange with constant returns, both in the usual sense of production sets with constant returns to scale and in the sense of "constant returns to agents" also pose no significant conceptual difficulties at this point in the development of economic theory. By "constant returns to coalitions" we mean that; for any disjoint coalitions  $S$  and  $S'$ ,  $Y(S) + Y(S') = Y(S \cup S')$  where  $Y(\cdot)$  is the production correspondence, i.e. the production correspondence is additive. Replication production economies with constant returns where all coalitions

had access to the same production sets were considered by Debreu and Scarf (1963). A continuum model of a coalition production economy where the production correspondence was additive was investigated by Hildenbrand (1968, 1970, 1974) and subsequently others.

It is when we consider production without constant returns that difficulties begin to arise. The questions of which agents have access to what production sets become critical. Leaving aside questions of ownership, one can associate a production technology set with each coalition of agents and consider the cooperative game form derived directly from the economic data. Other authors, cf. Ishiishi (1977), and Sondermann (1974) have made balancedness assumptions on the production correspondence to obtain non-emptiness of cores and existence of equilibria. In Shubik and Wooders (1982b) no balancedness assumptions are made. A sequence of replica coalition production economies is developed and it is shown that for all sufficiently large replications, approximate cores are non-empty.

Within the context of replication models of coalition production economies, no limiting core-equilibrium equivalence results have been obtained except for that of Boehm (1974) who makes an assumption of constant returns with respect to  $r$  to the productive coalition  $\{N_r\}$ . When there are "increasing returns to coalition size" (i.e. given a price system for private goods, the derived profit game is a convex game) with a continuum of agents, Sondermann shows that the core is in general larger than the set of competitive equilibrium payoffs.

Oddou (1982), also using a continuum of agents, presents a convincing argument that constant returns to coalitions is necessary for core-equilibria equivalence.

We remark that no formal development of a coalition production economy with a continuum of agents as the limit of a sequence of finite economies has been undertaken. For the model in Shubik and Wooders (1982b), for large economies there are approximately constant returns to scale to replication of those coalitions which are associated with "near-optimal" firms; the essential assumption is that positive outputs do not become virtually free as the economy is replicated. However, if one were to model the economies in Shubik and Wooders (1982b) as ones with a continuum of agents in the limit, then the "limit economy" would not necessarily have Oddou's constant returns property.

Some of the same observations as in the above apply to production economies with set-up costs and indivisibilities. In particular, the Shubik-Wooders result applies; for sufficiently large economies, the derived games have non-empty approximate cores.

For production and exchange economies without constant returns, including ones with set-up costs and capacities, under some conditions convergence of  $\epsilon$ -cores to equilibrium payoffs can be demonstrated (this result is not, at this point, in the literature but is fairly obvious).

In Wooders (1980) conditions are demonstrated under which economies with local public goods and endogenous jurisdiction formation have non-

empty approximate cores, and approximate "Tiebout-type" equilibria whose associated states of the economy are in the cores, and core-convergence results are obtained. This work is generalized in Wooders (1981b, 1982).

As we discussed in the introduction, the modeling of private goods exchange economies as games in characteristic function form is "natural" --the economies have the c-game property. For "coalition economies," cases 3-6, because production technologies are associated with groups and because exclusion is possible (and also may well be desirable from the point of view of optimality) in consumption of local public goods, we have the set-up of a c-game; from the point of view of the study of the core, the appropriate construction of the characteristic function is obvious.

For pure public goods, as we have argued, additional specifications must be made to construct the characteristic function. It needs to be specified who has access to what technology (one could use a coalition production model, for example) and what a coalition can consume of the public goods produced by the complementary coalition. We note that the case (7d), where the core has been most intensively studied, has a pessimistic characteristic function; the characteristic function is derived so that each coalition benefits only from the public goods it produces.

#### 4.1 The Firm and Replication

The discussion above covers cases one through six in Table 1 although cases 4, 6 and 7 are noted as posing modeling problems. The difficulties with these three cases can be illustrated by examining the alternative ways of replicating an extremely simple example.

We begin by utilizing a simple example to illustrate modeling difficulties finessed by the assumption of group production and how to construct a sequence of replicated economies given group production.

Suppose that an individual  $i$  has an initial endowment of  $(1, 0, s_i)$  where the first item is an individually owned good which can be used as an input to produce the second good. There is one firm which produces the second good with a production function given by  $y = \sqrt{x}$

where  $y$  is the output and  $x = \sum_{i=1}^n x_i$  is the total input. The utility

function of an individual  $i$  is given by  $U_i(x_i, y_i) = y_i$  where

$y = \sum_{i=1}^n y_i$ . The ownership share of the firm by individual  $i$  is  $s_i$  and

$\sum_{i=1}^n s_i = 1$ . Let  $p$  be the price of the output and fix the price of the

input as one.

The manager of the firm will attempt to maximize

$$\Pi = p\sqrt{x} - x \quad \text{where } 0 \leq x_i \leq 1$$

and each individual  $i$  will attempt to maximize

$$y_i \text{ subject to } x_i + s_i \Pi = p y_i$$

Thus  $x_i = 1$  and  $p = 2\sqrt{n}$ ; hence  $\Pi = n$  and  $y = \sqrt{n}$ .

Now  $0 \leq s_i \leq 1$ ; thus, depending upon the distribution of shares an individual  $i$  could obtain  $y_i$  where

$$\frac{1}{2\sqrt{n}} \leq y_i \leq \frac{n+1}{2\sqrt{n}}$$

Suppose that the original economy was of size  $n = 9$  with only one type of agent. If a type has both utility functions and endowments the same for all members we require that  $s_i = 1/9$ . At this point we have to decide how to describe the characteristic function for this game and furthermore we need to describe how to calculate the characteristic functions for the games arising from the replicated economy.

Alternative 1. Shares are ignored, any subgroup may use the technology. Here the game is immediately inessential, the characteristic function\* is given by  $f(s) = s$ ,  $s = 0, 1, \dots, 9$ .

This is equivalent to creating  $n$  independent technologies.

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\*For a type symmetric game instead of using the notation  $v(S)$  to stand for the amount obtained by the sets of agents we can use  $f(s)$  where  $s = |S|$  as all coalitions of the same size obtain the same amount.



Alternative 2. A simple majority at least is required to operate the firm. The game becomes

$$\begin{aligned} f(s) &= 0 & \text{for } s \leq n/2 \\ &= \sqrt{s} & \text{for } n/2 < s \leq n \end{aligned}$$

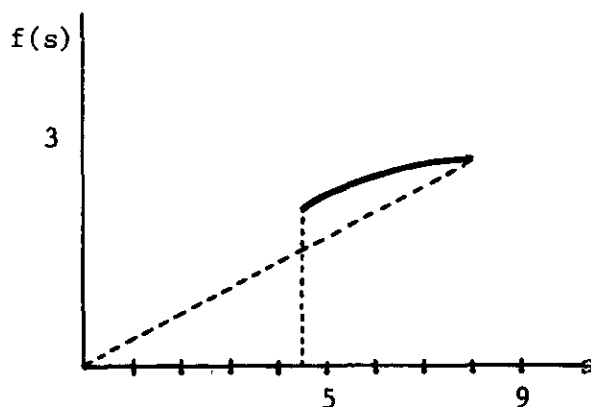


FIGURE 1

This game has no core as can be seen from Figure 1.

Alternative 3. Unanimity is required to operate the firm. The game becomes

$$\begin{aligned} f(s) &= 0 & \text{for } 0 \leq s \leq n-1 \\ &= \sqrt{n} & \text{for } s = n \end{aligned}$$

Here all imputations are in the core.

Alternative 4. Any simple majority can use the production function, but it can only take the product in proportion to its shares

$$\begin{aligned} f(s) &= 0 & \text{for } s \leq n/2 \\ &= \frac{s}{n}\sqrt{s} & \text{for } \frac{n}{2} < s \leq n \end{aligned}$$

It is easy to check that this has a core.

We now turn to replication. What do we mean by a replication of the 9 agents, 1 firm economy? We may consider at least two types of replication. In the first there is still one firm but we replicate the number of agents and shares. Thus the endowments in the  $r^{\text{th}}$  replication become  $(1, 0, 1/9r)$  where although all individuals may have the same number of shares as previously, the percentage ownership of the firm has gone down by  $1/r$ . In the second form of replication the firms are increased as well. At this point we may wish to consider the distinction between giving all agents a portfolio in each firm, or enlarging the number of types. For example, for one replication do we consider one type with  $(1, 0, 1/2n, 1/2n)$  where the last two entries indicate shares in the two firms or do we consider two types with  $(1, 0, 1/n, 0)$  and  $(1, 0, 0, 1/n)$  ?

Replication 1. There is only one firm, regardless of the replication number. The characteristic function becomes

$$\begin{aligned} f(s) &= 0 \quad \text{for } s \leq nr/2 \\ &= \sqrt{s} \quad \text{for } nr/2 < s \leq nr \end{aligned}$$

for the simple majority case. There is no core. Moreover, given any  $\epsilon > 0$  there is an  $n_0$  such that for all  $n$  greater than  $n_0$ , the game has an empty  $\epsilon$ -core. The sequence of games with  $n = 1, 2, \dots$  is not a sequence of replica games since the payoff to a given coalition eventually decreases as the set of players containing that coalition becomes large.

Replication 2a ("balanced portfolio")

$$f(s) = 0 \quad \text{for } s \leq nr/2$$

$$= r\sqrt{s/r} \quad \text{for } nr/2 < s \leq nr$$

This sequence of games has cores or  $\epsilon$ -cores and is a sequence of replica games.

Replication 2b (different types)

$$f(s_1, s_2, \dots, s_r) = p \sqrt{\frac{\sum_{j=1}^r \delta_j s_j}{p}} \quad \text{where } \delta_j = 1 \quad \text{if } s_j < n/2$$

$$= 0 \quad \text{if } s_j \leq n/2$$

$$\text{and } p = \sum_{j=1}^r \delta_j$$

This sequence of games has a core or  $\epsilon$ -core.

4.2. The Firm as a Local Public Good

We could reinterpret the example in 4.1 in terms of a local public goods and endogenous jurisdictions. Consider an economy of size  $n$  with a minimal size jurisdiction of 9 where each individual has an initial endowment of  $(1,0)$ , the first item is a private good which can be converted to a public good according to  $y = \sqrt{\sum x_i}$ . The utility function for each individual  $i$  is  $U_i(x_i, y, s) = y/s$  where  $s$  is the number

of individuals in a jurisdiction.

It is easy to see that this formulation leads to the formation of as many 9 person jurisdictions as possible. If  $n = 0 \pmod 9$  there is a core, otherwise an  $\epsilon$ -core. Furthermore implicit in this formulation is the idea that all individuals of the same type have an equal share of the jurisdiction to which they belong. Thus the analogue with the corporate economy can be completed by considering shares in jurisdictions.

#### 4.3. The Pure Public Good

The example noted in 5.2 can be extended to an example illustrating a pure public good by replacing  $U_i(x_i, y_i) = y_i$ , by  $U(x_i, y) = y$  where  $y = \sqrt{\sum_i x_i}$ .

Suppose that no subgroup smaller than 9 can produce. Then

$$\begin{aligned} f(s) &= 0 && \text{for } s < 9 \\ &= \sqrt{s} && \text{for } s \geq 9 \end{aligned}$$

This game has a large nonconverging core. Note also that the sequence of derived games does not satisfy the near-minimum efficient scale property.

## APPENDIX 1

In this appendix, the theorems in Section 2 of the paper are proven. Whenever possible, for the sake of brevity, we use results currently available for sequences of replica games. Also, throughout this appendix, the sequences of games are assumed to be sequences of per-capita bounded, superadditive replica games.

For ease in notation, given a payoff  $\alpha$  for a game  $(N_r, v_r)$  and  $S \subseteq N_r$  we write  $\alpha(S)$  for  $\sum_{tq \in S} \alpha^{tq}$  and, when  $S = \{(t,q)\}$  for some  $(t,q) \in N_r$ , we write  $\alpha(t,q)$  for  $\alpha(\{(t,q)\})$ .

Given  $r$  and a profile  $s$  of some subset  $S \subseteq N_r$ , a sub-profile  $s'$  of  $s$  is a member of  $I$  (the  $T$ -fold Cartesian product of the non-negative numbers) with  $s' \leq s$ . Let  $\underline{1}$  denote the vector of ones in  $I$ ; then, given  $r$ , we have  $\rho(N_r) = r\underline{1}$ .

When a sequence of replica games  $(N_r, v_r)_{r=1}^{\infty}$  is superadditive and per-capita bounded, it can be shown that for any  $r'$ , and any profile  $s \leq \rho(N_{r'})$ , the sequence of subgames  $(S_n, v_{nr'})_{n=1}^{\infty}$  is superadditive and per-capita bounded, where  $v_{nr'}$  is the characteristic function of the  $nr'$ th game and  $S_n$  is any subset of  $N_{nr'}$ , with profile equal to  $ns$ . Thus, to show that the sequence

$(N_r, v_r)_{r=1}^{\infty}$  is asymptotically totally balanced, we need only show that the sequence is asymptotically balanced, i.e.  $\lim_{r \rightarrow \infty} \frac{v_r(N_r)}{|N_r|} = \lim_{r \rightarrow \infty} \frac{\tilde{v}_r(N_r)}{|N_r|}$ .

Then every subsequence of subgames  $(S_n, v_{nr'})_{n=1}^{\infty}$  as above is asymptotically balanced.

Observe that from Wooders [(1981a), Lemma 8], the limits, as  $r$  goes to infinity, of both  $v_r(N_r)/r$  and  $\tilde{v}_r(N_r)/r$  exist and are equal. This, and our observations above, prove Theorem 1. Also, this theorem can easily be obtained as a consequence of Wooders [(1981a), Theorem 1].

### Proof of Theorem 2

Given  $\lambda$  and  $\delta$  greater than zero, select  $\epsilon^*$ ,  $r^*$ , and  $r' \leq r^*$  so that:

- (a)  $\frac{\epsilon^* r'}{r^*} < \frac{\lambda}{2}$  ;
- (b)  $\epsilon^* > 0$  and  $\epsilon^* < \min\left\{\frac{\lambda\delta}{4}, \frac{\delta}{2T}\right\}$  ;
- (c) for all  $r \geq r'$ ,  $\left\| \frac{v_r(N_r)}{rT} - \frac{\tilde{v}_r(N_{r'})}{r'T} \right\| \leq \epsilon^*$  .

Since  $\lambda > 0$  and  $\delta > 0$ , and since  $(N_r, v_r)_{r=1}^\infty$  is asymptotically balanced, such a selection is possible.

Select any  $r \geq r^*$  and let  $\alpha$  be in the  $\epsilon^*$ -core of  $(N_r, v_r)$ . For each  $t$ , define  $\bar{\alpha}_t$  by  $\bar{\alpha}_t = \alpha([t]_r)/r$  and let  $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_T)$ . It is easily verified that  $\bar{\alpha}$  represents an equal-treatment payoff in the  $\epsilon^*$ -core of  $(N_r, v_r)$  so  $r\bar{\alpha} \cdot \underline{1} \leq v(N_r)$  and  $\bar{\alpha} \cdot s \geq v_r(s) - \epsilon^* |s|$  for all subprofiles  $s$  of  $r\underline{1}$ .

Select a subset  $S'$  of  $N_r$  so that  $\rho(S') = \rho(N_r)$  and  $S'$  contains the "worst-off" players of each type; i.e. if  $(t, q) \in S'$ ,

then  $\alpha(t, q) \leq \alpha(t, q')$  for all  $(t, q') \notin S'$ . Suppose that  $|S' \cap \{(t', q) \in [t']_r : \alpha(t', q) < \bar{\alpha}_{t'} - \delta\}| = r'$  for some type  $t'$ ; i.e. all players of type  $t'$  in  $S'$  receive less than the average payoff for players of that type minus  $\delta$ .

$$\text{We then have } \tilde{v}_r(S') - \epsilon^* r' T \leq \alpha(S') < r' \bar{\alpha}_{t'} - \delta r' \leq r' \frac{v(N_r)}{r} - \delta r'$$

The first inequality follows from the fact that  $\alpha$  is in the  $\epsilon$ -core of  $(N_r, v_r)$ . The second follows from the facts that  $r' \bar{\alpha}_{t'} \geq \alpha(S' \cap [t']_r)$  for each  $t$  and  $\alpha(S' \cap [t']_r) \leq r' \bar{\alpha}_{t'} - r' \delta$ . The final inequality is from the feasibility of  $\alpha$ ; we have  $\alpha(N_r)/r = \bar{\alpha}_{t'} \leq v(N_r)/r$ .

We now have

$$\tilde{v}_r(S') - \epsilon^* r' T < r' \frac{v(N_r)}{r} - \delta r'.$$

Subtracting  $\tilde{v}_r(S')$  from both sides of the expression, adding  $\delta r'$  to both sides, and dividing by  $r' T$ , we obtain

$$\frac{\delta}{T} - \epsilon^* < \frac{v(N_r)}{rT} - \frac{\tilde{v}_r(S')}{r'T}.$$

From the fact that  $\rho(S') = \rho(N_r)$ , we have  $\tilde{v}_r(S') = \tilde{v}_r(N_r)$ .

From (b) above,  $\frac{\delta}{T} - \epsilon^* > \epsilon^*$ , which implies  $\epsilon^* < \frac{v(N_r)}{rT} - \frac{\tilde{v}_r(N_r)}{r'T}$ , a contradiction to (c). Therefore, for each  $t' = 1, \dots, T$  we have  $|S' \cap \{(t', q) \in [t']_r : \alpha(t', q) < \bar{\alpha}_{t'} - \delta\}| < r'$ .

From the facts that:

$$\alpha(S') \geq \tilde{v}_r(S') - r'\epsilon^* T ,$$

$$\frac{v_r(N_r)}{r} \geq \bar{\alpha} \cdot 1 ;$$

and

$$\frac{\tilde{v}_r(N_r)}{r'} = \frac{\tilde{v}_r(S')}{r'} \geq v_r \frac{(N_r)}{r} - \epsilon^* T$$

we have

$$0 \leq \bar{\alpha} \cdot 1 - \frac{\alpha(S')}{r'} \leq 2\epsilon^* T .$$

Informally, this expression says that, on average, players of type  $t$  in  $S'$  are receiving payoffs within  $2\epsilon^*$  of  $\bar{\alpha}_t$  for each  $t$ .

Define  $V = \{(t,q) \in N_r : \alpha(t,q) > \bar{\alpha}_t + \delta\}$  and

$W = \{(t,q) \in N_r : (t,q) \notin V \cup S'\}$  where  $S'$  is as defined above.

Observe that

$$\begin{aligned} \delta|V| &\leq \sum_{(t,q) \in V} (\alpha(t,q) - \bar{\alpha}_t) \\ &= \sum_{tq \in S' \cup W} (\bar{\alpha}_t - \alpha(t,q)) \text{ since} \\ \sum_{tq \in N_r} (\alpha(t,q) - \bar{\alpha}_t) &= 0 . \end{aligned}$$

From the preceding paragraph and the above expression,

$$\delta|V| \leq \epsilon^* r' T + \sum_{tq \in W} (\bar{\alpha}_t - \alpha(t,q)) .$$

Obviously, the larger the value of  $\sum_{tq \in W} (\bar{\alpha}_t - \alpha(t,q))$ , the larger

it is possible for  $|V|$  to be and still satisfy the above expression.

We claim that  $\sum_{tq \in W} (\bar{\alpha}_t - \alpha(t,q)) \leq 2\epsilon^* |W|$ . This follows from the fact

that the players in  $S'$  of type  $t$  are, on average, within  $\epsilon^*$  of  $\bar{\alpha}_t$  and  $S'$  contains the "worst-off" players of each type. Since



the players in  $W$  are all "better-off" than those in  $S'$ , we have

$$\sum_{tq \in W} (\bar{\alpha}_t - \alpha(t,q)) \leq \epsilon^* |W|. \text{ Therefore } \delta |V| \leq 2\epsilon^* r'T + 2\epsilon^* |W|. \text{ Since } |W| + r'T \leq rT, \text{ we have } \frac{|V|}{rT} \leq \frac{2\epsilon^*}{\delta}.$$

Then  $|\{(t,q) \in N_r : \|\alpha^{tq} - \bar{\alpha}_t\| > \delta\}|/rT \leq \frac{|S'|}{rT} + \frac{|V|}{rT} \leq \frac{\epsilon^* r'}{r} + \frac{2\epsilon^*}{\delta} < \lambda$  from (b) above.

The conclusion of the theorem follows from the observation that if  $\alpha$  is in the  $\epsilon$ -core of  $(N_r, v_r)$  for  $r \geq r^*$  and  $0 \leq \epsilon \leq \epsilon^*$ , then  $\alpha$  is in the  $\epsilon^*$ -core of  $(N_r, v_r)$ .

Q.E.D.

Before proving the following theorem, we require some additional notation. (See Hildenbrand (1974), p. 15 for the definitions of the terms used.) Let  $Li(B(\epsilon))$  be the lim inf of the sequence of sets  $(B_r(\epsilon))_{r=1}^{\infty}$  and let  $Ls(B(\epsilon))$  denote the lim sup. The closed limit  $L(B(\epsilon))$  exists if and only if  $Ls(B(\epsilon)) = Li(B(\epsilon))$ , in which case  $L(B(\epsilon)) = Li(B(\epsilon)) = Ls(B(\epsilon))$ . Similarly,  $Li(C)$  and  $Ls(C)$  denote the lim inf and the lim sup of  $C(r)$  respectively. Recall that for any sequence of the sets, the lim inf is contained in the lim sup.

Since the sets  $C(r)$  and  $B_r(\epsilon)$  are all contained in a separable metric space,  $\mathbb{R}^T$  with the sup norm, both  $Ls(C)$  and  $Ls(B(\epsilon))$ , for any  $\epsilon$ , are non-empty.\*

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\*See Hildenbrand (1974, p. 16). This can also be observed directly from Wooders [(1981a), the beginning of the proof of Theorem 2].

Theorem 6. Given  $\varepsilon > 0$ , the closed limit of the sequence  $(\mathcal{B}_r(\varepsilon))_{r=1}^{\infty}$  exists.

Proof. Let  $\alpha \in \text{Ls}(\mathcal{B}(\varepsilon))$ . It is easily verified that  $\lim_{r \rightarrow \infty} \frac{v(N_r)}{r} - \varepsilon T \leq \alpha \cdot \underline{1} \leq \lim_{r \rightarrow \infty} \frac{v(N_r)}{r}$ . We first consider the case where  $\alpha \cdot \underline{1} < \lim_{r \rightarrow \infty} \frac{v(N_r)}{r}$ .

Let  $(\mathcal{B}_{r_q}(\varepsilon))_{r_q=1}^{\infty}$  be a subsequence of  $(\mathcal{B}_r(\varepsilon))_{r=1}^{\infty}$  such that for some  $\alpha_{r_q} \in \mathcal{B}_{r_q}(\varepsilon)$  for each  $r_q$ , we have  $\alpha_{r_q} \rightarrow \alpha$  as  $r_q \rightarrow \infty$ . Since, for each  $r_q$ ,  $\alpha_{r_q} \cdot s \geq v_{r_q}(s) - \varepsilon |s|$  for all subprofiles  $s$  of  $r_q \underline{1}$  and for all  $r_q$ , we have  $\alpha \cdot s \geq v_r(s) - \varepsilon |s|$  for all subprofiles  $s$  of  $r \underline{1}$  for all  $r$ . Since  $\alpha \cdot \underline{1} < \lim_{r \rightarrow \infty} \frac{v(N_r)}{r}$ , for all  $r$  sufficiently large  $\alpha \cdot \underline{1} \leq v_r(N_r)$ . Therefore  $\alpha \in \mathcal{B}_r(\varepsilon)$  for all  $r$  sufficiently large so  $\alpha \in \text{Li}(\mathcal{B}(\varepsilon))$ .

Suppose  $\alpha \cdot \underline{1} = \lim_{r \rightarrow \infty} \frac{v(N_r)}{r}$ . Let  $\alpha' \in \text{Ls}(C)$ . Observe that given any  $\lambda \in (0,1)$ , we have  $\lambda(\alpha' - \varepsilon \underline{1}) \cdot \underline{1} + (1-\lambda)\alpha \cdot \underline{1} \leq \lim_{r \rightarrow \infty} \frac{v(N_r)}{r} - \lambda \varepsilon T$ . Thus, for all  $r$  sufficiently large,  $\lambda(\alpha' - \varepsilon \underline{1}) \cdot \underline{1} + (1-\lambda)\alpha \cdot \underline{1} \leq \frac{v_r(N_r)}{r}$ . Also, since  $\alpha' \cdot s \geq v_r(s)$  and  $\alpha \cdot s \geq v_r(s) - \varepsilon |s|$  for all subprofiles  $s$  of  $r \underline{1}$  and for all  $r$ , we have  $(\lambda(\alpha' - \varepsilon \underline{1}) + (1-\lambda)\alpha) \cdot s \geq \lambda \alpha' \cdot s + (1-\lambda)\alpha \cdot s - \lambda \varepsilon \underline{1} \cdot s \geq v_r(s) - \varepsilon |s|$  for all subprofiles  $s$  of  $r \underline{1}$  and for all  $r$ . Therefore given  $\lambda \in (0,1)$ , for all  $r$  sufficiently large,  $\lambda(\alpha' - \varepsilon \underline{1}) + (1-\lambda)\alpha$  is in  $\mathcal{B}_r(\varepsilon)$ . Consider the sequence  $(\lambda_n)_{n=1}^{\infty}$  where  $\lambda_n = 1 - \frac{1}{n}$ , so  $\lambda_n \rightarrow 1$  and  $\lambda_n(\alpha' - \varepsilon \underline{1}) + (1-\lambda_n)\alpha$  converges to  $\alpha$  as  $n \rightarrow \infty$ . For each  $n$ , select  $r(n)$  sufficiently large so

that all  $r \geq r(n)$ ,  $\lambda_n(\alpha' - \epsilon \underline{1}) + (1 - \lambda_n)\alpha$  is in  $B_r(\epsilon)$ . Define  $\alpha_r = \lambda_n(\alpha' - \epsilon \underline{1}) + (1 - \lambda_n)\alpha$  for all  $r$  with  $r(n) \leq r < r(n+1)$ . Then  $\alpha_r \in B_r(\epsilon)$  for each  $r$  and  $\alpha_r \rightarrow \alpha$  so  $\alpha$  is in  $Li(B(\epsilon))$ .

Q.E.D.

Since  $L(B(\epsilon)) \subseteq L(B(\epsilon'))$  whenever  $\epsilon' > \epsilon$  and each  $L(B(\epsilon))$  is non-empty and compact for  $\epsilon > 0$ , the collection of sets  $\{L(B(\epsilon)) : \epsilon > 0\}$  has the finite intersection property; therefore  $A^* \neq \emptyset$ .

### Proof of Theorem 3

First we show that  $L(C)$  exists. Let  $r^*$  denote an MES bound for the sequence of games.

Given  $r > r^*$ , let  $\alpha \in C(r+1)$ ; we claim  $\alpha \in C(r)$ . Since  $\alpha \cdot s \geq v(s)$  for all subprofiles  $s$  of  $(r+1)\underline{1}$ , we have  $\alpha \cdot s \geq v(s)$  for all subprofiles of  $s$  of  $r\underline{1}$ . Also, since  $\frac{v_r(N_r)}{r} = \frac{\tilde{v}_{r+1}(N_{r+1})}{r+1} = \alpha \cdot \underline{1}$ , we have  $\tilde{v}_r(N_r) = r\alpha \cdot \underline{1}$  so  $\alpha \in C(r)$ . Therefore  $C(r+1) \subset C(r)$ . Since  $C(r)$  is compact and non-empty for each  $r$ ,  $\bigcap_{r=1}^{\infty} C(r) \neq \emptyset$ . Given these observations, it is routine to verify that  $\bigcap_{r=1}^{\infty} C(r) = L(C)$  so  $L(C) \neq \emptyset$ .

Let  $\alpha \in L(C)$ . Given  $\epsilon > 0$ , observe that  $\alpha \cdot s \geq v_r(s) - \epsilon |s|$  for all subprofiles  $s$  of  $r\underline{1}$  and for all  $r$ . Since

$$\lim_{r \rightarrow \infty} \frac{\tilde{v}_r(N_r)}{r} = \lim_{r \rightarrow \infty} \frac{v_r(N_r)}{r} = \alpha \cdot \underline{1}, \text{ for all } r \text{ sufficiently large,}$$

$\alpha - \epsilon \underline{1} \in B_r(\epsilon)$  . Since this holds for any  $\epsilon > 0$  ,  $\alpha \in \bigcap_{\epsilon > 0} L(B(\epsilon)) = A^*$  .

Therefore  $L(C) \subseteq A^*$  .

Suppose the sequence  $(N_r, v_r)_{r=1}^{\infty}$  has the MES property with MES bound  $r^*$  . Let  $\alpha \in A^*$  . Since  $\frac{\tilde{v}_r(N_r)}{r} \geq \frac{v_r(N_r)}{r}$  for all  $r$  and since  $\frac{\tilde{v}_r(N_r)}{r} = \frac{\tilde{v}_{r'}(N_{r'})}{r'}$  for all  $r$  and  $r'$  sufficiently large,  $\lim_{r \rightarrow \infty} \frac{v_r(N_r)}{r} = \frac{\tilde{v}_{r'}(N_{r'})}{r'}$  for all  $r'$  sufficiently large. Therefore  $\alpha \cdot \underline{1} \leq \frac{\tilde{v}_r(N_r)}{r}$  for all sufficiently large  $r$  . Consequently, if  $\alpha \notin L(C)$  , for all  $r$  sufficiently large, for some  $S \subseteq N_r$  we have  $\alpha \cdot s < v_r(S)$  where  $s = \rho(S)$  . But then for some  $\epsilon$  sufficiently small,  $\alpha \cdot s < v_r(S) - \epsilon |S|$  which contradicts the assumption that  $\alpha$  is in  $A^*$  .

Q.E.D.

## APPENDIX II

In this appendix, Theorems 4 and 5 are proven.

Proof of Theorem 4

To prove the theorem, we need only prove that the sequence of derived games is superadditive and per-capita bounded. It is straightforward to verify that the sequence is superadditive; thus we omit the proof.

To prove per-capita boundedness of the sequence of games, we construct another sequence of economies, say the  $*$ -economies, so that the sequence of games derived from the  $*$ -economies, denoted by  $(N_r, v_r^*)_{r=1}^{\infty}$ , is per-capita bounded and has the property that for all  $r$  and for all non-empty subsets  $S$  of  $N_r$ , we have  $v_r(S) \leq v_r^*(S)$ .

For the  $*$ -economies, we let  $Y^*$  be the private goods production possibility set available to all coalitions  $S \subseteq N_r$  for all  $r$ . Observe that for any firm structures, say  $F_r(S)$  and  $F'_r(S)$  we have

$$\sum_{S' \in F_r(S)} Y^* = \sum_{S' \in F'_r(S)} Y^*$$

so the firm structure will be irrelevant. Of course,  $Y_r(S) \subseteq Y^*$

and for any firm structure  $F(S)$ ,  $\sum_{S' \in F(S)} Y^* \subseteq Y^*$ .

Given  $r$  and a subset  $S$  of  $N_r$ , define  $Z^*(S) = \{(x, z) \in \mathbb{R}^{m+l} : (|S|x, z) \in Z^*\}$ . Note that  $Z^*(S)$  is a closed convex cone with vertex

$\{0\}$  and  $\{0\} \times \mathbb{R}_+^l \subseteq Z^*(S)$  ; this follows from the assumptions on  $Z^*$  . Let  $\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') : S' \in J(S)\})$  be an  $S$ -public goods production plan. From assumption A.2, there is an  $(x, z) \in Z^*(S)$  such that

$$(a) \text{ for each } (t, q) \in S, \text{ we have } \sum_{\{S' \in J(S) : (t, q) \in S'\}} x_{S'} \leq x$$

and

$$(b) \sum_{S' \in J(S)} z_{S'} \leq z .$$

Informally, there is an  $(x, z)$  in  $Z^*(S)$  "at least as good as" any  $S$ -public goods production plan in the sense that with the production possibility set  $Z^*(S)$  , the agents can consume as much of the local public goods while using no more of the inputs. (Note, however, because agents do not necessarily have monotonic increasing preferences for the local public goods, they may not prefer to have more of them.)

For the purposes of this theorem, we can restrict our attention to states of the  $*$ -economies with associated jurisdiction structures  $\{(t, q) : (t, q) \in N_r\}$  since with this jurisdiction structure all agents can be made "at least as well-off" as with any other jurisdiction structure. To see this, given any  $r$  and any non-empty subset  $S$  of  $N_r$  , let  $(x, z) \in Z^*(S)$  so  $(|S|x, z) \in Z^*$  and  $(x, z/|S|) \in Z^*(\{(t, q)\})$  for each  $(t, q) \in S$  ; thus with the jurisdiction structure  $\{(t, q) : (t, q) \in N_r\}$  each agent in  $S$  can consume  $x$  and total inputs are unchanged.

In the  $*$ -economies, the agents will all have the same utility

functions. For each  $r$  and for all  $(t,q) \in N_r$ , define  $u^{*tq}(x,y) = L(x,y)$  where  $L$  is a linear function such that  $u^{tq}(x,y) \leq L(x,y) + c$  for some constant  $c$  for all  $(x,y) \in \mathbb{R}_+^m + \mathbb{R}_+^k$  and for all  $(t,q) \in N_1$ .

For each  $r$  and each non-empty subset  $S$  of  $N_r$ , let  $A_r^*(S)$  denote the set of  $S$ -attainable allocations for the  $*$ -economies.

For each  $r$  and all non-empty subsets  $S$  of  $N_r$ , define

$$v_r^*(S) = \sup_{(x,y) \in A_r^*(S)} \sum_{(t,q) \in S} L^*(x^{tq}, y^{tq}) .$$

The finiteness of this "sup" is ensured by the linearity of the function  $L^*$  and boundedness of  $A_r^*(S)$ . Also,

$$v_r^*(S) = \sup_{(x,y) \in A_r^*(S)} L\left(\sum_{tq \in S} x^{tq}, \sum_{tq \in S} y^{tq}\right) .$$

Let  $K$  be a real number such that  $K > v_1^*(N_1)$ . We will show that  $K$  is a per-capita bound for the sequence of games  $(N_r, v_r^*)_{r=1}^\infty$ . Suppose not. Then there is an  $r'$  and a feasible state of the  $r'$ <sup>th</sup>  $*$ -economy, say  $e^*(N_{r'}) = (\bar{y}, (J(N_{r'}), \{(x_{S^t}, z_{S^t}) \in Z^*(S^t) : S \in J(N_{r'})\}))$ ,  $(x,y)$ , such that  $L\left(\sum_{tq \in N_{r'}} x^{tq}, \sum_{tq \in N_{r'}} y^{tq}\right) > Kr'$ . We can assume without any loss that  $F(N_{r'}) = \{N_{r'}\}$  and  $J(N_{r'}) = \{(t,q) : (t,q) \in N_{r'}\}$ . Also, we can assume that  $x^{tq} = x^{t'q'}$  and  $y^{tq} = y^{t'q'}$  for all  $(t,q)$

and  $(t', q')$  in  $N_r$ . We claim that there is an  $(x', y') \in A_1^*(N_1)$  with  $x'^{tq} = x^{tq}$  and  $y'^{tq} = y^{tq}$  for all  $(t, q) \in N_1$ , which will yield a contradiction. Since  $e^*(N_r)$  is feasible (for the  $r^{\text{th}}$  \*-economy), we have  $\bar{y} \in Y^*$ . Observe that  $\bar{y}/r' \in Y^*$ . Also, for some  $z^{tq}$  for each  $(t, q) \in N_r$ , we have  $(x^{tq}, z^{tq}) \in Z^*({(t, q)})$ , and

$$\sum_{(t, q) \in N_r} (y^{tq} - w^{tq}) + \sum_{(t, q) \in N_r} z^{tq} = \bar{y} \quad \text{so} \quad \sum_{(t, q) \in N_1} (y^{tq} - w^{tq}) + \sum_{(t, q) \in N_1} z^{tq} = \bar{y}/r' . *$$

This proves our assertion that  $(x', y') \in A_1^*(N_1)$ . But then

$$L \left( \sum_{(t, q) \in N_1} x'^{tq}, \sum_{(t, q) \in N_1} y'^{tq} \right) = \frac{1}{r'} L \left( \sum_{(t, q) \in N_r} x^{tq}, \sum_{(t, q) \in N_r} y^{tq} \right) > K$$

which is a contradiction. Therefore the sequence  $(N_r, v_r)_{r=1}^{\infty}$  is per-capita bounded.

$$\text{Since } u^{tq}(x, y) \leq L(x, y) + c \quad \text{for all } (x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^l ,$$

and from the construction of the production possibilities set for the \*-economies, we have  $v_r(N_r) \leq v_r^*(N_r) + c|N_r|$  for all  $r$ . Therefore the sequence  $(N_r, v_r)_{r=1}^{\infty}$  is per-capita bounded.

Q.E.D.

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\*The vector  $z^{tq}$  can be taken equal to  $z_S$ , when  $\{(t, q)\} = S'$ , where  $S' \in J(N_r)$ .



Proof of Theorem 5

Given  $\delta > 0$ ,  $r^*$  satisfying the conditions of A3, and  $r' = nr^*$  for some positive integer  $n$ , there is an  $(x, y) \in A_{r'}(N_{r'})$  such that

$$v_{r'}(N_{r'}) \leq \sum_{tq \in N_{r'}} u^{tq}(x^{tq}, y^{tq}) + \delta .$$

From A3, there is an  $(x^*, y^*) \in A_{r^*}(N_{r^*})$  such that

$$\sum_{tq \in N_{r'}} u^{tq}(x^{tq}, y^{tq}) \leq n \sum_{tq \in N_{r^*}} u^{tq}(x^{tq}, y^{tq}) .$$

Therefore, it follows that

$$\frac{v_{r'}(N_{r'})}{r'} \leq \frac{v_{r^*}(N_{r^*})}{r^*} .$$

Since this holds for all positive integers  $n$ , we have

$$\lim_{r \rightarrow \infty} \frac{v_r(N_r)}{r} = \frac{v_{r^*}(N_{r^*})}{r^*}$$

Since

$$\lim_{r \rightarrow \infty} \frac{\tilde{v}_r(N_r)}{r} = \lim_{r \rightarrow \infty} \frac{v_r(N_r)}{r}$$

we have

$$\lim_{r \rightarrow \infty} \frac{\tilde{v}_r(N_r)}{r} = \frac{v_{r^*}(N_{r^*})}{r^*} .$$

Since  $\frac{v_r(N_r)}{r} \leq \frac{\tilde{v}_r(N_r)}{r}$  for all  $r$  and since  $\frac{\tilde{v}_r(N_r)}{r}$  is non-decreasing, we have

$$\frac{\tilde{v}_r(N_r)}{r} = \frac{\tilde{v}_{r^*}(N_{r^*})}{r} \text{ for } r \geq r^* .$$

Q.E.D.

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